Summary of Professional Accomplishments

Michał Grzegorz Studziński

Faculty of Mathematics, Physics and Informatics
University of Gdańsk
Gdańsk
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Contents

1 Name and surname

Michał Grzegorz Studziński

2 Diplomas, scientific degrees

1. PhD in Physics – 11.06.2015
   Institution: Faculty of Mathematics, Physics and Informatics, National Quantum Information Centre, University of Gdańsk, Poland
   PhD Thesis: Application of theory of groups and algebras representations to some quantum information processing problems
   Supervisor: prof. dr hab. Michał Horodecki
   Assistant Supervisor: dr Jarosław Korbicz
   Reviewers: prof. dr hab. Marek Kuś, prof. dr hab. Andrzej Jamiołkowski
   Funding source: International PhD project: Physics of future quantum-based information technologies, grant MPD/2009-3/4 from Foundation for Polish Science.

2. MSc degree in Astronomy – 01.07.2010
   Institution: Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University in Toruń, Poland
   Master's Thesis: Integrability analysis of rational homogeneous potentials
   Supervisor: dr hab. Maria Przybylska
   Grade: very good

3. BSc degree in Astronomy – 11.05.2009
   Institution: Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University in Toruń, Poland
   Bachelor's Thesis: Integrability analysis of certain classes of rational homogeneous potentials
   Supervisor: dr hab. Maria Przybylska
   Grade: very good

3 Information on previous employment in scientific institutions

1. 02.01.2022-present: Adiunkt (permanent position)
   Institution: Faculty of Mathematics, Physics and Informatics, University of Gdańsk, Gdańsk, Poland
   Supervisor: dr hab. Marek Krośnicki, prof. UG

2. 01.2019-01.01.2022: Adiunkt (post-doctoral position)
   Institution: Faculty of Mathematics, Physics and Informatics/National Quantum Information Centre, University of Gdańsk, Gdańsk, Poland
   Supervisor: dr hab. Marcin Marciniak, prof. UG/dr hab. Marek Krośnicki, prof. UG

3. 01.01.2016-31.12.2018: post-doctoral researcher
   Institution: Department of Theoretical Physics and Applied Mathematics, University of Cambridge, Cambridge, United Kingdom
   Supervisor: prof. Richard Jozsa

4. 06.2015-31.01.2016: post-doctoral researcher
   Institution: Faculty of Mathematics, Physics and Informatics/National Quantum Information Centre, University of Gdańsk, Gdańsk, Poland
   Supervisor: prof. Michał Horodecki

4 Description of the achievements according to Art. 219 Para 1 Point 2 of the Act

4.1 Title of the achievement

Single-themed series of publications, titled: *Groups and algebras representation theory as a tool for description and construction of novel quantum information processing protocols.*

4.2 List of selected publications

The list of publications related thematically (chronological order):

1. *Square-root measurements and degradation of the resource state in port-based teleportation scheme*
   M. Studziński, M. Mozrzymas, P. Kopszak
   Journal of Physics A: Mathematical and Theoretical **55** 375302 (2022)
   IF: 1.996 / MNiSW points: 100
   `https://arxiv.org/abs/2105.14886`

2. *Efficient multi-port teleportation schemes*
   M. Studziński, M. Mozrzymas, P. Kopszak, M. Horodecki
   IEEE Transactions on Information Theory **68(12)** 7892-7912 (2022)
   IF: 2.978 / MNiSW points: 200
   `https://arxiv.org/abs/2008.00984`

3. *Optimal Multi-port-based Teleportation Schemes*,
   M. Mozrzymas, M. Studziński, P. Kopszak
   Quantum **5**, 477 (2021)
   IF: 2.921 / MNiSW points: 140
   `https://arxiv.org/abs/2105.14886`

4. *Efficient Classical Simulation and Benchmarking of Quantum Processes in the Weyl Basis*,
   D. S. França, S. Strelchuk, M. Studziński
   Physical Review Letters **126** 210502 (2021)
   IF: 9.185 / MNiSW points: 200
   `https://arxiv.org/abs/2008.12250`

5. *Multiport based teleportation – protocol and its performance*
   P. Kopszak, M. Mozrzymas, M. Studziński, M. Horodecki
   Quantum **5**, 576 (2021)
   IF: 2.921 / MNiSW points: 140
   `https://arxiv.org/abs/2008.00856`

6. *Positive Maps From Irreducibly Covariant Operators*
   P. Kopszak, M. Mozrzymas, M. Studziński
   Journal of Physics A: Mathematical and Theoretical **53** 395306 (2020)
   IF: 1.996 / MNiSW points: 100
   `https://arxiv.org/abs/1911.13137`

