Summary of Professional Accomplishments

Dr Marcin Marek Markiewicz

December 14, 2021

Contents

1 Name and surname

Marcin Marek Markiewicz

2 Diplomas, scientific degrees

- **PhD in Physics** – September 2014.
  Institution: Faculty of Mathematics, Physics and Informatics, University of Gdansk.
  PhD Thesis: Characterization and detection of multipartite entanglement.
  Supervisor: prof. dr hab. Marek Żukowski.
  Reviewers: prof. dr hab. Marek Kuś, prof. dr hab. Antoni Wójcik.
  Date and place of diploma examination: 02.09.2014, Gdańsk.
  Date and place of the conferment of PhD degree: 25.09.2014, Gdańsk.
  Grant: International PhD project: *Physics of future quantum-based information technologies*, grant MPD/2009-3/4 from Foundation for Polish Science.

- **MSc in Physics** – July 2009.
  Institution: Faculty of Mathematics, Physics and Informatics, University of Gdansk.
  Master's Thesis: Transfer of quantum states through spin chains.
  Supervisor: prof. dr hab. Marek Żukowski.
  Grade: very good.

3 Information on previous employment in scientific institutions

- 01.01.2020 – present: **Adiunkt**
  Institution: International Centre for Theory of Quantum Technologies, University of Gdansk;
  Supervisor: prof. Marek Żukowski.
  Research topic: quantum optics, foundations of quantum mechanics;

- 01.10.2015 – 30.09.2018: **Adiunkt**
  Institution: Institute of Physics, Faculty of Physics, Astronomy and Applied Computer Science, Jagiellonian University in Kraków;
  Supervisor: prof. Karol Życzkowski.
  Research topic: quantum correlations in higher-level quantum systems;

- 01.07.2015 – 30.06.2016: **Adiunkt**
  Institution: Centre for Theoretical Physics, Polish Academy of Sciences, Warsaw;
  Supervisor: prof. Marek Kuś.
  Research topic: correlations in generalised probabilistic theories;

- 01.07.2014 – 30.06.2015: **Research Assistant**
  Institution: Institute of Theoretical Physics, Faculty of Physics, University of Warsaw;
  Supervisor: dr hab. Rafał Demkowicz-Dobrzański.
  Research topic: quantum metrology, quantum algorithms;

4 Description of the achievements according to Art. 219 Para 1 Point 2 of the Act

4.1 Title of the achievement

Single-themed series of publications, titled *Signatures of nonclassicality in quantum mechanical and optical systems*.

4.2 List of selected publications

The list of publications related thematically:

1. *Efficient linear optical generation of a multipartite W state*
   Paweł Błasiak, Ewa Borsuk, Marcin Markiewicz, Yong-Su Kim
   Physical Review A **104**, 023701, Aug 2021.

2. *Can single photon excitation of two spatially separated modes lead to a violation of Bell inequality via weak-field homodyne measurements?*
   Tamoghna Das, Marcin Karczewski, Antonio Mandarino, Marcin Markiewicz, Bianka Wołoncewicz, Marek Zukowski
   New Journal of Physics **23**(7), 073042, Jul 2021.

3. *Entangling three qubits without ever touching*
   Paweł Błasiak, Marcin Markiewicz
   Scientific Reports **9**(1), 20131, Dec 2019.

4. *From contextuality of a single photon to realism of an electromagnetic wave*
   Marcin Markiewicz, Dagomir Kaszlikowski, Paweł Kurzyński, Antoni Wójcik
   npj Quantum Information **5**(1), 5, Jan 2019.

5. *Unified approach to geometric and positive map-based nonlinear entanglement identifiers*
   Marcin Markiewicz, Adrian Kołodziejski, Zbigniew Puchała, Adam Rutkowski, Tomasz Tylec, Wiesław Laskowski
   Physical Review A **97**, 042339, Apr 2018.

6. *Generalized probabilistic description of noninteracting identical particles*
   Marcin Karczewski, Marcin Markiewicz, Dagomir Kaszlikowski, Paweł Kurzyński
   Physical Review Letters **120**, 080401, Feb 2018.

7. *Investigating nonclassicality of many qutrits by symmetric two qubit operators*
   Marcin Markiewicz, Kamil Kostrzewa, Adrian Kołodziejski, Paweł Kurzynski, Wiesław Laskowski
   Physical Review A **94**, 032119, Sep 2016.

8. *Probing the quantum–classical boundary with compression software*
   Hou Shun Poh, Marcin Markiewicz, Paweł Kurzynski, Alessandro Cerè, Dagomir Kaszlikowski, Christian Kurtsiefer
   New Journal of Physics **18**(3), 035011, Mar 2016.

9. *Correlation-based entanglement criterion in bipartite multiboson systems*
   Wiesław Laskowski, Marcin Markiewicz, Danny Rosseau, Tim Byrnes, Kamil Kostrzewa, Adrian Kołodziejski
   Physical Review A **92**, 022339, Aug 2015.

5 Presentation of significant scientific activity

My scientific achievement has a form of collective publications. My own contribution has been described in section I.2 of enclosed document "List of scientific or artistic achievements which present a major contribution to the development of a specific discipline", whereas the contribution statements of other co-authors are enclosed in a separate document "Contribution Statements".

In the presentation the following convention for references is used:

- the publications which belong to the presented scientific achievement are referred to as [H1]-[H9],

- other publications co-authored by me, but not included in the achievement, are cited as [O1]-[O24],

- external publications are cited as [E1]-[E73].

5.1 Introduction

Quantum theory significantly differs from classical physics both on the conceptual as well as mathematical level. The term *nonclassicality* refers to phenomena that arise in the context of quantum theory, and which do not have well defined counterparts in classical physics. The most important nonclassical phenomena, which will be discussed in this presentation of my scientific achievement are: *quantum entanglement*, *fundamental indistinguishability of particles*, *Bell nonclassicality* and *contextuality*. First two are explicitly encoded in quantum mechanical formalism, whereas the other two are formulated in a probabilistic language *outside* quantum theory, and characterise it in a negative way, by showing that some intuitive concepts stay in odds with quantum predictions. All these signatures of nonclassicality have a common trait, namely they are related to correlations of measurement outcomes. On the other hand correlations are crucial for information-theoretic investigations, and this is the reason why all these features are central object of interest for quantum information science. Below I present a brief description of all the four signatures of nonclassicality.

Quantum entanglement [E33] is a phenomenon, in which locally incompatible properties of two or more physical objects are strongly correlated. The fact that the correlated properties correspond to incompatible observables implies that their correlations have to be understood in a partially *counterfactual* sense, namely had the observers measured another tuple of observables that they *actually* had done, they would have also observed strong correlations. This counterfactual character of quantum correlations is a consequence of quantum complementarity. On the formal level entanglement is encoded in the inseparability of the entire state describing a multipartite system. The consequence of this inseparability is that physical systems in entangled states have well defined global properties (correlations), and at the same time their local properties are blurred (the local outcomes exhibit intrinsic randomness).

Indistinguishability of quantum particles [E7] is a fundamental concept in quantum mechanics, in which quantum particles with the same values of all internal degrees of freedom are *fundamentally* indistinguishable. This means that they cannot be individually addressed neither experimentally nor theoretically, by means of any hidden labeling. In the context of quantum optics indistinguishability of photons gives rise to strongly nonclassical interference phenomena. The most typical phenomenon is the Hong-Ou-Mandel effect

[E30], which happens if two indistinguishable photons enter the two input ports of a symmetric beamsplitter. One observes that at the output of the beamsplitter the photons come always in pairs, and no coincidence detection is observed. This effect is also referred to as photon bunching, and happens, because the quantum amplitudes corresponding to two photons going each through different output ports cancel each other. Another type of interference effect due to indistinguishability of photons has been proposed by Yurke and Stoler [E73, E72], in which an interference of photons from completely independent sources leads to correlations equivalent to the ones of entangled states.

Bell nonclassicality [E12] (often confusingly referred to as *nonlocality*) is a well established concept in the field of foundations of quantum mechanics, which states that correlations arising due to local measurements on a multipartite quantum system distributed in space cannot be reconstructed using classical probabilistic description. This description assumes: *realism* (measured physical quantities can be represented as classical random variables), *locality* (no communication between the local subsystems) and *free choice* of the measurement settings (choice of settings is decorrelated with any physical process in the common past of the experiment). Bell nonclassicality is typically demonstrated by showing violation of some Bell inequality. A Bell inequality is satisfied by any correlations arising due to processes that fulfill *joint* assumptions of realism, locality and free choice.

Contextuality [E13] is a similar but more general concept than Bell nonclassicality. It can be thought of as equivalent to a failure of the assumption of realism in a Bell nonclassicality approach, with the difference that it can be formulated with respect to a single system, without the need for partitioning it into subsystems. A physical system describable by a set of observables $\{A_i\}$ is *noncontextual*, if there exists a joint probability distribution for outcomes of these observables, which operationally means that they are jointly commeasureable. Otherwise it is *contextual*, which means that there are only subsets of observables $\{C_i\}$, called *measurement contexts* for which there exists a joint probability distribution, however these distributions cannot be extended to a single one, of whom they would be marginals.

5.2 Motivation and goals

A significance of presented nonclassical features of quantum theory manifests itself in two different aspects, the foundational investigations about quantum theory and practical applications known under the term *quantum technologies*. What is very interesting, these two aspects mutually intertwine, and this intertwining states behind the establishment of a new interdisciplinary branch known as *quantum information science*. In this way presence of *quantum entanglement* between physical systems is a necessary condition for observing Bell nonclassicality, on the other hand it is also necessary in the field of quantum computing for implementing quantum algorithms [E41] which offer exponential quantum speedup, and in the field of quantum metrology [E26] for achieving quantum gain in precision of estimation of unknown parameters, and even these two seemingly different applications turn out to be mutually connected [O6]. *Indistinguishability* of quantum particles serves as a tool to create entanglement between physical systems [E73, E72, E36], and on the other hand it is an object of fundamental investigations, enclosed in this presentation, concerning the structure of quantum theory [H4] and optical quantum-to-classical transition [H6]. *Bell nonclassicality* is a central object of investigation for foundations of quantum theory, which indicates, that quantum mechanical correlations cannot be reproduced by any hidden variable theory fulfilling intuitive classical assumptions, on the other hand it is an important tool in the field of quantum technologies, as it guarantees the security of quantum key dis-

tribution [E54] and enables generation of intrinsic randomness [E47]. Finally *contextuality* is a main tool in foundational investigations of nonclassicality of single physical systems, but on the other hand it has been identified as a crucial resource for quantum computing [E34].

The aim of presented research is to further investigate the described interplay between the four notions of nonclassicality: entanglement, indistinguishability, Bell nonclassicality and contextuality in the context of both fundamental and practical aspects of quantum information science. Detailed aims of the undertaken research can be summarised as follows:

- developing the theory of experimentally friendly entanglement detection and the theory of entanglement generation, which utilises indistinguishability of photons,

- investigating the structural aspects of quantum theory related with interference due to indistinguishability of photons,

- investigating weakly understood aspects of Bell nonclassicality like: mismatch between maximal entanglement and maximal Bell nonclassicality, and the problem of formulation of Bell nonclassicality outside probabilistic framework,

- investigating nonclassical properties of a single photon.

5.3   Summary

Presented achievement in a form of a series of thematically related publications focuses on investigation of four aspects of nonclassical phenomena across physical systems considered in quantum information theory.

5.3.1   Detection of quantum entanglement based on experimentally accessible local observables

A correlation tensor is a representation of a density matrix of a multipartite quantum state in a product Hermitian basis. The theory of entanglement detection based on local measurements of correlation tensors can be traced back to the work of Badziag et. al. [E6], in which a necessary and sufficient condition for rejecting full separability of a quantum state has been proposed. This condition has a geometric origin since the space of correlation tensors can be treated as an Euclidean space with naturally defined scalar product. These type of geometric conditions was further developed in our former works [O11, O16, O12] in order to effectively characterise *partial* separability of multipartite quantum states.

Despite successful extensions of geometrical criteria for entanglement detection several open questions have remained. The first question, theoretical in nature, concerns relation between the geometrical criteria and the Peres-Horodecki positive map-based criteria for bipartite entanglement [E31]. Both of them are necessary and sufficient, and determine the same set of states, therefore the question about their relation seems natural. The second question, more applied in nature, concerns the possibility of adjusting the geometrical criteria for detecting entanglement of multipartite states of indistinguishable particles.

Both these questions have been addressed in my Habilitation works [H7, H5]. In the work [H7] we have solved the problem of relation between the geometrical and the map-based criteria for bipartite entanglement by providing a generalisation of the geometrical criterion to a purely algebraic one. Such a generalised criterion can be easily transformed

into the one based on a positive map. Apart from purely theoretical issues provided generalisation of the geometrical criterion turns out to be much more effective in entanglement detection, which shows that the original geometric formulation [E6], [O11] was too restrictive.

The problem of extending geometrical criteria of entanglement to systems of indistinguishable particles has been partially solved in the work [H5], in which a criterion for two-component two-mode multipartite bosonic systems has been proposed. It is assumed that the two components are mutually distinguishable, however the particles within each component are fully indistinguishable. Such a hybrid scenario allows for construction of effective entanglement identifiers which can be in principle used to detect entanglement in experimentally accessible systems, like two-component Bose-Einstein condensates.

5.3.2 New insights about Bell nonclassicality of low-dimensional quantum systems

My Habilitation works [H9, H8] concern two weakly understood aspects of Bell nonclassicality. The first issue, discussed in [H8], is the problem of relation between Bell nonclassicality and entanglement. It is well known that for the two-qubit system, Bell inequalities are maximally violated by maximally entangled states of two qubits. However for higher-dimensional quantum systems the relation between Bell nonclassicality and entanglement is more complicated: namely optimal Bell inequalities are maximally violated by non-maximally entangled states. The work [H8] explains this phenomenon in the context of two-qutrit quantum systems, and gives partial explanation for the three-qutrit scenario.

The second issue, investigated in the work [H9], concerns the question, whether it is possible to express Bell nonclassicality beyond the framework of statistical analysis of the observed correlations. In [H9] a new approach to Bell inequalities is proposed, which is based on the idea of algorithmic complexity. Algorithmic complexity is an abstract concept from the theory of computing, in which a complexity of some string of data is defined by the length of the shortest algorithm (encoded using some universal model of computation) which reproduces the data. Although it is an uncomputable quantity, it can be well approximated using compression algorithms. In [H9] a Bell inequality is introduced, which involves normalised lengths of compressed strings of aggregated outcomes of a Bell experiment and an experiment is reported, which shows violation of this inequality using polarisation-entangled photons. This result opens up the possibility of purely algorithmic analysis of Bell nonclassicality, both on the theoretical as well as experimental level.

5.3.3 Nonclassical properties of interference due to indistinguishability of particles

The Habilitation works [H4, H2, H1] focus on deeper understanding of two aspects of quantum interference of indistinguishable particles: Hong-Ou-Mandel [E30] type interference, which leads to bunching (grouping) of photons at the output ports of an optical multiport, and Yurke-Stoler [E73, E72] type interference of particles from independent sources. In [H4] we investigate the bunching phenomenon for optical three-port (tritter). It is well known that the bunching effect in a tritter is weaker than in a beamsplitter, namely three photons entering a tritter through different modes leave the tritter together with probability at most $\frac{2}{3}$ [E17]. The question arises whether there exists a deeper physical reason for lack of perfect bunching in this case. In [H4] a generalised probabilistic model for optical multiports is proposed. A series of reasonable physical assumptions from outside the

quantum formalism are found which allow to recover the quantum bunching probability for a tritter.

