Abstract

Equivariant degree $\deg_G$ is a useful tool that detects zeros of equivariant local maps. It works similarly to the classical Brouwer degree: if its value is non-zero, then a given map has zeros. While the evaluation of equivariant degree can sometimes be complicated, its structure is richer in comparison with the classical degree. The values of $\deg_G$ are not integers, but elements of the Burnside ring, and they consist of multiple coefficients. In many cases, $\deg_G$ is more efficient than the Brouwer degree, taking non-zero values where the latter is equal to zero. Moreover, the analysis of non-zero coefficients of $\deg_G$ allows one to draw conclusions about specific orbit types occuring in the set of zeros. For the even more narrow class of maps that are both equivariant and gradient, the equivariant gradient degree theory was developed, with degree $\deg_G^\nabla$ having even richer structure and taking values in the Euler-tom Dieck ring. Over the last few decades, numerous applications and generalizations of both of these degrees have been explored.

The aim of the thesis is to develop two selected aspects from theory of these degrees. The first one is related to the product property of degree $\deg_G$. This property was proved in the general case of a compact Lie group by Kazimierz Gęba, Wiesław Krawcewicz and Jianhong Wu in a paper from 1994, but the proof introduced there was rather formal and sketchy in some parts. We present a new, complete proof of the product formula in the case where the compact Lie group G is finite or abelian. The second aspect discussed in the thesis is the introduction of a new version of equivariant gradient degree in a Hilbert space. The main motivation behind development of this invariant was the possibility of applying it to various problems of nonlinear analysis, such as finding periodic solutions of Hamiltonian systems.

Chapter 1 includes preliminaries: basic definitions about equivariant maps and the degree $\deg_G$ along with elements of vector bundle theory. These will be applied in further chapters focused on the product formula. In Chapter 2 we present the proof of that formula for the degree $\deg_G$ in the case where G is a finite group. It is based on

a new notion of polystandard maps and on a Hopf type theorem for equivariant local maps, proved by P. Bartłomiejczyk in his 2017 paper. In Chapter 3 we prove the product property under the action of a compact abelian Lie group, using a similar approach with necessary adjustments.

The next two chapters regard a new infinite-dimensional version of the degree. Namely, we present a construction of a degree in a Hilbert space for equivariant, gradient, compact perturbations of an equivariant, unbounded, self-adjoint operator with purely discrete spectrum. The invariant, denoted by Deg_G^∇, takes values in the Euler-tom Dieck ring of G. In Chapter 4 we introduce the degree definition based on finite-dimensional approximations, and then we show that Deg_G^∇ meets the usual conditions required from degree-type invariants, such as additivity, otopy invariance, existence property, normalization and product property. Chapter 5 starts with a discussion of the problem of finding periodic orbits of Hamiltonian systems, along with the historical background and a description of previous results from that area of research. Later in the chapter we explain how to transform the ODE problem to the issue of determining zeros of a certain nonlinear operator in Hilbert space. Such transformation opens a possibility of applying our equivariant gradient degree Deg_G^∇.

In the following, final chapter, we propose several hypotheses naturally arising from the previously discussed results. They pave the ways of possible future research on the problems presented in the thesis.