7. *Simplified formalism of the algebra of partially transposed permutation operators with applications*
   M. Mozrzymas, M. Studziński, M. Horodecki
   Journal of Physics A: Mathematical and Theoretical **51** 125202 (2018)
   IF: 1.996 / MNiSW points: 100
   `https://arxiv.org/abs/1708.02434`

8. *Optimal Port-based Teleportation*
   M. Mozrzymas, M. Studziński, S. Strelchuk, M. Horodecki
   New Journal of Physics **20.5** (2018): 053006
   IF: 3.539 / MNiSW points: 140
   `https://arxiv.org/abs/1707.08456`

9. *Port-based teleportation in arbitrary dimension*
   M. Studziński, S. Strelchuk, M. Mozrzymas, M. Horodecki
   Scientific Reports **7**: 10871 (2017)
   IF: 3.998 / MNiSW points: 140
   `https://arxiv.org/abs/1612.09260`

10. *Structure of irreducibly covariant quantum channels for finite groups*
    M. Mozrzymas, M. Studziński, N. Datta
    Journal of Mathematical Physics **58**, 052204 (2017)
    IF: 1.488 MNiSW points: 70
    `https://arxiv.org/abs/1610.05657`

5 Presentation of significant scientific activity

My scientific achievements have a form of collective publications. My own contribution has been described in section I.2 of the enclosed document "List of scientific or artistic achievements which present a major contribution to the development of a specific discipline", whereas the contribution statements of other co-authors are enclosed in a separate document "Contribution Statements". In addition, the following reference convention is used in this presentation:

- the publications which belong to the presented scientific achievement are referred to as [H1]-[H10],
- other publications co-authored by me, and not included in the achievement are cited as [P1]-[P15],
- external publications are cited as [E1]-[E107].

5.1 Introduction

The phenomenon of quantum entanglement is believed to be most amazing and eluding the schemes of classical thinking. That fact was noticed directly after the mathematical principles of non-relativistic quantum mechanics had been formulated [E1]. Scientists investigating mentioned quantum-mechanical effects have realized if we could control and apply them it would open completely new, not accessible for classical implementations areas [E2]. Today we exploit these findings and develop the theory of secure quantum algorithms [E3, E4], quantum cryptography [E5] or quantum computing [E6, E7] - we have cited here only a few classical results. Such promising perspectives of practical implementation of quantum features as a resource, clearly demonstrate the importance of undertaken efforts to improve the theoretical comprehension of this phenomenon as well as its experimental implementation, and finally commercialization which can be observed in today's boost for developing quantum technologies. Besides the enormous progress in the field in the last decades, still, quantum information is a rich area for new and important results, coming from both directions – theoretical and practical implementations. One way to make the progress in the area can be by exploiting and developing the mathematical approach. In our case, it was done by considering the internal symmetries of the considered systems. It is well-known fact, that whenever a system possesses symmetries of some kind, its description becomes much easier, and frequently closed formulas describing its certain properties can be obtained. To identify and exploit underlying symmetries we can use a powerful tool which is group and algebra representation theory, which in most cases allows us to reduce the underlying complexity of the problem. This is the reason why a deep understanding of abstract tools together with their 'applicable' translation is so important.

The aim of this habilitation series is to deliver *new advanced mathematical tools* arising from the representation theory and provide new insights on the set of *covariant quantum channels* and their applications for problems related to *noisy quantum circuits and their classical simulability* and a better understanding of one of the most important primitives in quantum information exploiting quantum entanglement- *quantum teleportation*. More detailed motivation for undertaking such a research program is included in every section dedicated to a certain particular research task.

All research results have been obtained with close collaboration with Polish and foreign scientific centers – *The University of Wrocław, The University of Cambridge (United Kingdom)* and *The University of Copenhagen*. Part of the results has been obtained by the applicant during his postdoc stays at The University of Cambridge (papers [H1],[H2],[H3],[H4]) and at The University of Gdńsk (papers [H5],[H6],[H7],[H8]). The last two papers [H9],[H10] have been published after getting a faculty position at The University of Gdańsk.

The detailed achievements of the undertaken research problems in the habilitation series can be summarised in the list below:

1. Classification and construction of linear irreducibly covariant maps with respect to finite groups and chosen compact groups. This is covered in the series of two papers [H1], [H5].

2. Developments in randomized benchmarking protocols and classical simulation of quantum circuits by involving new classes of irreducibly covariant quantum channels. This is covered in paper [H6].