The works [H2, H1] focus on generalisation of the idea of Yurke and Stoler [E73, E72] of interference due to indistinguishability of photons to a general multiphoton scenario. An interferometric scheme is proposed, which allows to prepare entangled quantum states using only passive optical devices starting from a product state of indistinguishable photons. The most interesting feature of this scheme is the no-touching property, which means that photons, which finally give rise to entangled correlations have never met during the preparation procedure. One can look at the phenomenon of indistinguishability in two differnt ways. It can be treated as a fundamental feature of quantum particles without any underlying structure (*second quantisation* picture), or it can be treated as arising due to symmetrisation (or antisymmetrisation) of the particle's wavefunctions (*first quantisation* picture). If one takes the second viewpoint, than the entanglement due to symmetrisation seems to be a mathematical artefact, since one cannot address single indistinguishable particle in the experiment. The entanglement found at the output of the no-touching protocol can be thought of as arising due to unlocking the hidden symmetrisation entanglement. In [H2] concrete schemes for creation of three-photon GHZ and W states are presented, whereas in [H1] a general protocol for optical generation of a W state for arbitrary number of photonic qubits is shown.

5.3.4 Nonclassical properties of a single photon

The question whether a single photon possesses nonclassical properties is a matter of long standing debate in the quantum foundations community, and still lacks definite answer. It is known that a single photon in an interferometer gives rise to a contextual behaviour [E5]. A subtle issue is that this behaviour can be simulated by a local hidden variable model, in which locality means that the information propagates forward through the optical paths [E9, E10]. One of the interesting in-build aspects of quantum optics is that the probability amplitudes of a single photon in an interferometer propagate in the same way as classical electromagnetic waves. This correspondence has motivated a series of works, which aim at showing, that classical electromagnetic waves also reveal contextual behaviour [E65, E66, E57, E4, E28]. In the Habilitation work [H6] we show, that such claims are unfounded. It turns out that electromagnetic waves fail to reveal contextual behaviour, since they cannot simulate the structure of single detector clicks characteristic to quantum optical experiments with few-photon states.

The problem of Bell nonclassicality of a single photon has been introduced in a seminal work by Tan, Walls and Collett (TWC) [E67], in which a violation of a Bell inequality by a superposition of a single photon in two spatial modes has been reported. The authors of [E67] used a measurement setup based on homodyne detection and intensity measurements. This result has been for many years treated as a solid proof of nonclassicality of a single photon, and already appeared in quantum optical textbooks (see [E71], page 264). In recent work [O5] we have proposed a complete local hidden variable model for the TWC setup, which exactly recovers all quantum mechanical probabilities. Therefore the TWC setup does not reveal any Bell nonclassicality. In Habilitation work [H3] we explain what is the source of the TWC mistake and show how one can modify the setup in order to obtain indisputable violation of local realism by a single photon superposition measured locally with weak-field homodyne detection. The conclusion is however, that observed nonclassicality cannot be attributed to the single photon itself, but is a consequence of quantum interference due to indistinguishability between the single photon from the input

state and the photons coming from local oscillators.

5.4 Detection of quantum entanglement based on experimentally accessible local observables

5.4.1 Relation between two entirely different necessary and sufficient criteria for bipartite entanglement

The issue of complete characterisation of entanglement has been one of the main problems in quantum information science from its beginnings. The problem has been solved in the context of bipartite systems in 1996 in the seminal paper by Horodecki [E31] in which the following condition has been stated:

$$\rho \text{ is entangled } \iff \exists_\Lambda (\mathbb{1} \otimes \Lambda)[\rho] \ngeq 0, \tag{1}$$

which means that a bipartite state ρ is entangled if and only if there exists a positive but not completely positive map Λ such that a local action of this map on one of the subsystems transforms the density matrix of the state to a non-positive definite matrix. In the case of 2×2 and 2×3 systems (qubit-qubit and qubit-qutrit) there exists a universal map, namely the transposition $\mathcal{T}$, which is sufficient for the criterion [E53]:

$$\rho \text{ is entangled } \iff (\mathbb{1} \otimes \mathcal{T})[\rho] \ngeq 0, \tag{2}$$

whereas even in the 3×3 case there exists no finite family of positive maps characterising entanglement [E63]. The entanglement condition (2) is commonly known as the *PPT condition* (Positive Partial Transpose).

Despite its formal simplicity the map-based condition (1) could be difficult to utilise both theoretically and experimentally, therefore there were attempts to find conditions which are more feasible. Such a condition has been proposed in the seminal work of Badziag et. al. [E6]:

$$\rho \text{ is entangled } \iff \exists_G \max_{\sigma_{\text{prod}}} \text{Tr}(\rho G[\sigma_{\text{prod}}]) < \text{Tr}(\rho G[\rho]), \tag{3}$$

in which the maximisation is performed over all bipartite pure product states and the map G is a linear Hermiticity-preserving map which fulfills a specific positivity condition:

$$\text{Tr}(\rho G[\rho]) \geq 0, \tag{4}$$

which should hold for arbitrary quantum state ρ. Due to this condition the expression $(\rho_1, \rho_2)_G = \text{Tr}(\rho_1 G[\rho_2])$ can be treated as a scalar product which gives rise to a seminorm $||\rho||_G = (\rho, \rho)_G = \text{Tr}(\rho G[\rho])$, and the map G can be treated as a pseudo-metric. The semi-character of the norm $||\alpha||_G$ means that it does not *separate points*, namely $||\alpha||_G = 0 \nRightarrow \alpha = 0$ for some matrix α. All these features imply, that the condition (3) has a geometric interpretation: to refute that a vector $\vec{a}$ belongs to a convex set S it suffices to show that: $\max_{\vec{b} \in S} \vec{a} \cdot \vec{b} < \vec{a} \cdot \vec{a}$. The condition (3) is directly applicable experimentally due to the following representation. Any bipartite state ρ can be represented in the correlation tensor form:

$$\rho = \frac{1}{4} \sum_{ij} T_{ij} \sigma_i^A \otimes \sigma_j^B, \tag{5}$$

where $\{\sigma_i^A\}$ and $\{\sigma_j^B\}$ denote Hermitian operator bases on the two subsystems. The elements $T_{ij} = \text{Tr}(\rho \sigma_i^A \otimes \sigma_j^B)$ of the correlation tensor are directly measurable quantities.

The entire condition (3) can be expressed solely in terms of correlation tensors, since for any linear map G the operator $G[\rho]$ is also a function of the correlation tensor elements via:

$$G[\rho] = \frac{1}{4} \sum_{ijkl} T_{ij} G_{kl}^{ij} \sigma_k^A \otimes \sigma_l^B, \tag{6}$$

in which G_{kl}^{ij} is a tensorial representation of the operator G in chosen local bases.

Having defined all the basic concepts we can proceed to the description of relation between the criteria (1) and (3) described in my Habilitation work [H7]. The construction starts by providing a functional generalisation of the condition (3) by removing the positivity condition (4) and keeping only the Hermiticity preservation assumption. Let us denote such a generalised map by $\mathcal{G}$. Since $\mathcal{G}$ preserves Hermiticity, the maximisation of the overlap $\mathrm{Tr}(\rho \mathcal{G}[\sigma_{\mathrm{prod}}])$ over the set of product states is still meaningful, and the following condition holds:

$$\rho \text{ is entangled } \iff \exists_{\mathcal{G}} \max_{\sigma_{\mathrm{prod}}} \mathrm{Tr}(\rho \mathcal{G}[\sigma_{\mathrm{prod}}]) < \mathrm{Tr}(\rho \mathcal{G}[\rho]). \tag{7}$$

This purely algebraic condition, proposed in [H7], although similar to (3), provides much more efficient entanglement detection criteria, which shall be discussed at the end of this section. Note that in the condition (7) the map $\mathcal{G}$ need not be in principle linear. However, the action on a state by a general nonlinear map may have no representation in terms of locally measurable quantities like in (6). In order to preserve experimentally friendly character of the criterion (7) we restrict the class of allowed nonlinear maps to:

$$\mathcal{G}[\rho] = \frac{1}{4} \sum_{kl} \left(\sum_{ij} f(T_{ij}) \mathcal{G}_{kl}^{ij} \right) \sigma_k^A \otimes \sigma_l^B, \tag{8}$$

in which $f(x)$ is any real-valued function.

In order to find a relation of this generalised condition with the map-based condition (1), we first transform it into an entanglement-witness-like form [E31, E68]:

$$\mathrm{Tr}(\rho W_{\mathcal{G}}[\rho]) < 0, \tag{9}$$

where the witness operator is defined as:

$$W_{\mathcal{G}}[\rho] = \max_{\sigma_{\mathrm{prod}}} \mathrm{Tr}(\rho \mathcal{G}[\sigma_{\mathrm{prod}}]) \mathbb{1} - \mathcal{G}[\rho]. \tag{10}$$

We have shown that due to Choi-Jamiolkowski isomorphism [E40, E20] the operator $W_{\mathcal{G}}[\rho]$ (treated not as a map but as a transformed state) can be equivalently represented in the positive-map-based version as follows:

$$W_{\mathcal{G}}[\rho] = (\mathbb{1} \otimes \Lambda_{\mathcal{G},\rho} \circ \mathcal{T})[\rho_{\Phi^+}], \tag{11}$$

where $\mathcal{T}$ denotes the transposition map, ρ_{Φ^+} is a projector onto an unnormalised maximally entangled state $|\Phi^+\rangle = \sum_i |ii\rangle$ and finally the positive map $\Lambda_{\mathcal{G},\rho}$ is specified by:

$$\Lambda_{\mathcal{G},\rho}[\lambda] = \sum_{i,j} \frac{1}{4} w_{ij}^{\mathcal{G},\rho} \mathrm{Tr}(\sigma_i^A \lambda) \sigma_j^B. \tag{12}$$

In the above formula $w_{ij}^{\mathcal{G},\rho}$ is the matrix element of the operator $W_{\mathcal{G}}[\rho]$ in local bases: $W_{\mathcal{G}}[\rho] = \frac{1}{4} \sum_{ij} w_{ij}^{\mathcal{G},\rho} \sigma_i^A \otimes \sigma_j^B$.

Having all the ingredients we can present the main result of the work [H7]:

Theorem 1 *If an entanglement identifier* (7) *detects bipartite entanglement of a state* ρ *for some Hermiticity-preserving map* $\mathcal{G}$, *then a positive map* $\mathcal{T} \circ \Lambda^{\mathrm{d}}_{\mathcal{G},\rho}$ *also detects entanglement of* ρ *via the condition:*

$$(\mathbb{1} \otimes \mathcal{T} \circ \Lambda^{\mathrm{d}}_{\mathcal{G},\rho})[\rho] \ngeq 0, \tag{13}$$

where the map $\Lambda^{\mathrm{d}}_{\mathcal{G},\rho}$ *is a dual map to* (12) *and reads explicitly:*

$$\Lambda^{\mathrm{d}}_{\mathcal{G},\rho}[\lambda] = \sum_{i,j} \frac{1}{4} w^{\mathcal{G},\rho}_{ij} \operatorname{Tr}(\sigma^B_j \lambda) \sigma^A_i. \tag{14}$$

The above theorem is a one-way relation between the generalised condition (7) and the map-based condition (1), namely for a functional entanglement identifier (7) generated by a given map $\mathcal{G}$ there always exists a positive map (14) which detects bipartite entanglement of at least the same set of states that the functional condition does. In general, the corresponding map-based condition (13) can detect entanglement of a broader class of states. One important comment is necessary here: although the map (14) is linear, the entire condition is not linear, since the map (14) explicitly depends on the state it acts on.

As we indicate in [H7] the inverse construction, namely finding a map $\mathcal{G}$ which via the relation (14) gives rise to a *fixed* map Λ for every state ρ, is intractable in general. Interestingly, it can be easily done for a two-qubit system in the case of the transposition map, for which the map-based condition (2) is known as the positive partial transpose (PPT) condition. We have shown that the PPT criterion for two-qubit system can be restored by taking the map (8) with the tensorial representation $\mathcal{G}^{ij}_{kl} = -\delta_{ij}\delta_{jl}\delta_{ik}[\{i,j,k,l\} \neq 0]$, in which $[p]$ denotes Boolean value of the proposition p, and a constant function $f(x) = 1$, for which we have:

$$\mathcal{G}_{\mathrm{PPT}}[\rho] = -\frac{1}{4}\sum_{i=1}^{3} \sigma_i \otimes \sigma_i \tag{15}$$

for any two-qubit state ρ. Then we have $\Lambda^d_{\mathcal{G}_{\mathrm{PPT}},\rho}[\lambda] = \lambda$, and the condition (13) becomes the PPT condition. The map (15) generates a functional criterion (7) for entanglement, which has the following simple form in the correlation tensor representation:

$$\sum_{i=1}^{3} T_{ii} < -1. \tag{16}$$

The above condition detects entanglement of a singlet state, for which the left hand side reads -3, but fails for other three Bell states. Therefore we see that the correspondence between positive map criterion and functional criterion is effectively one way: the Theorem 1 assures that a positive map based criterion corresponding to a fixed functional criterion is at least as effective as the functional one, however the example with the PPT criterion shows that the functional criterion corresponding to a fixed positive map-based one can be less effective in detecting entanglement.

As a last remark it is worth to mention that the generalised functional condition (7) is much more effective in the entanglement detection that the original geometric condition (3). The original condition allows only for linear maps G, whereas in the generalised case we can utilise nonlinear maps of the form (8). As we show in [H7] a choice of (8) with $\mathcal{G}^{ij}_{kl} = \delta_{ik}\delta_{jl}[i,k \neq 0]$ and $f(x) = \operatorname{sgn}(x)$ gives rise to an explicitly non-linear map $\mathcal{G}$, which

via the condition (7) perfectly detects entanglement of all the states from two families of two qubit states, namely the Bell diagonal states:

$$\rho_{BD} = a\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + b\left|\Phi^-\right\rangle\left\langle\Phi^-\right| + \frac{(1-a-b)}{4}\mathbb{1}, \tag{17}$$

where $\left|\Phi^\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle \pm \left|11\right\rangle)$, and the Weyl states [E32], specified by the condition that the only nonvanishing elements of their correlation tensor are $T_{11} = p, T_{22} = q, T_{33} = r$.

5.4.2 Detecting entanglement in bipartite multiboson systems with geometric entanglement identifiers

The original geometric entanglement condition (3) can be expressed entirely using correlation tensor representations (5) of the states:

$$\rho \text{ is entangled} \iff \exists_G \max_{T_{\text{prod}}}(T, T_{\text{prod}})_G < (T, T)_G, \tag{18}$$

in which the scalar product is understood as $(X, Y)_G = \sum_{ijkl} X_{ij}G^{ij}_{kl}Y^{kl}$. The condition can be simplified by taking a metric tensor G_S which removes all local terms of the correlation tensor (terms containing one index equal to zero), since then the LHS of the condition becomes the highest Schmidt coefficient of the correlation tensor (the highest singular value of the tensor T treated as a matrix): $\max_{T_{\text{prod}}}(T, T_{\text{prod}})_{G_S} = T_{\text{max}}$. Let us denote the RHS as $\mathcal{T} = (T, T)_{G_S}$. Then the criterion (18) can be transformed into a simple entanglement identifier of the form:

$$\epsilon = \frac{\mathcal{T}}{T_{\text{max}}} > 1 \implies \rho \text{ is entangled.} \tag{19}$$

Note that by fixing the choice of the metric tensor G_S we obtain a condition, which is sufficient, but already not necessary. The above condition holds for any two d-level quantum systems. The question which we addressed in the Habilitation work [H5] is whether this condition can be applied to detecting entanglement of indistinguishable particles. We found that it can be adapted to a two-component system of N two-mode identical bosons in order to detect bipartite entanglement between the components, treated as distinguishable subsystems. A general state of such a system can be represented as:

$$\left|\Psi\right\rangle = \sum_{k,l=0}^{N} c_{kl} \left|N-k\right\rangle_A^{a_1} \left|k\right\rangle_A^{a_2} \otimes \left|N-l\right\rangle_B^{b_1} \left|l\right\rangle_B^{b_2} \tag{20}$$

In the above formula c_{kl} denote arbitrary amplitudes of the corresponding states specified in a Fock basis, in which A and B denote two spatial modes, which determine the two components, whereas a_1, a_2 and b_1, b_2 denote two additional bosonic modes within each of the component. Note that the bosons are indistinguishable within each of the components, however the components are mutually distinguishable by spatial modes A and B. It is worth to emphasise, that the above state appears in several realistic physical situations, which we will discuss further on. The crucial idea found in [H5] relies on defining an effective $(N + 1)$-element basis for both two-mode subsystems A and B by the following identification:

$$\left|k\right\rangle_X = \left|N-k\right\rangle_X^{x_1} \left|k\right\rangle_X^{x_2}, \ X = A, B, \ x = a, b. \tag{21}$$

The newly defined vectors $\{\left|k\right\rangle_X\}_{k=0}^{N}$ can be treated as eigenvectors of the diagonal element of an Hermitian basis of the $(N + 1)$-level system, typically chosen as generalised Gell-Mann matrices. This convention is analogous to defining the standard basis for a qubit

as eigenbasis of the σ_z operator (diagonal Pauli matrix). In this way the state of the two-component two-mode bosons (20) is mapped into a state of two $(N+1)$-level systems:

$$|\Psi\rangle = \sum_{k,l=0}^{N} c_{kl}|k\rangle_A \otimes |l\rangle_B, \tag{22}$$

which allows for direct application of the geometric entanglement condition (19).

We tested the efficiency of the entanglement criterion (19) for three physical implementations of the state (20), two for Bose-Einstein Condensates (BEC's), and one for the optical state arising from parametric down-conversion. In this short summary we present two of them: one for BEC and the optical one.

The first example is a two-component BEC entangled by the spin $S^z S^z$ interaction, which is known to reveal exotic structure of entanglement with fractal properties [E14, E43]. Such an entangling interaction can be implemented experimentally in several ways [E35, E69, E56, E1]. Each component is described by a spin-coherent state of a general form:

$$|\alpha, \beta\rangle\rangle_X = \sum_{k=0}^{N} \sqrt{\binom{N}{k}} \alpha^{N-k} \beta^k |N-k\rangle_X^{x_1} |k\rangle_X^{x_2}, \tag{23}$$

in which two local modes are denoted by x_1 and x_2 and X numbers the component. Taking Schwinger representation of the spin operator S^z on the X-th component $S_X^z = x_1^\dagger x_1 - x_2^\dagger x_2$ one can prepare an entangled state of the two-component condensate prepared in an initial state with $\alpha = \beta = \frac{1}{\sqrt{2}}$ [E14]:

$$|\psi_{BEC}\rangle = e^{-iS_A^z S_B^z t} \left| \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle\rangle_A \left| \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle\rangle_B = \sum_{k,l=0}^{N} b_{kl}(t) |N-k\rangle_A^{a_1} |k\rangle_A^{a_2} \otimes |N-k\rangle_B^{b_1} |k\rangle_B^{b_2}, \tag{24}$$

where the time-dependent coefficients read: $b_{kl}(t) = \frac{1}{2^N} \sqrt{\binom{N}{k}\binom{N}{l}} e^{-i(N-2k)(N-2l)t}$. Since the interaction is periodic, we restrict to $t \in [0, \frac{\pi}{2})$. In [H5] we have numerically calculated the value of the entanglement identifier (19) for the state (24) as a function of the interaction time for N up to eight bosons (see Fig. 1). It can be seen that the identifier (19) detects entanglement between the components of this state for each value of the interaction time, and the effectiveness of the identifier grows with increasing N.

The second example is the bright squeezed vacuum state resulting from spontaneous parametric down conversion [E62]:

$$|\psi_{SPDC}\rangle = \frac{1}{\cosh^2 K} \sum_{N=0}^{\infty} \tanh^N K \sqrt{N+1} |\psi^N\rangle, \tag{25}$$

where the N-photon component has the following form:

$$|\psi^N\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=0}^{N} |N-m\rangle_A^H |m\rangle_A^V |N-m\rangle_B^H |m\rangle_B^V, \tag{26}$$

in which H and V denote polarisation modes and A, B denote spatial modes. The entanglement identifier (19) can be applied to the projection of the state (25) onto a fixed photon number component, which is exactly the state (26), for which the identifier (19) reads $\epsilon^{(N)} = N+2$. This confirms entanglement between the components in spatial modes A and B for arbitrary N.
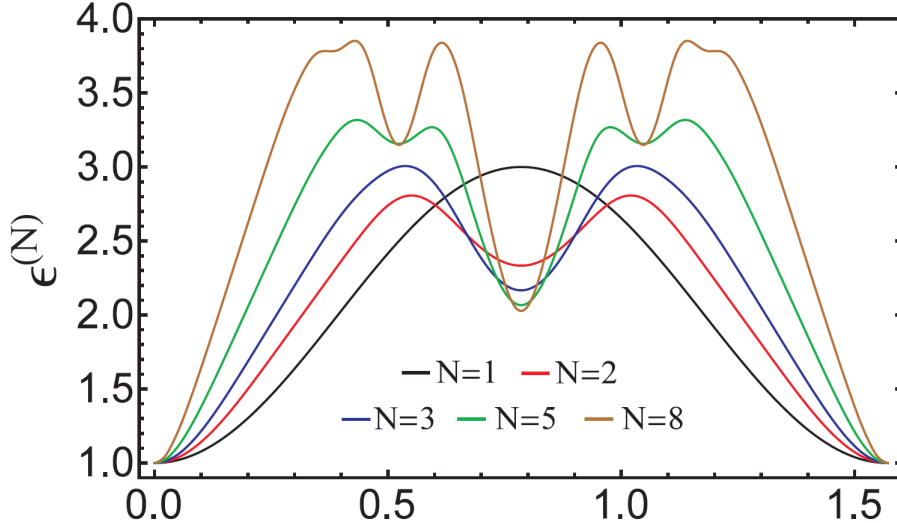
Figure 1: The value of the entanglement identifier (19) ϵ as a function of the interaction time for a two-component Bose-Einstein condensate (24), which undergoes an $S^z S^z$-type entangling interaction. The identifier is calculated for several values of the total number N of bosons per component. It is clear that the identifier $\epsilon^{(N)}$ exceeds the value of one for each value of N and for each value of the interaction time (taken within one period of evolution), which confirms the presence of entanglement between the two components in the state (24).

5.5 New insights about Bell nonclassicality of low-dimensional quantum systems

5.5.1 Maximal Bell nonclassicality vs maximal entanglement for qutrit quantum systems

The optimal Bell inequality, the violation of which captures two-qutrit Bell nonclassicality, is the CGLMP (Collins, Gisin, Linden, Massar, Popescu) inequality [E24, E48], which applies to a scenario with two observers, two measurement settings and three outcomes. Let us label the local outcomes for the two observers as $\{A_i\}_{i=1,2}$ and $\{B_i\}_{i=1,2}$, where i is the number of the local setting. Each of the outcomes A_i and B_i takes the values $\{0,1,2\}$. Then the CGLMP expression reads:

$$
\begin{aligned}
I_3 = {} & P(A_1 = B_1) + P(A_2 + 1 = B_1) + P(A_2 = B_2) + P(A_1 = B_2) \\
& - P(A_1 = B_1 - 1) - P(A_2 = B_2) - P(A_2 = B_2 - 1) - P(A_1 - 1 = B_2),
\end{aligned}
\tag{27}
$$

where $P(\cdot)$ denotes probability of specified events. The CGLMP inequality has the form:

$$
-4 \le I_3 \le 2.
\tag{28}
$$

Let us denote the standard qutrit basis as $\{|i\rangle\}_{i=0}^2$. Then the CGLMP inequality is maximally violated by the state [E3]:

$$
|\psi_{\mathrm{opt}}\rangle = a\,|00\rangle + b\,|11\rangle + a\,|22\rangle,
\tag{29}
$$

where $a \approx 0.617$ and $b \approx 0.489$, for which the CGLMP expression reads $I_3(|\psi_{\mathrm{opt}}\rangle) = 2.915$. Surprisingly maximally entangled two-qutrit state $|\psi_{\mathrm{ME}}\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^2 |ii\rangle$ leads to a slightly

weaker violation: $I_3(|\psi_{\text{ME}}\rangle) = 2.873$. Comparison between the two cases indicates an inconsistency between the notions of maximal entanglement and maximal Bell nonclassicality. This inconsistency persists for two higher-level quantum systems [E3], and for three-qutrit systems [E2]. There were proposed geometrical arguments to explain this discrepancy [E27, E64], however the problem remained unclear.

In order to provide a deeper understanding of the discrepancy between maximal non-classicality and maximal entanglement in my Habilitation work [H8] a Bell-operator analysis of the CGLMP inequality has been conducted. In simple words a Bell operator is an observable, the mean value of which reproduces the value of the Bell expression corresponding to some inequality for arbitrary quantum state under fixed measurement settings. In the context of CGLMP inequality this correspondence reads:

$$I_3(\rho) = \text{Tr}(\hat{B}(\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2)\rho), \tag{30}$$

where ρ denotes arbitrary two-qutrit quantum state and $\hat{B}(\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2)$ is the Bell operator for fixed measurement settings corresponding to the choice of local observables as $\{\hat{A}_i\}$ and $\{\hat{B}_j\}$. Therefore the largest eigenvalue of the Bell operator for optimal measurement settings provides the maximal quantum value of the Bell expression, which corresponds to the maximal violation of the Bell inequality. The clue of the analysis in [H8] relies on representing the two-qutrit Bell operator of the CGLMP inequality as a four-qubit observable, build up from symmetric two-qubit observables. In order to understand this correspondence let us introduce it in more generality.

Any operator acting on a state space of n qutrits (that is $(\mathbb{C}^3)^{\otimes n}$) can be represented in local Hermitian bases as follows [E8]:

$$\hat{B} = \sum_{i_1,\ldots,i_n=1}^{9} B_{i_1,\ldots,i_n} \gamma_{i_1} \otimes \ldots \otimes \gamma_{i_n}, \tag{31}$$

where the basis $\{\gamma_i\}$ consists of Gell-Mann matrices and the identity. However there exists another hermitian basis for the operators acting on $\mathbb{C}^3$, namely the one based on spin-1 operators $\tilde{S}_x, \tilde{S}_y, \tilde{S}_z$ (tilde denotes operators acting on $\mathbb{C}^3$). The correspondence between the two bases is provided by the following relation [O10]:

$$\begin{aligned}
&\gamma_1 = \tilde{S}_x, \gamma_2 = \tilde{S}_y, \gamma_3 = \tilde{S}_z, \\
&\gamma_4 = \mathbb{1}_3 - (\tilde{S}_x)^2, \gamma_5 = \mathbb{1}_3 - (\tilde{S}_y)^2, \gamma_6 = \mathbb{1}_3 - (\tilde{S}_z)^2, \\
&\gamma_7 = \tilde{A}_x, \gamma_8 = \tilde{A}_y, \gamma_9 = \tilde{A}_z,
\end{aligned} \tag{32}$$

where $\tilde{A}_i = \{\tilde{S}_j, \tilde{S}_k\}$, for $i, j, k = x, y, z$ and cyclic permutations, is an anticommutator of the spin operators and $\mathbb{1}_3$ is the identity on $\mathbb{C}^3$. Finally note that spin-1 operators can be represented by symmetric two-qubit operators, which is related to a well-known fact from the representation theory, that the state space of two spin-half particles decomposes into spin-1 subspace (the triplet subspace) and spin-0 subspace (the singlet subspace). The direct correspondence between spin-1 operators in the fundamental representation (on $\mathbb{C}^3$) and in the symmetric two-qubit representation (on $\mathbb{C}^2 \otimes \mathbb{C}^2$) reads:

$$\begin{aligned}
\tilde{S}_x &\mapsto \frac{1}{2}(\mathbb{1}_2 \otimes \sigma_x + \sigma_x \otimes \mathbb{1}_2) = \delta_1, \\
\tilde{S}_y &\mapsto \frac{1}{2}(\mathbb{1}_2 \otimes \sigma_y + \sigma_y \otimes \mathbb{1}_2) = \delta_2, \\
\tilde{S}_z &\mapsto \frac{1}{2}(\mathbb{1}_2 \otimes \sigma_z + \sigma_z \otimes \mathbb{1}_2) = \delta_3,
\end{aligned} \tag{33}$$

where $\{\sigma_i\}$ for $i = x, y, z$ are Pauli matrices, whereas $\mathbb{1}_2$ is the identity matrix on $\mathbb{C}^2$. Using the formulas (32) and the above correspondence we obtain a full mapping from the basis $\{\gamma_i\}$ to $\{\delta_i\}$ (whose first three elements are already listed in formulas (33)). Finally any n-qutrit operator (31) can be represented as $2n$-qubit operator:

$$\hat{B} = \sum_{i_1,\ldots,i_n=1}^{9} B_{i_1,\ldots,i_n} \delta_{i_1} \otimes \ldots \otimes \delta_{i_n}. \tag{34}$$

One can also obtain the relation between the state space of a qutrit and a symmetric subspace of two-qubit system, which corresponds to a standard Clebsch-Gordan decomposition, in which the coupled basis corresponds to a qutrit system. Namely, one obtains the following mapping:

$$\begin{aligned}
|0\rangle &\mapsto |++\rangle, \\
|1\rangle &\mapsto \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \\
|2\rangle &\mapsto |--\rangle,
\end{aligned} \tag{35}$$

where the basis $\{|+\rangle, |-\rangle\}$ denotes here the standard qubit basis (denoted differently to distinguish it from a qutrit basis).