3. Developing novel mathematical toolkit inspired by the Schur-Weyl duality with applications to develop and describe new covariant quantum teleportation schemes. All developed tools are spread among papers on port-based teleportation and multi port-based teleportation schemes [H2], [H10] [H3], and [H9].

4. Group theoretic description of port-based teleportation protocols in every variant and arbitrary dimension of the port, together with detailed efficiency analysis. Description and analysis of the recycling scheme for the deterministic port-based teleportation scheme. This is covered in the series of three papers [H3], [H4], and [H9].

5. Construction and efficiency analysis of novel covariant teleportation protocols with mild correction for transmission of a large amount of quantum information. Investigation on ultimate limitations imposed on port-based type protocols by quantum mechanics. This is covered in the series of three papers [H7], [H10], and [H8].

The results regarding mathematical tools and (multi) port-based teleportation schemes have been presented as contributed talks at the most prestigious annual conference in the field - Quantum Information Processing (2018: `https://qipconference.org/2018/qutech.nl/qip2018/qip-2018-program-details/index.html`, 2021: `https://www.mcqst.de/qip2021/program/friday.html`).

5.2 Primer on representation theory used in this summary

To make the material presented in the habilitation series more accessible for readers we deliver here a short introduction to some aspects of group representation theory. We focus here only on the main definitions, notations, and theorems used later in this manuscript. For more details, we encourage readers to have a look at references within this text or habilitation papers. The pieces of information contained in this subsection are not original results of the applicant, except the presentation and some notation.

Let S_n be a symmetric group over a finite set of n symbols (here number of systems). A permutational representation is a map $V : S_n \to \operatorname{Hom}((\mathbb{C}^d)^{\otimes n})$ of the symmetric group S_n in the space $\mathcal{H} \equiv (\mathbb{C}^d)^{\otimes n}$ defined by its action on orthonormal basis $\{|e_k\rangle\}_{k=1}^d$ of $d-$dimensional space $\mathbb{C}^d$:

$$\forall \pi \in S_n \qquad V_\pi.|e_{i_1}\rangle \otimes |e_{i_2}\rangle \otimes \cdots \otimes |e_{i_n}\rangle := |e_{i_{\pi^{-1}(1)}}\rangle \otimes |e_{i_{\pi^{-1}(2)}}\rangle \otimes \cdots \otimes |e_{i_{\pi^{-1}(n)}}\rangle. \tag{1}$$

Since the representation V is defined with respect to a given basis in the space $\mathbb{C}^d$ it is a matrix representation, and operators V_π just permute basis vectors according to the given permutation $\pi \in S_n$. The representation V naturally extends to the representation of the group algebra $\mathbb{C}[S_n]$ defined as:

$$\mathbb{C}[S_n] \equiv \mathcal{A}_n(d) := \operatorname{span}_{\mathbb{C}}\{V_\pi : \pi \in S_n\} \subset \operatorname{Hom}((\mathbb{C}^d)^{\otimes n}). \tag{2}$$

For the further reasons by writing $V_{(a,n)}$ we understand the following operator $\mathbf{1}_{1\ldots\bar{a}\ldots n} \otimes V_{(a,n)}$ permuting systems according to the permutation $\pi = (a, n)$ with suppressed identity. The identity operator $\mathbf{1}_{1\ldots\bar{a}\ldots n}$ acts on the first $n-1$ particles but a. We will use the notion of suppressing the identity operator whenever it is clear from the context.

To work with irreducible representations (irreps) of the symmetric group S_n we need to introduce a notion of a *partition*. A partition α of a natural number n, which we denote as $\alpha \vdash n$, is a sequence of positive numbers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$ and $\sum_{i=1}^r \alpha_i = n$. Every partition can be visualised as a *Young frame* which a collection of boxes arranged in left-justified rows – see panel A in Figure 1. For a fixed number n, the