Using the above relation between qutrit operators and symmetric two-qubit operators we can express the CGLMP Bell operator as an operator acting on a state space of four qubits. It has the following form:

$$\hat{B}_{\mathrm{CGLMP}} = \frac{1}{4}\left(\frac{2\sqrt{3}}{3}\left(\hat{B}_{\mathrm{CHSH}_{13}} + \hat{B}_{\mathrm{CHSH}_{14}} + \hat{B}_{\mathrm{CHSH}_{23}} + \hat{B}_{\mathrm{CHSH}_{24}}\right) + \hat{B}_{\mathrm{Mermin}} \right). \tag{36}$$

It consists of four Bell operators for the CHSH Bell inequality for the following qubit pairs: $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ and of the Mermin inequality Bell operator [E50]. A Bell operator for a CHSH inequality for qubits 1 and 3 reads:

$$\hat{B}_{\mathrm{CHSH}_{13}} = \sigma_x \otimes \mathbb{1}_2 \otimes \sigma_x \otimes \mathbb{1}_2 - \sigma_y \otimes \mathbb{1}_2 \otimes \sigma_y \otimes \mathbb{1}_2$$

and similarly for other pairs, whereas the Mermin Bell operator reads:

$$\begin{aligned}
\hat{B}_{\mathrm{Mermin}} &= \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y - \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \sigma_x \\
&- \sigma_x \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y - \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \sigma_x - \sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y \\
&- \sigma_x \otimes \sigma_y \otimes \sigma_x \otimes \sigma_y - \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_x.
\end{aligned} \tag{37}$$

In this way we obtain a decomposition of a CGLMP Bell operator into Bell operators for known Bell inequalities, which is schematically represented in the Figure 2. As stated earlier, the maximal value of a Bell expression (which corresponds to a maximal violation of the corresponding Bell inequality) is specified by the highest eigenvalue of the Bell operator, and the optimal state is the corresponding eigenvector. The main result presented in [H8] is the analysis of the optimal state for violation of the CGLMP inequality from the point of view of CGLMP Bell operator in the decomposition (36). Namely the highest eigenvalue of the CHSH part of the operator (36) corresponds to the eigenvector in a form of a superposition of two Bell states and a four-qubit GHZ state:

$$|\psi\rangle_{4\mathrm{CHSH}} = \frac{1}{\sqrt{2}}|\mathrm{GHZ}\rangle + \frac{1}{\sqrt{2}}|\psi^+\rangle|\psi^+\rangle, \tag{38}$$
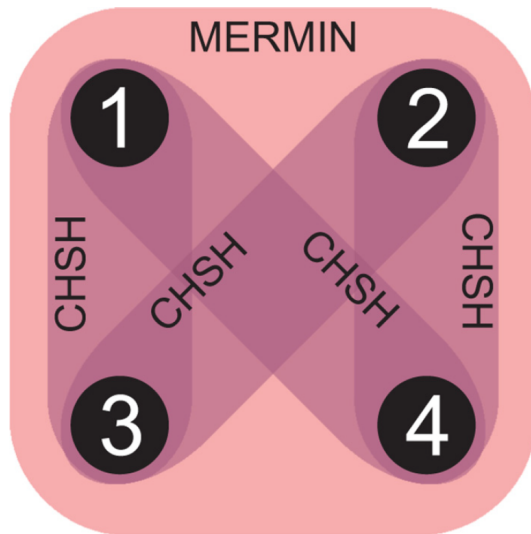
Figure 2: Schematic representation of the CGLMP Bell operator as an operator acting on a four-qubit space (36). The operator consists of five terms, one corresponding to a four-qubit Mermin inequality, and four corresponding to two-qubit CHSH inequalities for different pairs of qubits.

where $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|+ + ++\rangle + |- - --\rangle)$ and $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$. At the same time the highest eigenvalue of the Mermin Bell operator (37) corresponds to the eigenstate in a form of a four-qubit GHZ state. Therefore the eigenvector of the entire CGLMP Bell operator corresponding to the highest eigenvalue should be of the form:

$$|\psi(p)\rangle_{\text{CGLMP}} = \sqrt{p}\,|\text{GHZ}\rangle + \sqrt{1-p}\,|\psi^+\rangle\,|\psi^+\rangle\,. \tag{39}$$

Indeed, it turns out that the maximal eigenvalue corresponds to $p_{\text{opt}} \approx 0.761$. The optimal state $|\psi(p_{\text{opt}})\rangle_{\text{CGLMP}}$ when transformed to its two-qutrit counterpart via local transformation (35) gives exactly the optimal state (29) which maximally violates CGLMP inequality, as expected. On the other hand the maximally entangled two-qutrit state corresponds via transformation (35) to a four-qubit state of the form $\left|\psi\left(\frac{2}{3}\right)\right\rangle_{\text{CGLMP}}$, which is not an optimal eigenstate of the Bell operator (36). The above analysis shows that the optimal Bell nonclassicality revealed by the CGLMP inequality can be related with a tradeoff between Mermin-type correlations and CHSH-type correlations, and the optimal balance between the two does not correspond to a maximally entangled state of two qutrits.

5.5.2 Algorithmic approach to Bell nonclassicality

Standard approach to Bell nonclassicality is based on purely probabilistic reasoning [E12]. In derivation of a Bell inequality one assumes that all observables can be represented as random variables on a common sample space. Additionally one introduces constraints on functional dependencies of these variables, namely that they depend only on the local measurement settings and a hidden variable (locality assumption). Finally one assumes that in a Bell test the probability distribution of the measurement settings does not depend on the hidden variable, which means that the stochastic process, which leads to the choice of the settings is free of any influences from the common past of the experiment (the *free choice*

assumption, sometimes called provocatively the *free will* assumption). A Bell inequality can be always represented as a function of conditional probabilities of the outcomes of the random variables representing observables conditioned on the measurement settings. In the work [H9] we state the question whether one can express Bell nonclassicality without resort to the probabilistic framework.

Historically the first attempt to redefine the framework for Bell-type analysis were information-theoretic entropic Bell inequalities by Braunstein and Caves [E11] based on conditional entropies, further modified into Bell inequalities based on mutual information [E18]. However any Shannon-type entropy is a function of a probability distribution of outcomes of a random variable, therefore this approach is still founded on the probabilistic analysis.

In the Habilitation work [H9] a completely different approach is proposed, which entirely goes beyond the probabilistic framework. Let us assume the simplest Bell scenario with two observers, two local measurement settings denoted by 0 and 1 and two binary (0 or 1) outcomes of each of the two locally measured observables, denoted as x and y for Alice and Bob respectively. Let the number of repetitions of the single experiment be N. Then we can group the outcomes from the entire experiment involving N repetitions into four pairs of bit strings corresponding to coincidence detections for the fixed settings: $\{x_0, y_0\}, \{x_0, y_1\}, \{x_1, y_0\}, \{x_1, y_1\}$ (subscript indices denote settings). These are the outcomes of the experiment which will be tested towards violation of classicality, the concrete meaning of which will be specified later. Note that this approach stands in a sharp contrast to the standard Bell approach, in which single outcome corresponds to single measurement. Here, the single outcome is an aggregated object, corresponding to many repetitions of the measurement procedure.

In order to test the nonclassical character of such prepared outcomes a distance-type approach to Bell inequalities is applied [E60, E45, E70], in which a Bell inequality is derived using triangle inequalities for distances between the outcomes. Since the idea is to perform a probability-free analysis of the data, a notion of a distance based on Kolmogorov complexity is used in [H9]. The Kolmogorov complexity $K(x)$ of a bit string x is a purely algorithmic measure of randomness of the string, and is defined as the length of the shortest program (with respect to some universal model of computation) which reproduces the string. The string is called algorithmically random if its Kolmogorov complexity is at least as large as the length of the string, which means that the string cannot be compressed. Although Kolmogorov complexity is uncomputable, it can be well upper bounded by the length of a compressed string with respect to some compression algorithm, which will be utilised in further reasoning.

Let us introduce the notion of Normalised Information Distance (NID) [E46] between two strings x and y of length n:

$$\text{NID}(x, y) = \frac{K(x, y) - \min(K(x), K(y))}{\max(K(x), K(y))}, \tag{40}$$

where $K(x, y)$ is the Kolmogorov complexity of a concatenated string of x and y, whereas $K(x), K(y)$ are respective Kolmogorov complexities of single strings. It is proved to be a metric up to corrections of the order of $(\log n)/n$. It is equal to zero for identical strings, and equal to one for completely independent ones.

In order to construct a nonclassicality test we have to define some classical model reproducing the experimental data, which could be experimentally falsified. For this reason we assume two spatially separated universal Turing machines (see Figure 3), each having access to the local input strings determining the settings and to a common program Λ
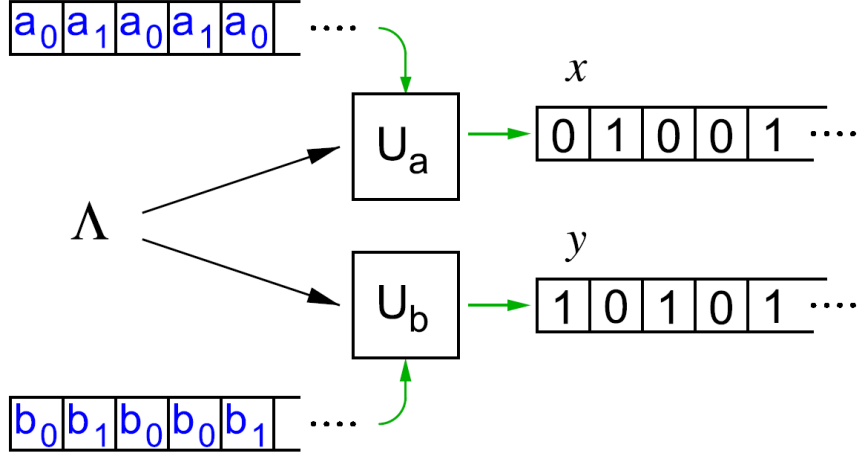
Figure 3: Local realistic computational model aiming at reproduction of correlated bit strings arising from quantum measurements. Two local Turing machines U_a and U_b are fed with local bit strings $\{a_i\}$ and $\{b_i\}$ encoding the settings and a common program Λ, the counterpart of a shared hidden variable in probabilistic models. The machines produce local output strings x and y encoding simulated outcomes of quantum measurements.

representing the classical description of the physical system (analogue of the local hidden variable in the standard Bell scenario). Each local machine produces the output string which should mimic the experimental results obtained from real measurements. Finally we could prepare four pairs of strings by taking substrings of the output strings corresponding to coincident input values (the settings). We derive a constraint on the level of correlation of such obtained output bit strings and verify whether quantum mechanical measurements can violate this constraint. The starting point to formulate the constraint is the fact that NID obeys triangle inequality, therefore we can write:

$$\text{NID}(x_0, y_0) + \text{NID}(y_0, y_1) \geq \text{NID}(x_0, y_1),$$
$$\text{NID}(x_1, y_0) + \text{NID}(x_1, y_1) \geq \text{NID}(y_0, y_1). \tag{41}$$

Note that the pair of strings $\{y_0, y_1\}$ in general cannot be obtained experimentally, since it may correspond to coincidence outcomes of incompatible observables. However, we assume a version of realism, which states that the outcomes of unperformed measurements are still well defined, therefore one can meaningfully assign a Kolmogorov complexity to a bit string corresponding to a sequence of unperformed measurements. This assumption is in fact a counterfactual definiteness. Let us assume that the string y_0 has been measured. Then the string y_1 could not have been measured, however the assumption assures it is well defined. Combining the two triangle inequalities one obtains the following quadrangle inequality:

$$S' = \text{NID}(x_0, y_1) - \text{NID}(x_0, y_0) - \text{NID}(x_1, y_0) - \text{NID}(x_1, y_1) \leq 0. \tag{42}$$

Note that each of the strings occurs twice, and each occurrence of the string corresponds to different experimental realisations. We postulate additional assumption of uniform complexity, which means that the complexity of a given string corresponding to a fixed setting is constant for different runs of the experiment (when by experiment we mean the entire set of N runs of a single measurement procedure). The final issue is the problem with uncomputability of the Kolmogorov complexity. In order to make the condition (42)

experimentally testable, we replace the Normalised Information Distances with Normalised Compression Distances. For this reason let us fix the choice of a compression algorithm and define [E21]:

$$\text{NCD}(x, y) = \frac{C(x, y) - \min(C(x), C(y))}{\max(C(x), C(y))}, \tag{43}$$

where $C(x)$ is the length of the compressed string x and $C(x, y)$ is the length of the compressed concatenated string of x and y. In this way we obtain the following algorithmic Bell inequality:

$$S = \text{NCD}(x_0, y_1) - \text{NCD}(x_0, y_0) - \text{NCD}(x_1, y_0) - \text{NCD}(x_1, y_1) \leq 0. \tag{44}$$

In order to test the inequality (44) a Bell experiment with polarisation entangled photon pairs prepared in a state:

$$|\psi_{\text{exp}}\rangle = \frac{1}{\sqrt{2}}(|H\rangle |V\rangle - |V\rangle |H\rangle) \tag{45}$$

has been performed. A single run of the experiment has produced output bit strings of average length of the order of 10^5. The inequality (44) has been tested using LZMA (Lempel–Ziv–Markov) compression algorithm in order to evaluate compression distances (43). Although the inequality itself is purely algorithmic, in order to assess the uncertainty in estimation of S, the entire measurement procedure has been repeated eight times, giving rise to $S = 0.0494 \pm 0.0076$. This result shows that outcomes of quantum mechanical measurements significantly violate the algorithmic Bell inequality (44), which means that quantum predictions cannot be reproduced by a classical computational model fulfilling joint assumptions of realism (counterfactual definiteness), locality and uniform complexity.

5.6 Nonclassical properties of interference due to indistinguishability of particles

5.6.1 Generalised probabilistic description of multiphoton interferometry and recovery of the quantum bunching probability

When two identical photons enter symmetric beamsplitter through different input ports, they always leave the beamsplitter bunched together, via either of the output ports with equal probability $\frac{1}{2}$. This property fails to hold for higher number of modes [E17]. Indeed, when single photons enter each of the entry port of a multiport with more than two modes, the probability of fully bunched output is strictly less than one. This fact can be easily derived using the formalism of linear quantum optics, however the question arises whether it can be attributed to some physical principles, which can be formulated outside quantum formalism. In order to answer this question in [H4] we introduce a generalised probabilistic description of linear quantum optics.

Generalised probabilistic approach [E19, E25, E49] relies on defining abstract states, transformations and measurements in terms of quasi-probabilistic relations, which do not necessarily need to fulfill all Kolmogorovian aspects of probability (for example there need not exist a common sample space for the probability distributions). The aim of this approach is to single out quantum description *from outside* by introducing reasonable information-theoretic or structural assumptions (e.g. no-signalling conditions). In [H4] we propose a generalised probabilistic description of linear optics. The model is parametrised

by the number of modes K and the number of particles N. Since the particles are identical, the number of distinct configurations is:

$$d = \frac{(K + N - 1)!}{N!(K - 1)!}. \tag{46}$$

The states are specified by the probability distributions $\mathbf{\Pi}$ over the configurations of mode occupations by hypothetical identical particles. For example for $N = K = 2$ we have three possible configurations $\{2, 0\}, \{1, 1\}, \{0, 2\}$ therefore a general state is specified by the probability vector $\mathbf{\Pi} = [\mathbf{p_{20}}, \mathbf{p_{11}}, \mathbf{p_{02}}]$. Transformations are specified by the stochastic $d \times d$ transfer matrices $\mathbb{S}$. For example the transformation corresponding to a symmetric beamsplitter has the form:

$$\mathbb{S}_{BS} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}. \tag{47}$$

The elements of the transfer matrix $\mathbb{S}$ are transition probabilities of the form $P_x^{(y)}$, where y denotes initial configuration whereas x denotes final configuration. For example an element $[\mathbb{S}_{BS}]_{11}$ of the matrix (47) represents $[\mathbb{S}_{BS}]_{11} = P_{20}^{(20)} = \frac{1}{4}$.

We impose two structural restrictions on the introduced model:

- **double stochasticity**: the transfer matrix should be doubly stochastic, which guarantees that the entropy of the probability distribution does not decrease under the physical transformations,

- **consistency between transformation and reduction of the distribution**: if we first reduce N-partite distribution to the $M < N$-partite one, and then evolve the reduced state, we obtain the same as if we first evolved the N-partite state and then reduced it to the M-partite one. To put it formally, let us introduce a reduction stochastic operation $\mathbb{D}^{(N)}$, which transforms N-partite state into an $(N - 1)$-partite one by randomly removing one particle. For example for two particles in two modes ($N = K = 2$) the reduction from the bipartite to single-particle distribution is specified by:

$$\mathbb{D}^{(2)} = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}. \tag{48}$$

  Then arbitrary reduction can be specified by a composition:

$$\mathbb{D}^{(N \mapsto M)} = \mathbb{D}^{(M+1)} \dots \mathbb{D}^{(N-1)} \mathbb{D}^{(N)}.$$

  The consistency condition can be formalised as follows:

$$\forall_{1 \leq M < N} \ \mathbb{D}^{(N \mapsto M)} \mathbb{S}^{(N)} \mathbf{\Pi}^{(N)} = \mathbb{S}^{(M)} \mathbb{D}^{(N \mapsto M)} \mathbf{\Pi}^{(N)}. \tag{49}$$

Note that both the above properties are fulfilled in linear quantum optical systems. Unitary quantum evolution, when mapped to the transfer-matrix model, is represented by unistochastic matrices, which belong to a subclass of doubly-stochastic matrices. Secondly, note that the consistency condition taken for $M = 1$ means that the entire evolution of arbitrary state is in fact determined by a single-particle evolution. This is exactly the case of linear quantum optics: since photons are not interacting, single-photon creation operators are transformed independently on each other via the relation: $a_i'^{\dagger} = \sum_j U_{ij} a_j^{\dagger}$ [E51].

The above discussion shows that generalised probabilistic model for linear optics captures essential features of the quantum evolution of non-interacting identical particles. The question arises whether the model reproduces quantum behaviour in the context of bunching properties of the photons. Namely let us formally define bunching probability for a three-mode three-particle case ($N = K = 3$):

$$B_3 = P_{300}^{(111)} + P_{030}^{(111)} + P_{003}^{(111)}, \tag{50}$$

in which $P_{300}^{(111)}$ denotes transition probability from the configuration $\{1, 1, 1\}$ to $\{3, 0, 0\}$ and similarly for others. It turns out that hypothetical particles whose dynamics is governed by the proposed general probabilistic model have stronger than quantum tripartite bunching probability. Namely the maximal quantum value for the bunching probability (50) equals $B_3^{QM} = \frac{2}{3}$, and is realised by a symmetric quantum tritter. In the generalised probabilistic description it is represented by a unistochastic matrix obtained by taking the modulus squares of the entries of the quantum tritter matrix. On the other hand a generalised probabilistic model fulfilling the consistency and double-stochasticity constraints allows for higher bunching probability $B_3^{GP} = \frac{3}{4}$. The question arises whether some additional constraint on the generalised probabilistic model can restore the maximal quantum bound for bunching probability.

The main result of the work [H4] is that in the case of three modes such an additional constraint is the product character of the evolution of a two-partite configuration in which the two particles are initially in the same mode. The first stage of the argument is showing, that due to double stochasticity and consistency conditions (49) the 3-partite bunching probability (50) is upper-bounded by the sum of two-partite transition probabilities:

$$B_3 \le 1 - \frac{1}{3} \left(P_{200}^{(200)} + P_{200}^{(020)} + P_{200}^{(002)} + P_{020}^{(200)} + P_{020}^{(020)} + P_{020}^{(002)} + P_{002}^{(200)} + P_{002}^{(020)} + P_{002}^{(002)} \right). \tag{51}$$

Next, we postulate that due to the lack of interaction between the particles, the product two-particle states:

$$\mathbf{\Pi_{200}} = [\mathbf{p_{200} = 1, p_{020} = 0, p_{002} = 0, p_{110} = 0, p_{101} = 0, p_{011} = 0}],$$

and analogously defined $\mathbf{\Pi_{020}}, \mathbf{\Pi_{002}}$ should evolve according to a product evolution:

$$\mathbb{S}^{(2)}\mathbf{\Pi_{200}} = \mathbb{S}^{(1)}\mathbf{\Pi_{100}} \times \mathbb{S}^{(1)}\mathbf{\Pi_{100}}, \tag{52}$$

where $\mathbf{\Pi_{100}} = [\mathbf{p_{100} = 1, p_{010} = 0, p_{001} = 0}]$, and the product in the above equation is understood as a Kronecker product of probability vectors with additional shrinking of dimensions from nine to six due to indistinguishability of the particles. Formula (52) implies that for a symmetric tritter all the probabilities in (51) equal to $\frac{1}{9}$, and therefore one obtains the bound $B_3 \le \frac{2}{3} = B_3^{QM}$. In this way the maximal quantum bunching probability for three photons in an optical tritter is recovered within the generalised probabilistic model of linear optics, with three assumptions: double stochasticity of the evolution, consistency between reduction and evolution (49) and product evolution of states corresponding to two particles in the same initial mode (52).

5.6.2 *No-touching* interferometric schemes for generation of multipartite entangled states

Interference due to indistinguishability of photons is typically related with the Hong-Ou-Mandel effect [E30], in which two photons impinge on a beamsplitter through different

input ports, and the quantum probability amplitudes corresponding to the coincidence outputs cancel each other. Therefore the two photons always leave the beamsplitter bunched together. This effect is inevitably related with the commutation relations of the photonic creation operators, namely if we consider fermionic particles with anticommuting operators, the observed effect would be the opposite – the particles will leave the beamsplitter always in coincidence. Therefore the Hong-Ou-Mandel-type interference strongly depends on the statistical properties of the particles. However there exists another type of interference due to indistinguishability of particles, which does not depend on the commutation relations of the particle's operators. It occurs due to indistinguishabiliy of pairs of paths of the particles coming from independent sources. It was firstly proposed by Yurke and Stoler in two seminal works [E73, E72]. The characteristic feature of this type of experiments is the necessity of postselection of the detection events in order to obtain nonclassical correlations. After the works of Yurke and Stoler further development of this branch of interferometry went another directions towards *event ready* schemes [E36], therefore the idea of Yurke-Stoler-type interference due to indistinguishability was almost entirely abandoned. It has been reintroduced and substantially extended in the Habilitation works [H2, H1], in which the notion of no-touch interferometric schemes has been proposed. Although the schemes work for arbitrary indistinguishable particles (bosonic or fermionic) we keep the convention of optical description, which is closest to achievable experimental implementations of multiparticle interferometry.

The most general *no-touching* scheme, proposed in [H2], is an interferometric scheme based on an N-mode interferometer, in which the modes are grouped into K (M_i)-tuples of modes, interpreted as subsystems (the number of modes M_i can differ from subsystem to subsystem). The entire protocol consists of three steps (see Figure 4):

- preparation of single photon inputs for each group of modes A_i, $i = 1, \ldots, K$ and application of local unitaries U_i implemented by optical M_i-ports,

- global permutation of modes $\sigma \in S_N$, where S_N is the permutation group of N elements,

- application of final local unitaries V_i and postselection within each M_i-mode output subsystem B_i on single-photon events in a group $\bar{B}_i$ of $d \leq M_i$ local output modes.

Using the idea of multirail encoding, in which a d-level quantum system is represented by a single photon impinging on a general optical d-port realising arbitrary $SU(d)$ transformation on a single photon state [E58], one can promote the following interpretation of the no-touching scheme: a product state of K M_i-level systems is transformed into a (possibly entangled) state of K d-level systems. The name *no-touching scheme* comes from the fact that if we apply a realistic viewpoint in which a photon can be thought of as a localised point particle, then we can say with certainty that the photons which pass the postselection condition have never met in the same point at any stage of the protocol. Therefore if at the output stage of the protocol one obtains an entangled multiphoton state, one can interpret this fact as extraction of nonclassical correlations from pure indistinguishability of photons, as in the protocol there is no place for any sort of interaction. There is even no place for Hong-Ou-Mandel-type interference, which depends on commutation properties of creation operators in a single point in space (which in the realistic viewpoint corresponds to a situation in which the particles meet). Therefore the only resources we utilise in the protocol are: single particle superposition and indistinguishability of particles.

The presence of postselection procedure in the protocol raises doubts about the possibility of using the proposed scheme for Bell nonclassicality tests. This is because postselection
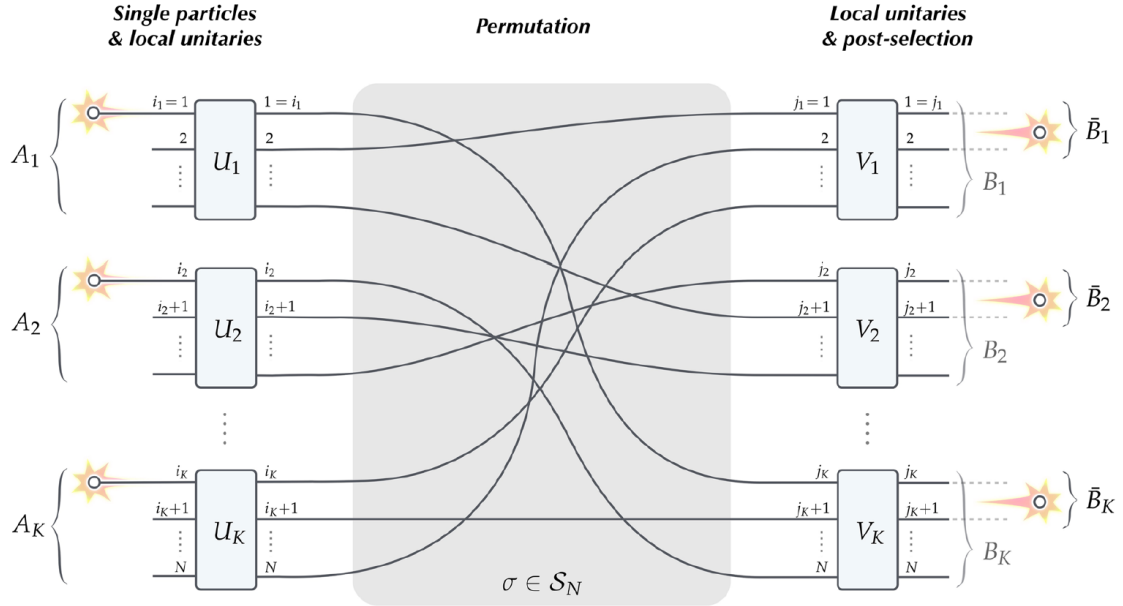
Figure 4: General *no-touching* scheme for generation of entangled multipartite states. K independent and indistinguishable particles impinge on K $M_i \times M_i$ multiports U_i, $i = 1, \ldots, K$, such that each particle enters the first mode of each multiport. At the next stage the paths are permuted according to some fixed permutation of N objects. At the final stage second group of local unitaries V_i, $i = 1, \ldots, K$ is applied to M_i-tuples of local modes in subsystems B_i, and a postselection on single photon counts in local d-mode subsystems $\bar{B}_i \subset B_i$ is performed. Finally after the postselection one effectively obtains measurement results, which are equivalent to the ones arising from a measurement performed on some quantum state of K d-level systems, the concrete form of which depends on $\{U_i\}, \sigma$ and $\{V_i\}$. In the case of non-demolition postselection one physically obtains such a state at the output stage of the protocol.
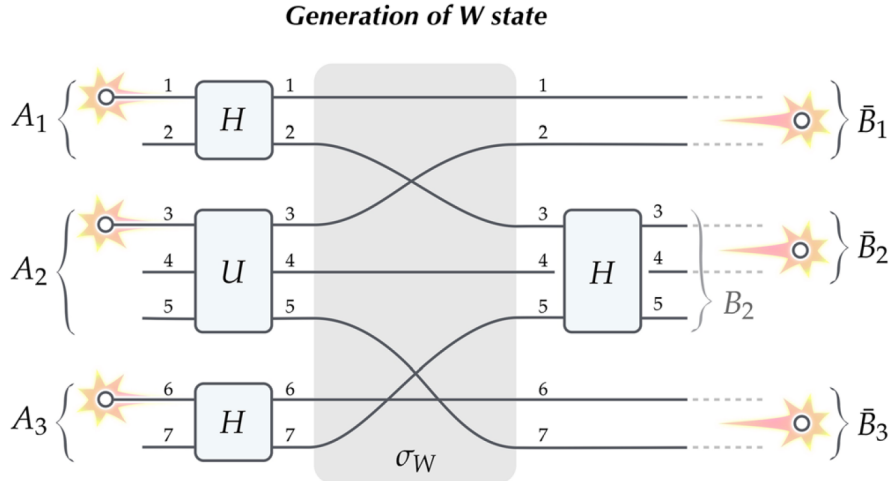
Generation of W state

Figure 5: *No-touching* scheme for three-qubit W state (53) generation presented in the Habilitation work [H2]. Three independent photons are injected into three local subsystems A_1, A_2, A_3 and enter three initial unitaries, where H denotes a Hadamard gate, whereas U is a specifically chosen tritter transformation. After permutation of modes and final application of a Hadamard gate to modes 3 and 5 a postselection on a single photon in groups of modes $\{1, 2\}$, $\{3, 4\}$ and $\{6, 7\}$ is performed.

of the output events may introduce additional classical correlations, which may spoil the correctness of the Bell test (so called *selection bias*) [E55, E37, E61]. However, in our recent work [O1] we have shown that the postselection of the type used in the no-touching scenarios is indeed safe for Bell tests.

The no-touching scenario can be thought of as a general scheme for entanglement generation using only product input Fock states, linear optical devices and postselection at the detection stage. In [H2] a three-photon scheme has been proposed for creation of a three-qubit W state (see Figure 5):

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \tag{53}$$

whereas in Habilitation work [H1] a general N-photon protocol for creation of a W state of arbitrary number of N qubits has been presented (see Figure 6):

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|10\ldots0\rangle + |010\ldots0\rangle + \ldots + |0\ldots01\rangle). \tag{54}$$

The efficiency of this protocol scales inversely quadratically with the number of qubits as $\text{Eff}_N \sim \frac{1}{N^2}$, which makes it currently the most efficient protocol for W state generation without additional feed-forward correction techniques (using correcting unitary operations dependent on measurement results on auxiliary particles during performance of the protocol). A protocol with similar efficiency using quantum erasure techniques has been recently proposed [E42], however this protocol uses $N + 1$ photons in order to create N-qubit W state, and its efficiency in a version without feed-forward active correction techniques is slightly worse than the efficiency of our protocol proposed in [H1].
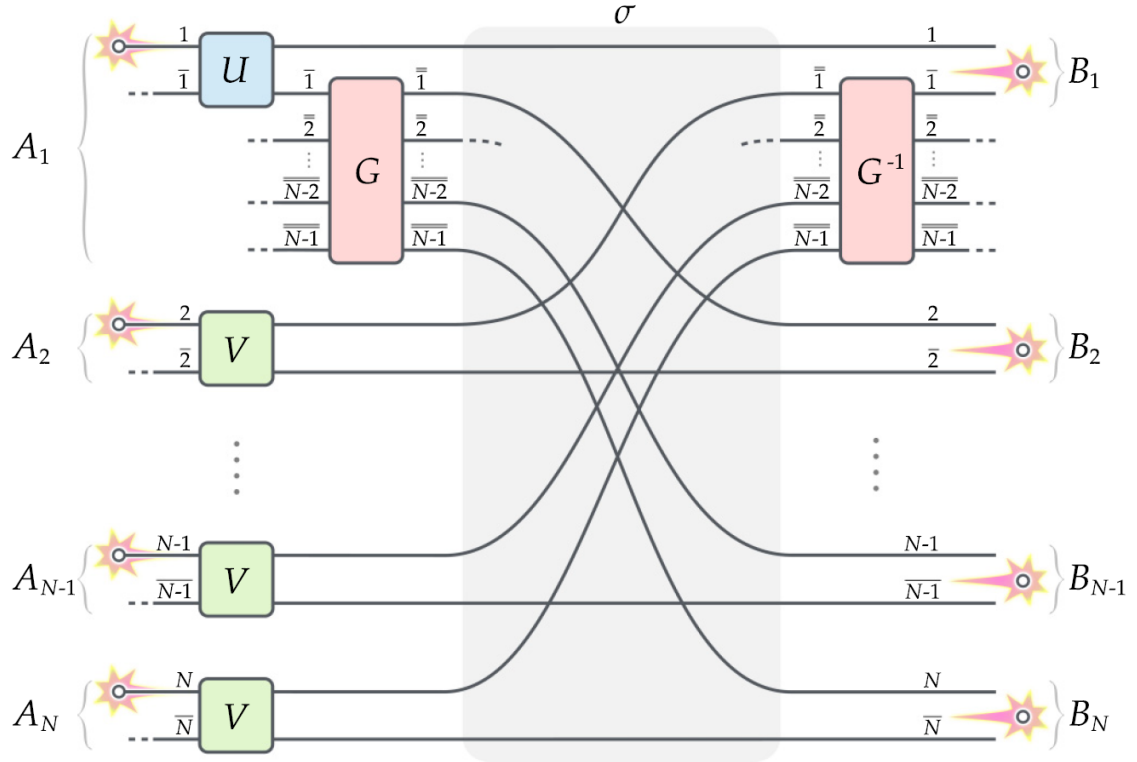
Figure 6: *No-touching* scheme for N-qubit W state (54) generation presented in the Habilitation work [H1]. N independent photons are injected into local subsystems $A_1, \ldots, A_N$ and enter local unitaries $U \circ G, V, \ldots, V$, specified in [H1]. Further a permutation of modes σ is performed and final unitary G^{-1} for all but the first mode in the first output subsystem is applied. Finally one performes a postselection on single photon counts in two-mode subsystems B_i, $i = 1, \ldots, N$.