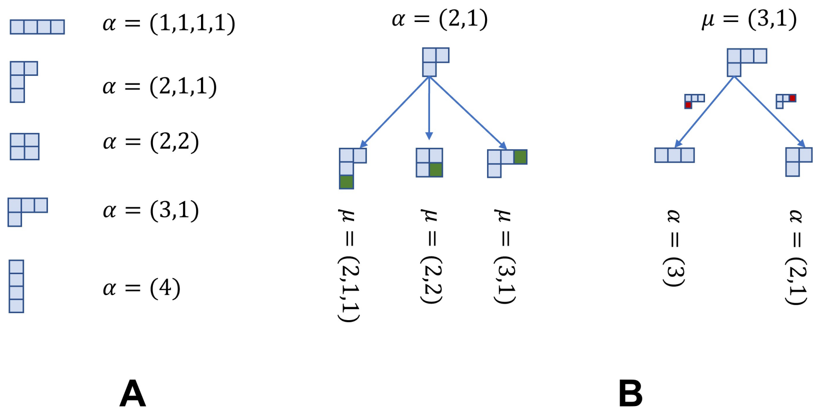

Figure 1: **Panel A** depicts five possible Young frames for $n = 4$ corresponding to all possible abstract irreducible representations of the group S_4. If the representation space is $(\mathbb{C}^d)^{\otimes 4}$ the only irreps that appear are those for which the height $h(\cdot)$ (the length of the first column) of the corresponding Young frames is no larger than d. In particular, when one considers qubits ($d = 2$) we have only three admissible frames: $(4), (3,1), (2,2)$. **Panel B** presents all possible Young frames $\mu \vdash 4$ satisfying relation $\mu \in \alpha$ for $\alpha = (2,1)$. Green squares depict boxes that are added to the initial frame α. On the most right-hand side, we present all possible Young frames $\alpha \vdash 3$ satisfying relation $\alpha \in \mu$ for $\mu = (3,1)$. Boxes subtracted from the initial Young frame $\mu = (3,1)$ are shown in red.

number of Young frames determines the number of nonequivalent irreps of S_n in an abstract decomposition. The set of all Young diagrams, with up to n boxes, is denoted as $\mathbb{Y}_n$. However, working in the representation space $\mathcal{H} \equiv (\mathbb{C}^d)^{\otimes n}$, when decomposing of S_n into irreps we take Young frames α whose height $h(\alpha)$ is at most d. This restricts the set of Young diagrams to the set of Young diagrams with no more than d rows, and it is denoted as $\mathbb{Y}_{n,d}$. We endow $\mathbb{Y}_n, \mathbb{Y}_{n,d}$ with a structure of a partially ordered set by setting

$$\alpha \preceq \mu, \tag{3}$$

if $\mu_i \geq \alpha_i$ for all $i = 1, 2, \ldots, l$. If $\alpha \preceq \mu$ we denote by μ/α a skew Young shape, obtained by removing from the Young frame μ the boxes of the Young frame of α. Now suppose we have $\alpha \vdash n-1$ and $\mu \vdash n$. When writing $\mu \in \alpha$ we mean Young frames μ generated from α by adding a single box. Similarly, $\alpha \in \mu$ denotes Young frames α generated from μ by removing a single box. Panel B in Figure 1 illustrates these processes. The described procedure, of course, can be generalized to more boxes being added/removed. We use the same symbols $\mu \in \alpha$, $\alpha \in \mu$ independently from the number k of boxes added/removed, since k will be always clear from the context. For any $\alpha, \mu \in \mathbb{Y}_n$ we say that μ covers α, or α is covered by μ if $\alpha \preceq \mu$ and

$$\alpha \preceq \nu \preceq \mu, \quad \nu \in \mathbb{Y}_n \Rightarrow \nu = \alpha \text{ or } \nu = \mu. \tag{4}$$

In other words, μ covers α if and only if $\alpha \preceq \mu$ and μ/α consists of at least a single box. Having the concept of Young diagram and sets $\mathbb{Y}_n, \mathbb{Y}_{n,d}$ we define Young's lattice and its reduced version. The Young's lattice of $\mathbb{Y}_n$ is the non-oriented graph with vertex set $\mathbb{Y}_n$ and an edge from λ to μ if and only if λ covers μ. The same definition applies to Young's lattice of $\mathbb{Y}_{n,d}$, but we remove all Young diagrams with more than d rows. This idea is illustrated in Figure 2.

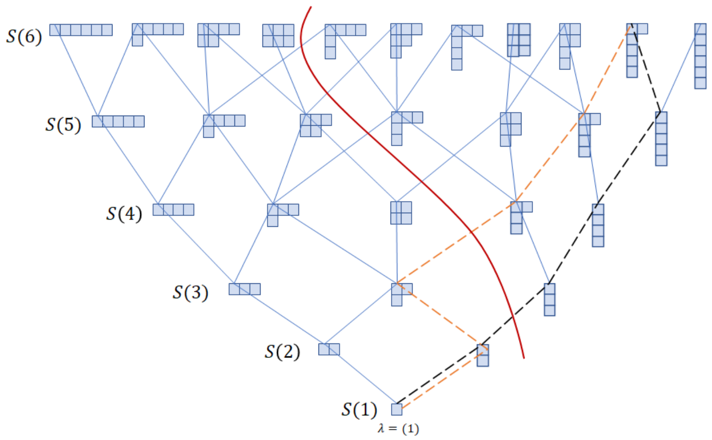

Figure 2: The Young's lattice $\mathbb{Y}_6$ composed of six consecutive layers labelled by permutation groups from S_1 to S_6. By orange and black dashed lines, we depict two possible paths from irrep $\lambda = (1)$ of S_1 to irrep $\lambda' = (2, 1, 1, 1, 1)$ of S_6. The reduced Young's lattice $Y_{6,2}$, i.e. for $d = 2$ is defined by all diagrams on the left-hand side of the red line.