5.7 Nonclassical properties of a single photon

5.7.1 Contextuality of a single photon and classicality of an electromagnetic wave

The simplest correlational Bell-type experiment consists of a source of a pair of particles in an entangled state (for example two polarisation- or path-entangled photons) and two spatially separated measurement stations which implement local measurements of two incompatible dichotomic ± 1 observables A_0, A_1 and B_0, B_1 (see Figure 7 a). Any local realistic description of the arising correlations has to fulfill Clauser-Horne-Shimony-Holt (CHSH) inequality [E23]:

$$CHSH = |\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle| \leq 2. \tag{55}$$

For a maximally entangled initial state and properly chosen local measurements quantum mechanical correlations can achieve $CHSH = 2\sqrt{2}$, which confirms that the local realistic description is not possible, and the correlations are truly Bell-nonclassical.

On the other hand the CHSH Bell-type scenario can be realised in a sequential way on a single four-level quantum system (see Figure 7 b). Then the violation of the CHSH inequality indicates contextuality of the correlations rather than Bell nonclassicality. Interestingly such a scenario can be implemented with a single photon, which can occupy four modes. For simplicity of presentation we will assume that the four modes are two spatial modes and two polarisation modes. The scheme goes as follows (see Figure 7 b). A source emits a photon prepared in an arbitrary superposition of two polarisation and two spatial modes a_H, a_V, b_H and b_V, in which the subscript indices denote orthogonal polarisation states. The photon enters a tunable beamsplitter A, which can be in two configurations corresponding to two possible settings (specified for example by the transmittivity). The output modes of the first beamsplitter are linked with input ports of two polarisation beamsplitters B, which could implement a polarisation measurement in two different bases. In each experimental run the settings of the first and the two second beamsplitters are chosen randomly, with the constraint that the two polarisation beamsplitters B implement the same measurement in a single run. In this way one can implement a measurement of the correlation $\langle A_i B_j \rangle$, where A_i denotes dichotomic measurement of path degrees of freedom, whereas B_j denotes dichotomic polarisation measurement. Indeed, at the final stage one obtains four possible detection events, which we can label as $(++|ij)$, $(+-|ij)$, $(-+|ij)$ and $(--|ij)$, and the result e.g. $(++|ij)$ represents the event $A_i = B_j = +1$, and similarly for other events (see Figure 7 b). Note that in a single run one obtains just a single detector click corresponding to the one of the four mentioned events. However after many experimental rounds one can estimate the probabilities $p(++|ij)$, $p(+-|ij)$, $p(-+|ij)$ and $p(--|ij)$, which can be used to determine the correlation functions:

$$\langle A_i B_j \rangle = p(++|ij) - p(+-|ij) - p(-+|ij) + p(--|ij). \tag{56}$$

By preparing a proper initial state of the single photon and choosing appropriate measurements $\{A_i\}$ and $\{B_j\}$ one can obtain a violation of the CHSH inequality (55), which demonstrates contextuality of observed correlations.

An interesting situation happens if instead of a single photon one uses N identically prepared photons or a classical beam of light. Because the photons transform independently and because a classical electromagnetic wave is transformed by linear optical devices in the same way as the single photon probability amplitude, the observed probabilities at the output would be proportional to respectively $Np(++|ij)$, $Np(+-|ij)$, $Np(-+|ij)$ and
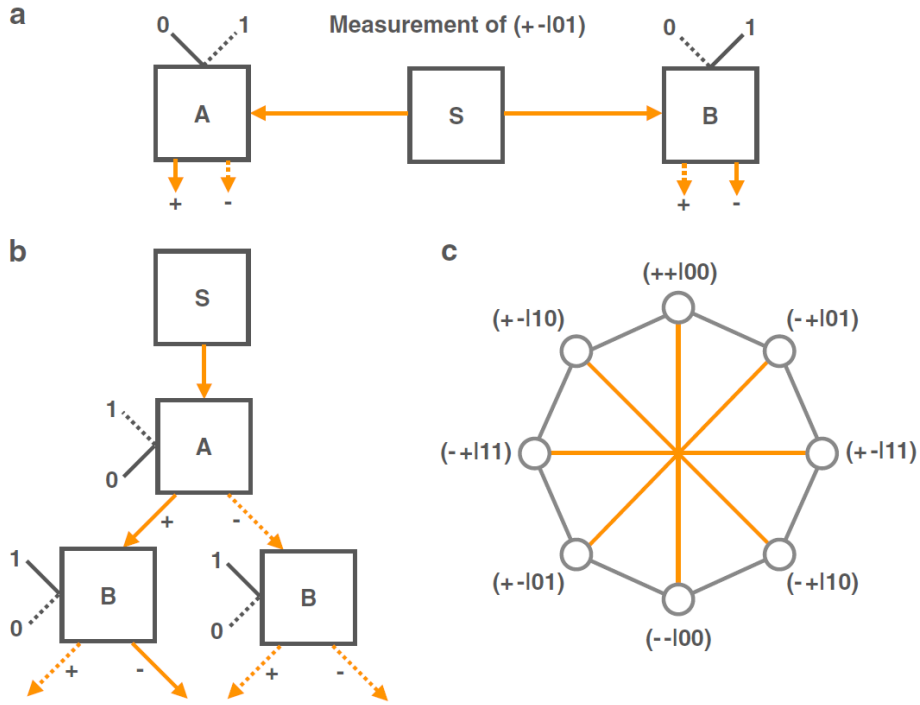
Figure 7: **a)** Standard Bell-type scenario with a source S distributing two correlated particles to two spatially separated stations of Alice and Bob. It is assumed that both parties have two local settings denotes by numbers $\{0, 1\}$ and register binary outcomes $\{+, -\}$. In the picture a detection of the event $(+ - |01)$ is depicted. **b)** Sequential realisation of the same measurement scheme, in which one assumes that the properties A and B are compatible. The outcome of the observable A is now encoded in a path degree of freedom, and can be determined only after the final measurement of B, by checking whether the click was observed in the left (" $+$ " outcome) or right (" $-$ " outcome) B device. In the picture it is assumed that the " $-$ " detector of the left B device has clicked, therefore the joint event corresponds once more to $(+ - |01)$. **c)** Exclusivity graph for the CHSH inequality (58). The graph consists of eight vertices representing joint detection events specified in the formula (58) and twelve edges joining mutually exclusive events.

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$Np(--|ij)$, and $I(++|ij)$, $I(+-|ij)$, $I(-+|ij)$ and $I(--|ij)$, where N is the number of photons and I denotes measured intensity at the appropriate output port. One can define two random variables:

$$\begin{aligned}
\chi_{ij}^{(N)} &= \frac{1}{N}(n(++|ij) - n(+-|ij) - n(-+|ij) + n(--|ij)), \\
\chi_{ij}^{(cl)} &= \frac{1}{I}(I(++|ij) - I(+-|ij) - I(-+|ij) + I(--|ij)),
\end{aligned} \tag{57}$$

in which $n(++|ij)$ is the number of photons detected in the output port $++$, N is the total number of photons, $I(++|ij)$ is the intensity at the output port $++$ and I is the total intensity. Finally one can prepare a classical state of light, for which the corresponding CHSH inequality (55) with the correlators defined via (57), $\langle A_i B_j \rangle = \langle \chi_{ij}^{(cl)} \rangle$, can be maximally violated. This raises a question of whether a classical light can give rise to a contextual behaviour due to a violation of a Bell inequality. In a series of works similar scenarios were discussed [E65, E66, E57, E4, E28], which aim at showing, that it is indeed the case, and that such notions like entanglement and contextuality are not restricted to quantum physics. All these scenarios share the same trait, namely one simulates quantum detection probabilities with the relative intensities of a classical wave, and constructs a correlation function that violates some Bell inequality.

In Habilitation work [H6] we show that such an approach is entirely invalid due to a subtle issue connected with the structure of observed events. Let us start with an informal presentation. If we perform described experiment with a single photon input state, then all the four output events are exclusive, since at each run we can observe only a single detector click. However, if we perform the same experiment with more photons, or with a classical wave, the detection events are no longer exclusive: in the limiting case of a classical wave we detect all the relative intensities corresponding to the probabilities of the events simultaneously. It turns out that this lack of exclusivity of events prevents Bell inequality violation, namely the classical bound for the CHSH expression turns out to be equal to the algebraic bound, hence no place for any violation occurs, and no contextual effects can be observed. In order to show it formally, in [H6] we utilise the exclusivity graph approach to contextuality.

The starting point of this analysis is the graph-theoretical model for description of correlations in physical systems based on exclusivity structure of detection events, proposed by Cabello, Severini and Winter in [E15]. Let us describe this model in the case of a CHSH scenario. The CHSH inequality can be rewritten in terms of coincidence detection probabilities as follows:

$$\begin{aligned}
CHSH &= p(+-|11) + p(-+|11) + p(+-|01) + p(-+|01) + p(+-|10) \\
&+ p(-+|10) + p(++|00) + p(--|00) \le 3.
\end{aligned} \tag{58}$$

Probabilities appearing in the above formula may originate from different probabilistic models (classical, quantum, super-quantum), however they have to fulfill the exclusivity rule, which means that the sum of probabilities of mutually exclusive events cannot exceed one. The upper classical bound can be found purely from the exclusivity structure of the events. Let us represent the eight joint events in the above version of the CHSH inequality by a vertices, and lets assume that two vertices are connected by an edge if and only if they are mutually exclusive. In this way we obtain the exclusivity graph for the given correlational experiment (see Figure 7 c). As shown in [E15] the maximal value of the sum of the probabilities corresponding to the vertices of an exclusivity graph, which arise
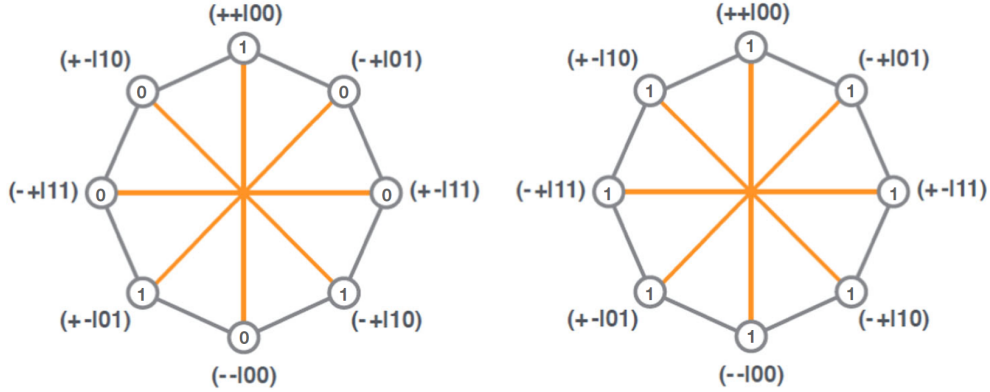
Figure 8: Non-contextual assignments of photons to events in the CHSH exclusivity graph, which maximize the non-contextual bound for the CHSH inequality (59). The left picture corresponds to one photon case (at most one photon can be assigned to a context), whereas the right picture represents two photon case (at most two photons per context).

from a non-contextual model, is equal to the independence number of the graph, defined as the size of the maximal subset of mutually non-adjacent vertices. Therefore for a given measurement scenario one can test the contextuality of the correlations by finding the independence number of the corresponding exclusivity graph and checking whether the physical probabilities violate the bound. In order to discuss the problem of contextuality of correlations for a transition from a single photon scenario to a classical wave we utilise a modified exclusivity-graph approach [E44], in which one assigns to each event (vertex) the number of detected photons. The noncontextual model assigns a definite photon number to each event in a way which respects exclusivity structure. Namely, if at most N photons can be assigned to a single event, then any measurement context, which is a subset of mutually exclusive events, can be assigned at most N photons in total. The CHSH inequality for the above model has an analogous form:

$$
\begin{aligned}
CHSH \;=\; & \frac{1}{N}\big(n(+-|11) + n(-+|11) + n(+-|01) + n(-+|01) \\
& + \; n(+-|10) + n(-+|10) + n(++|00) + n(--|00)\big) \leq \mathcal{C}(N),
\end{aligned}
\tag{59}
$$

where instead of probabilities one has (relative) photon numbers assigned to corresponding events. The bound $\mathcal{C}(N) = \frac{C_N}{N}$, where C_N is the upper bound on the sum of non-contextually assigned photon numbers in accordance with the exclusivity relations, directly depends on the total number of photons N. In the case of a single photon it equals to $\mathcal{C}(1) = 3$, as in the case of the original model. However if two photons are considered, one can non-contextually assign a single photon to each event in consistency with the exclusivity structure, therefore $\mathcal{C}(2) = \frac{8}{2} = 4$ (see Figure 8). This is in fact the maximal possible value achievable by any model which fulfills the exclusivity relations. Let us now discuss the classical limit of this inequality. By a classical limit we mean a macroscopic limit of a strong beam of quantum particles (e.g. photons), which fulfills two properties: (i) the occupation number ratios tend to intensity ratios, (ii) the ratio of the standard deviation of a particle number in a given mode to the average particle number tends to zero in the limit. Due to the second assumption macroscopic intensities can be treated as deterministic quantities. The single photon to electromagnetic wave transition fulfills these two

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assumptions, since a classical wave can be treated as a limit of a strong beam of photons in a coherent state, for which the standard deviation of the photon number behaves as a square root of the mean photon number (Poisonnian statistics). In the classical limit, the CHSH inequality (59) tends to the following intensity-based inequality:

$$
\begin{aligned}
CHSH \;\; = \;\; & \frac{1}{I}\big(I(+-|11) + I(-+|11) + I(+-|01) + I(-+|01) \\
+ \;\; & I(+-|10) + I(-+|10) + I(++|00) + I(--|00)\big) \leq \mathcal{C}_{cl} = 4,
\end{aligned}
\tag{60}
$$

for which the non-contextuality bound is equal to $\mathcal{C}_{cl} = 4$, the same as in the case of two photons. Since this bound is the maximal bound for any theory (classical, quantum, super-quantum) respecting the exclusivity structure, there is no place for violation of this inequality by any real or hypothetical physical system, and therefore the classical light cannot give rise to any contextual behaviour.

Three additional comments are necessary at this point. Firstly, the maximal physically attainable value of the CHSH inequality (60) equals to $2 + \sqrt{2} \approx 3.41$ [E16], which is the same as in the case of any implementation with photon-number states. This is because the quantum probability amplitudes of a single photon transform in the same way under the action of linear optical devices as the macroscopic electromagnetic waves. Note however, that only the single photon implementation gives rise to a contextual behaviour, as the classical bound for higher number of photons and for the classical wave is shifted above the physically attainable limit. Secondly, one has to emphasise that this shift of the classical (non-contextual) limit arises due to relaxation of the exclusivity structure of events in the transition from a single photon to a classical wave. This fact indicates, that the nonclassicality of correlations is not solely defined by the strength of correlation functions, but also by the specific exclusivity structure of detection events. As shown in the discussed example, it may happen that in a transition from a quantum to classical system the strength of the correlations remains the same, however the nonclassical character of these correlations disappears due to relaxation of the exclusivity structure of detection events. Finally, the example discussed here is only one of very many possible contextuality tests. However, as shown in [H6], the same reasoning applies to *arbitrary* contextuality scenario: in the classical limit the non-contextuality bound for any inequality designed to test contextuality of a system is equal to the bound for the most general probabilistic model, which satisfies the exclusivity relations specified by the experiment. Therefore classical electromagnetic waves cannot give rise to contextuality in any experimental scenarios.

5.7.2 Bell nonclassicality of a single photon

In this section I will focus on the Bell nonclassicality which can be induced by a single photon excitation in two spatially separated modes. The experimental setup discussed in the Habilitation work [H3] dates back to the first proposal for Bell nonclassicality of a single photon by Tan, Walls and Collett (TWC) [E67], see Figure 9. It consists of a source, which sends a single photon onto a balanced $50:50$ beamsplitter, which produces a superposition state in output modes b_1 and b_2 of the form:

$$
|\psi\rangle_{b_1,b_2} = \frac{1}{\sqrt{2}}\left[|01\rangle_{b_1,b_2} + i\,|10\rangle_{b_1,b_2}\right].
\tag{61}
$$

Although this state is just a single-photon excitation in a superposition of two modes, in the Fock representation it is mode-entangled, therefore the question arises whether this
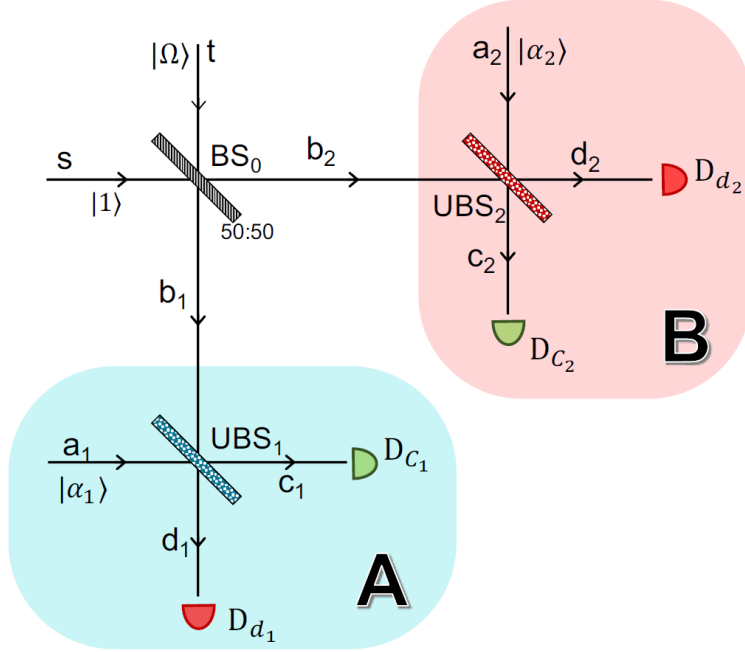
Figure 9: General scheme for testing nonclassical properties due to a single photon in a superposition of two spatial modes with homodyne detection. A single photon in mode s impinges on a beamsplitter BS_0. The two output modes b_1 and b_2 go to two measurement stations of Alice and Bob, and are superposed with local oscillators $\left|\alpha_i e^{i\theta_i}\right\rangle$ on beamsplitters UBS_1 and UBS_2. Finally photon numbers in output modes c_j and d_j of local beamsplitters are measured. In the original scheme due to [E67] beamsplitters UBS_1 and UBS_2 are symmetric, and the amplitudes $\alpha_i = \alpha$ of local oscillators are the same for both observers and both local measurement settings. In the modified scheme proposed by us in [H3] beamsplitters UBS_1 and UBS_2 have arbitrary tunable transmittivities, and the amplitudes of local oscillators α_i are also tunable and may differ for both observers and both local settings.

state can give rise to Bell-nonclassical correlations. In the original proposal of TWC, the two modes b_1 and b_2 are further measured by two local observers, Alice and Bob, via weak-field homodyne detection. The local measurement setup consists of a balanced $50:50$ beamsplitter BS_i, which superposes the input mode b_i with a weak local oscillator in a form of a coherent state $\left|\alpha e^{i\theta_i}\right\rangle$. Finally the output field in local modes c_i and d_i is measured by detectors D_{c_i} and D_{d_i}. In the TWC proposal the two different local measurement settings are specified solely by the phases of local oscillators $\{\theta_i, \theta_i'\}$. In order to check Bell nonclassicality of the correlations induced by the state (61), the authors of [E67] use CHSH-like inequalities [E59] of the form:

$$|E_T(\theta_1, \theta_2) + E_T(\theta_1', \theta_2) + E_T(\theta_1, \theta_2') - E_T(\theta_1', \theta_2')| \leq 2, \tag{62}$$

where the theoretical correlation functions for local intensities are specified as:

$$E_T(\theta_1, \theta_2) = \frac{\int d\lambda \rho(\lambda) \prod_{j=1,2} \left(I_{c_j}(\theta_j, \lambda) - I_{d_j}(\theta_j, \lambda)\right)}{\int d\lambda \rho(\lambda) I_1(\lambda) I_2(\lambda)}. \tag{63}$$

In the above equation $I_{x_j}(\theta_j, \lambda)$ denotes the hidden variable model for the local intensity measured at mode x_j for $x = c, d$ and $j = 1, 2$, whereas $I_j(\lambda)$ is the total local intensity. The quantum optical implementation of this correlation function reads:

$$
\begin{aligned}
E_T(\theta_1, \theta_2) &= \frac{\langle\Psi(\alpha)|(\hat{n}_{c_1} - \hat{n}_{d_1})(\hat{n}_{c_2} - \hat{n}_{d_2})|\Psi(\alpha)\rangle}{\langle\Psi(\alpha)|(\hat{n}_{c_1} + \hat{n}_{d_1})(\hat{n}_{c_2} + \hat{n}_{d_2})|\Psi(\alpha)\rangle} \\
&= A_T(\alpha)\sin(\theta_1 - \theta_2),
\end{aligned} \tag{64}
$$

where $A_T(\alpha) = \frac{1}{1+\alpha^2}$, and $|\Psi(\alpha)\rangle$ is the total initial state including local oscillators:

$$|\Psi(\alpha)\rangle = \frac{1}{\sqrt{2}} |\alpha e^{i\theta_1}\rangle_{a_1} (|01\rangle_{b_1 b_2} + i|10\rangle_{b_1 b_2}) |\alpha e^{i\theta_2}\rangle_{a_2}. \tag{65}$$

Substituting (64) into the inequality (62) and optimising over the local phases $\{\theta_1, \theta_1'\}$ and $\{\theta_2, \theta_2'\}$ one obtains a violation of the inequality (62) for the range of α's fulfilling $0 < \alpha^2 < 0.414$. However in our work [O5] it has been proven that for this range of α the output quantum probability distribution of the TWC setup can be exactly reproduced by a local hidden variable model. In the Habilitation work [H3] we emphasise, that the source of the failure of the TWC analysis is a hidden assumption in the CHSH-like inequality (62), namely that the total local intensity is independent of the local measurement setting:

$$I_j(\lambda) = I_{c_j}(\theta_j, \lambda) + I_{d_j}(\theta_j, \lambda). \tag{66}$$

This assumption is significantly restrictive with respect to possible local hidden variable models, and is manifestly violated by the model presented in [O5]. In the Habilitation work [H3] we show explicitly how to construct a proper Bell inequality for the TWC correlations. For this aim we use the intensity rates, introduced in [E39, E38]. They are defined via the relation:

$$R_{x_j}(\theta_j, \lambda) = \frac{I_{x_j}(\theta_j, \lambda)}{I_{c_j}(\theta_j, \lambda) + I_{d_j}(\theta_j, \lambda)}, \tag{67}$$

with an additional convention that $R_{x_j} = 0$ whenever the total intensity in the denominator equals to 0. Using intensity rates one can define the analogue of the correlation function (64):

$$E_R(\theta_1, \theta_2) = \left\langle \prod_{j=1}^{2} \left(R_{c_j}(\theta_j, \lambda) - R_{d_j}(\theta_j, \lambda)\right)\right\rangle_{HV}, \tag{68}$$

where the subscript HV indicates that the averaging is performed over the local hidden variable. Plugging the above correlation function into a CHSH expression one obtains a proper Bell inequality for the TWC setup:

$$|E_R(\theta_1, \theta_2) + E_R(\theta_1', \theta_2) + E_R(\theta_1, \theta_2') - E_R(\theta_1', \theta_2')| \le 2. \tag{69}$$

Note that due to a modified form of the correlator in (68) the inequality (69) does not depend on the additional assumption (66), as the total local intensity in the denominator of the intensity rate definition (67) explicitly depends on the settings. Quantum mechanical implementation of intensity rates reads as follows:

$$\hat{R}_{x_j} = \hat{\Pi}_{c_j d_j} \frac{\hat{n}_{x_j}}{\hat{n}_{c_j} + \hat{n}_{d_j}} \hat{\Pi}_{c_j d_j}, \tag{70}$$

where the operators $\hat{\Pi}_{c_j d_j} = \mathbb{1}_{c_j d_j} - \left|\Omega_{c_j d_j}\right\rangle \left\langle\Omega_{c_j d_j}\right|$ are projectors onto a non-vacuum sector in modes c_j and d_j. With the above definition one defines quantum correlation function:

$$
\begin{aligned}
E_R(\theta_1, \theta_2) &= \langle\Psi(\alpha)| \hat{H}_1(\theta_1)\hat{H}_2(\theta_2) |\Psi(\alpha)\rangle \\
&= A_R(\alpha)\sin(\theta_1 - \theta_2),
\end{aligned} \tag{71}
$$

where the operators $\hat{H}_i$ are defined as:

$$\hat{H}_j(\theta_j) = \hat{R}_{c_j} - \hat{R}_{d_j} = \hat{\Pi}_{c_j d_j} \frac{\hat{n}_{c_j} - \hat{n}_{d_j}}{\hat{n}_{c_j} + \hat{n}_{d_j}} \hat{\Pi}_{c_j d_j}, \tag{72}$$

and the amplitude $A_R(\alpha)$ reads:

$$A_R(\alpha) = \frac{e^{-2\alpha^2}(e^{\alpha^2} - 1)^2}{\alpha^2}. \tag{73}$$

In [H3] we show that the Bell inequality (69) is never violated for the TWC scheme, which is fully concurrent with the existence of the local hidden variable model. On the other hand we show, that both the inequalities (62), (69) can be utilised as entanglement witnesses.

In order to show an indisputable Bell nonclassicality induced by a single photon excitation in two modes via weak-field homodyne detection we modify the TWC scheme by relaxing the assumptions of fixed local 50:50 beamsplitters and fixed local oscillator strength. We assume that the two local settings are specified by three parameters: the strenght of the local oscillator α_i, and two parameters (χ_i, θ_i) of a general beamsplitter BS_i realising an $SU(2)$ transformation on input modes:

$$U_{BS_i}(\chi_i, \theta_i) = \begin{pmatrix} \cos\chi_i & e^{-i\theta_i}\sin\chi_i \\ -e^{i\theta_i}\sin\chi_i & \cos\chi_i \end{pmatrix}. \tag{74}$$

Note that $\cos^2\chi_i$ is the transmission coefficient of the beamsplitter whereas θ_i is the phase of the reflected beam. Instead of CHSH inequality, we use the probability-based Clauser-Horne inequality [E22]:

$$
\begin{aligned}
-1 \le P(A, B) + P(A, B') + P(A', B) - P(A', B') \\
-P(A) - P(B) \le 0,
\end{aligned} \tag{75}
$$

in which $P(\cdot, \cdot)$ denotes joint probability, $P(\cdot)$ denotes local probability, A, A' denote fixed events on Alice's side, whereas B, B' denote respective fixed events on the Bob's side. In

order to test the Bell nonclassicality in an extended TWC setup, we utilise CH inequlity for intensity rates, in which the probabilities in (75) are replaced with the corresponding rate operators:

$$
\begin{aligned}
-1 \leq \quad & K(\vec{v}_1, \vec{v}_2) + K(\vec{v}_1', \vec{v}_2) + K(\vec{v}_1, \vec{v}_2') - K(\vec{v}_1', \vec{v}_2') \\
& - S_1(\vec{v}_1) - S_2(\vec{v}_2) \leq 0,
\end{aligned}
\tag{76}
$$

in which the correlation functions and the local terms read:

$$
\begin{aligned}
K(\vec{v}_1, \vec{v}_2) &= \langle R_{d_1}(\vec{v}_1) R_{d_2}(\vec{v}_2) \rangle_{HV} , \\
S_j(\vec{v}_j) &= \langle R_{d_j}(\vec{v}_j) \rangle_{HV} .
\end{aligned}
\tag{77}
$$

In the above equations the vectors $\vec{v}_i$ specify local settings. We assume the so called *on/off* detection scheme, inspired by the work of Hardy [E29], in which in one of the local settings the local oscillator is turned *off* and the local beamsplitter is set to identity operation:

$$
\begin{aligned}
\vec{v}_1 &= (0,0,0) \\
\vec{v}_1' &= (\chi_1', \alpha_1', \theta_1') \\
\vec{v}_2 &= (0,0,0) \\
\vec{v}_2' &= (\chi_2', \alpha_2', \theta_2').
\end{aligned}
\tag{78}
$$

In [H3] we report a violation of the left-hand side of the inequality (76) with minimal achievable quantum value equal to -1.0239. The almost optimal settings are:

$$
\begin{aligned}
\vec{v}_1' &= \left(\chi_1' = \frac{3\pi}{20}, \alpha_1' = \frac{\sqrt{2}}{2}, \theta_1' = 0 \right) , \\
\vec{v}_2' &= \left(\chi_2' = \frac{3\pi}{20}, \alpha_2' = \frac{\sqrt{2}}{2}, \theta_2' = -\frac{\pi}{2} \right) ,
\end{aligned}
\tag{79}
$$

which correspond to beamsplitters with transmittivity around 79% and local oscillators with average photon number equal to $\frac{1}{2}$. We have also shown that one can obtain violation of the rate-based CH inequality (76) for non-perfect *on/off* settings, however the deviation from the *on/off* scheme must not be too high, see Figure 10. For this aim we assumed the following form of local settings:

$$
\begin{aligned}
\vec{v}_1 &= (\chi_1, \alpha, \theta_1), \\
\vec{v}_1' &= (\chi_1', \alpha', \theta_1'), \\
\vec{v}_2 &= (\chi_2, \alpha, \theta_2), \\
\vec{v}_2' &= (\chi_2', \alpha', \theta_2'),
\end{aligned}
\tag{80}
$$

in which the amplitudes of the local oscillators corresponding to a given setting are the same for both observers.

To sum up in the Habilitation work [H3] we have shown how to construct proper Bell inequalities for detection of Bell nonclassicality induced by a single-photon superposition in two modes. We have explained the failure of the TWC approach [E67] and shown that using local oscillators with strength varying from setting to setting is necessary for obtaining an indisputable Bell nonclassicality.

Two aspects need additional comments. Firstly, one may ask the question, why to utilise homodyne measurements for the task of single-photon Bell nonclassicality. The answer is
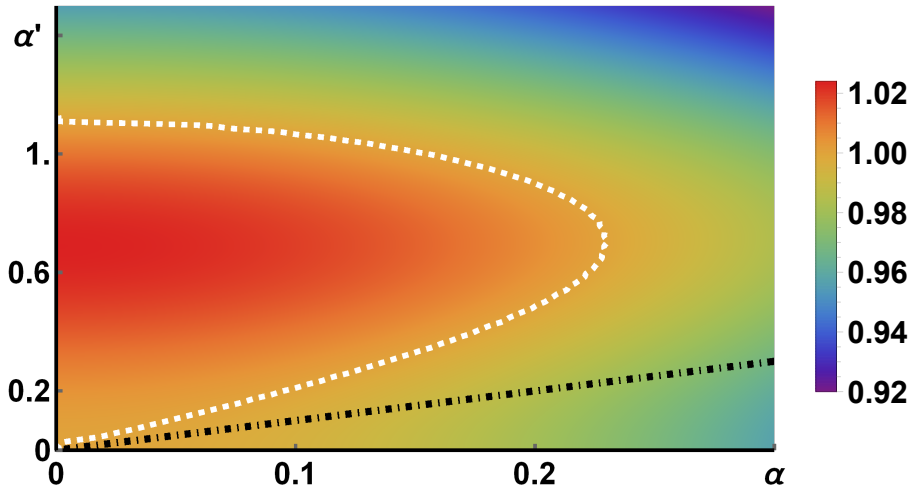
Figure 10: Plot of the CH expression for rates (76) for general measurement settings (80) as a function of the strengths of the local oscillators corresponding to two different local settings. We assume, that both observers use the same set of settings $\{\alpha, \alpha'\}$. The sign of the CH value is sign-flipped for the matter of clarity of presentation. Presented CH value for a given set of settings $\{\alpha, \alpha'\}$ corresponds to a CH expression minimised over the remaining settings, the transmittivity and phase of the local beamsplitter. The dotted white line is an envelope of the CH inequality (76) violation region, whereas the dotted black line corresponds to $\alpha = \alpha'$. The optimal violation is reached for perfect *on/off* arrangement, when $\alpha = 0$. It can be seen that the violation occurs also for non-perfect *on/off* settings, when both α and α' are non-zero, however it can never happen for constant local oscillator strengths $\alpha = \alpha'$ for both settings.

that in order to confirm Bell nonclassicality of the state (61) for one of the measurement settings one needs to perform a measurement in a basis complementary to the photon number basis. However this is impossible without the use of additional photons serving as a reference frame [E52], hence the presence of local oscillators (or their low-photon-number approximations) is necessary.

Second aspect is the apparent single-photon nature of the experiment. In [H3] we emphasize, that the discussed experiment is in fact a multiphoton one, since the observed nonclassicality comes from an interference due to indistinguishability of photons coming from local oscillator and the photon coming from the input source. At the detection stage, the distinction between these photons has no meaning, and therefore observed correlations cannot be attributed solely to a single photon.

6 Presentation of teaching, organisational, and 'popularisation of science' achievements

6.1 Teaching achievements

Academic teaching:

- Leading tutorials in *Linear Algebra* for 1st year Physics students, 90 hours, University of Gdańsk, Oct 2011 – Feb 2012.

Non-academic teaching:

- Leading tutorials in Physics for Students Physics Association in 3rd High School in Gdynia, March – June 2014.

6.2 Organisational achievements

- Member of the local organising committee of the conference: 9th Biennial IQSA Meeting, Quantum Structures, Brussels - Gdańsk 2008.
  Conference publications can be found at:
  https://doi.org/10.1007/s10773-010-0513-0

6.3 Popularisation of science achievements

- Public lecture *Quantum Information and Teleportation*, presented in 3rd High School in Gdynia, March 2014.

7 Other scientific achievements

7.1 Bibliometric data

Source: **Google Scholar** (14.12.2021)

- Number of peer-reviewed publications: 32 (20 after PhD)

- Number of unpublished arXiv preprints: 8

- Total number of citations: 601

- H-index: 14

- I10-index: 16

Source: **Web of Science** (14.12.2021)

- Number of peer-reviewed publications: 32 (20 after PhD)

- Total number of citations: 373

- H-index: 10

7.2 Awards

- **START Scholarship** from Foundation for Polish Science (FNP) in 2016/2017.

7.3 Track record before PhD

7.3.1 Research included in my PhD thesis:

- **Quantum precision gain in noisy metrology**. After it was realised that quantum gain in precision of estimation of unknown parameters disappears under presence of noise, many efforts have been undertaken to overcome this difficulty. Our work [O4] was one of the pioneering publications tackling this problem. We have proposed a frequency estimation protocol allowing for superclassical scaling of precision of estimation despite the presence of dephasing noise. The protocol is based on the idea of adjusting the estimation time interval to the total number of particles involved in the estimation process. The work [O4] has been an important motivation for further research in overcoming the influence of decoherence on metrological precision.

- **Correlation-tensor criteria for entanglement detection**. In a series of works [O11, O16, O12] we have developed correlation-tensor approach to detection of multipartite entanglement. In [O11] a necessary and sufficient geometrical criterion for partial separability has been presented. In [O16] a condition for detection of multipartite entanglement of pure states has been proposed which uses solely bipartite correlations, whereas in [O12] a criterion based on summing non-negative functions of correlations has been stated, which demonstrates special experimentally friendly features.

- **Hidden variable models for different levels of correlations**. In the work [O13] we show that local hidden-variable (LHV) models can exist for a given state for correlations involving fixed numbers of observers, however it may happen that these models cannot be extended to a single model for the entire state. Therefore we observe incompatibility between LHV models for different levels of correlations.

- **Quantum extensions of round-based models in distributed computing**. Round-based local models are graph-based models of computation, in which several computers distributed in space are connected according to some input graph, and aim at computing some target function in minimal number of rounds, each involving *communication with nearest neighbours*. In this model there are no limits on local computational capabilities or on the amount of communication between the adjacent nodes. In the work [O7] we have introduced quantum extensions of local round-based models, including sharing entangled states between the nodes and using quantum communication channels.

7.3.2 Research not included in my PhD thesis:

- **Quantum information processing on spin chains**. In a series of works [O21, O23, O22, O9] we have investigated information processing protocols implementable on linear spin chains. In the work [O21] a perfect state transfer protocol has been presented, which demands neither initialisation of the state of the chain nor any exchange of information between the sender and receiver. In [O22] we have discussed the impact of several models of noise on the state transfer through spin chains, whereas in [O23] we analysed the possibility of detecting imperfections in the structure of the chains. Finally in [O9] we have discussed chains of bipartite states, each of which do not violate Bell inequalities. We have shown that after performing some number of entanglement swapping operations Bell-nonclassical correlations appear in the system.

- **Unified approach to several different aspects of nonclassicality**. In the work [O15] we highlight the existence of a joint probability distribution as the common underpinning assumption behind Bell-type, contextuality, and Leggett-Garg-type tests. We then present a procedure to translate contextual scenarios into temporal Leggett-Garg-type and spatial Bell-type ones. To demonstrate the generality of this approach we construct a family of spatial Bell-type inequalities. We show that in Leggett-Garg scenario a necessary condition for contextuality in time is given by a violation of consistency conditions in Consistent Histories approach to quantum mechanics.

- **Multi-point temporal quantum correlations**. It has been thought for a long time that only spatial quantum correlations can reveal multi-point character in contrast to at most two-point temporal quantum correlations. In the work [O19] we have shown that this is not the case by presenting a constructive procedure of changing multipartite spatial correlations into multi-point temporal ones corresponding to sequential POVMs (Positive Operator Value Measurements). This procedure allows new architectures for quantum computing. For example measurement-based quantum computing, typically implemented on 2D lattices, can be realised on 1D lattices undergoing sequential adaptive POVM measurements.

7.4 Additional track record after PhD

Research carried out by me after PhD, which is not a part of this Habilitation achievement, includes the following projects:

- **Safe postselection in Bell experiments**. Postselection can be a harmfull procedure when performing Bell experiments, since it can introduce non-causal correlations, which can fake the nonclassicality test. This is particularly important in optical

experiments, in which postselection is very often a part of a preparation procedure of entangled quantum states. In the work [O1] we have introduced a condition called *all-but-one* principle, which assures safe postselection for multipartite Bell tests. A given postselection procedure conforms to the all-but-one condition if it can be resolved without one party being involved in the procedure, and this property must hold for any division of the parties (any observer can be excluded). Safety of such a postselection scheme is proved using diagramatic tools of causal inference called *d-separation* rules.

- **Quantum random walk models of multipartite bound states**. Borromean states are bound states of three or more particles which fall apart whenever one or more particles is removed. The main result presented in [O14] is the existence of three- and four-partite Borromean states in 1-dimensional discrete-time quantum walk with a coin operator being a generalised Grover one. The impact of this result relies on the fact that previously Borromean-type states have been found only in nuclear and atomic physics, whereas the construction presented in [O14] can serve as a toy model for investigating the structure of interactions in multipartite quantum systems, which lead to formation of Borromean states.

- **Multiparameter quantum metrology**. The main result of the work [O17] is the proof that a generalised three- and four-mode Mach-Zehnder interferometer can be utilised in order to perform estimation of respectively two and three unknown phases (placed arbitrarily among the three or four modes of the interferometer) with Heisenberg-like scaling of precision. The estimation procedure assumes entirely fixed interferometric setup, namely the same initial state and the same set of measurements are used in order to estimate arbitrary subset of unknown phases (the remaining not estimated phase in one mode serves as a phase reference).

- **Finite averaging sets for averaging over non-compact symmetry groups**. The project on averaging physical quantities over non-compact symmetry groups has been partially completed by a publication [O18], in which a construction of finite averaging sets for the symmetry group $SL(2, \mathbb{C})$ has been provided. The construction can be applied to build finite averaging sets for averaging multiqubit quantum states over SLOCC operations (Stochastic Local Operations and Classical Communication), which can be treated as a generalisation of the concept of unitary t-designs to averaging over non-unitary operations. The construction is based on the apparatus of the Lie algebras theory and the theory of generalised Gauss quadratures based on orthogonal polynomials.

- **Reconsideration of newest Wigner's-Friend-type paradoxes**. The main message of the work [O8] is the no-go theorem for existence of outcomes of so-called pre-measurements in quantum measurement theory. A pre-measurement is the first stage of the measurement process in which a system gets entangled with the pointer degrees of freedom of a measuring device. In [O8] it has been shown that one cannot ascribe any notion of outcome to such a process, since then, assuming correctness of quantum mechanical predictions, one obtains direct contradiction. This invalidates the recent claims of internal inconsistency of quantum mechanical description of the measurement process involving many observers presented in: D. Frauchiger and R. Renner , Nat. Comm. **9**, 3711 (2018), and in the works of the followers.

- **Multi-path coherence and interaction-free measurements**. In the work [O24] we discuss a relation between the existence of genuine multi-path coherence and perfect interaction-free measurements. We point out that in quantum theory only two-path coherence is allowed, which corresponds to the fact that any interferometric multi-slit experiment can be described as a probabilistic mixture of double-slit experiments. We show that this fact precludes existence of perfect interaction-free measurements in quantum theory. Nevertheless such measurements are possible in hypothetical theories, which allow genuine multi-path coherence, like the density-cube theory.

- **Random unitary noise generated by local random Hamiltonians**. In the work [O20] we investigate how random unitary noise affects multipartite quantum states. We assume that the noise is generated by random Hamiltonians. This provides a characterisation of robustness of quantum states against random unitary noise acting over sufficiently short time intervals. It turns out that such robustness is fully characterised by mean Quantum Fisher Information (QFI) of a given state, namely the lower the mean QFI the more robust a given state is. Moreover we have shown that such robustness very weakly depends on the probability distribution of the generators of noise.

- **New geometrical representation of a three-level quantum system**. In the work [O10] we present a three-dimensional Bloch-sphere-like representation of arbitrary state of a qutrit. We show that any such state can be represented in a 3-dimensional Euclidean space by a vector lying inside an ellipsoid of definite shape and orientation of the axes.

- **Quantum communication complexity advantage vs Bell nonclassicality**. Reduction of communication complexity and Bell nonclassicality are two different aspects of nonclassical properties of correlations arising due to measurements on quantum systems. Their general mutual relation has remained unclear for a long time. In the work [O3] we provide a significant step towards understanding this relation, by showing that statistics obtained from communication complexity protocols, which assures sufficiently significant reduction in complexity, leads to a violation of some Bell inequality. The proof is performed using the notion of port-based teleportation protocol.

- **New definition of nonclassicality of temporal correlations**. The notion of nonclassicality for temporal correlations arising from sequential measurements on a single physical system is not unique and not well justified by physical intuitions, which stays in sharp contrast to spatial correlations in Bell-type scenarios. In the work [O2] we propose a new definition of nonclassicality of temporal correlations inspired by the ideas of communication complexity. We define non-classical temporal correlations as the ones which cannot be simulated by propagating in time a classical information content of a quantum system. We present a sequence of POVM measurements on a single m-level quantum system that cannot be explained by propagating in time m-level classical system and using classical computers with unlimited memory.

- **Relation between quantum computation speedup limits and metrological precision bounds**. In the work [O6] we have introduced a new method of analyzing the performance of quantum search algorithms, by pointing out an analogy between Grover-type search algorithms and quantum phase estimation procedures in quantum

metrology. We have proposed a general conjecture, that the class of noises which preclude quantum speedup in search algorithms is the same as the class of noises which invalidate quantum gain in precision of phase estimation.

Habilitation Series

[H1] Pawel Blasiak, Ewa Borsuk, Marcin Markiewicz, and Yong-Su Kim. Efficient linear-optical generation of a multipartite W state. *Phys. Rev. A*, 104:023701, Aug 2021.

[H2] Pawel Blasiak and Marcin Markiewicz. Entangling three qubits without ever touching. *Scientific Reports*, 9(1):20131, Dec 2019.

[H3] Tamoghna Das, Marcin Karczewski, Antonio Mandarino, Marcin Markiewicz, Bianka Woloncewicz, and Marek Żukowski. Can single photon excitation of two spatially separated modes lead to a violation of Bell inequality via weak-field homodyne measurements? *New Journal of Physics*, 23(7):073042, Jul 2021.

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