We can introduce also the lexicographic ordering in the set of all Young diagrams for fixed n. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be partitions of n. The lexicographical ordering is defined so that if the first i for which $\mu_i \neq \lambda_i$, if any, has $\mu_i \leq \lambda_i$, then we write $\mu \leq \lambda$. In particular, we use lexicographic ordering for arranging Young diagrams in every row of the (reduced) Young lattice from Figure 2 and further for labeling rows and columns of the teleportation matrix described in Section 5.6.1.

Recall the celebrated Schur-Weyl duality [E8, E9], which states that the diagonal action of the unitary group $\mathcal{U}(d)$ of unitary complex matrices and action of the symmetric group S_n on $(\mathbb{C}^d)^{\otimes n}$ commute:

$$[V_\pi, U \otimes \cdots \otimes U] = 0, \tag{5}$$

where $\pi \in S_n$ and $U \in \mathcal{U}(d)$, and the space $(\mathbb{C}^d)^{\otimes n}$ can be decomposed as follows:

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\substack{\alpha \vdash n \\ h(\alpha) \leq d}} \mathcal{H}_{\mathcal{U}}^{\alpha} \otimes \mathcal{H}_{\mathcal{S}}^{\alpha}. \tag{6}$$

The symmetric group S_n acts non-trivially on the space $\mathcal{H}_{\mathcal{S}}^{\alpha}$ and the unitary group $\mathcal{U}(d)$ acts non-trivially on the space $\mathcal{H}_{\mathcal{U}}^{\alpha}$, labelled by the same partitions α. From the decomposition given in equation (6) we deduce that for a given irrep α of S_n, the space $\mathcal{H}_{\mathcal{U}}^{\alpha}$ is the multiplicity space of dimension m_α (multiplicity of irrep α), while the space $\mathcal{H}_{\mathcal{S}}^{\alpha}$ is the representation space of dimension d_α (dimension of irrep α). Every operator commuting with the diagonal action of $U^{\otimes n}$, due to Schur's lemma must be proportional to identity on spaces $\mathcal{H}_{\mathcal{U}}^{\alpha}$ and admits non-trivial parts on spaces $\mathcal{H}_{\mathcal{S}}^{\alpha}$. Conversely, any operator commuting with the action of the permutation group S_n is non-trivially supported on irreducible spaces $\mathcal{H}_{\mathcal{U}}^{\alpha}$. In very irreducible space $\mathcal{H}_{\mathcal{S}}^{\alpha}$ one can construct an orthonormal irreducible basis $\{|\alpha, i\rangle\}$, where $i = 1, \dots, d_\alpha$, for example by exploiting Young-Yamanouchi construction [E9, E10]. With the constructed basis vectors we associate an irreducible basis operators E_{ij}^{α}, for $i, j = 1, \dots, d_\alpha$ admitting the following form:

$$E_{ij}^{\alpha} := \mathbb{1}_{\mathcal{H}_{\mathcal{U}}^{\alpha}} \otimes |\alpha, i\rangle\langle\alpha, j|_{\mathcal{H}_{\mathcal{S}}^{\alpha}}. \tag{7}$$

Sine the basis $\{|\alpha, i\rangle\}_{i=1}^{d_\alpha}$ is orthogonal the above operators fulfil the following relations:

$$E_{ij}^{\alpha} E_{kl}^{\alpha'} = \delta^{\alpha\alpha'} \delta_{jk} E_{il}^{\alpha}, \qquad \operatorname{Tr} E_{ij}^{\alpha} = \delta_{ij} m_\alpha. \tag{8}$$

From the operators E_{ij}^{α}, by using the completeness relation $\sum_i |\alpha, i\rangle\langle\alpha, i|_{\mathcal{H}_{\mathcal{S}}^{\alpha}} = \mathbb{1}_{\mathcal{H}_{\mathcal{S}}^{\alpha}}$, we can construct so-called Young projectors P^α on irreducible components labelled by α:

$$P^{\alpha} := \sum_{i=1}^{d_\alpha} E_{ii}^{\alpha} = \mathbb{1}_{\mathcal{H}_{\mathcal{U}}^{\alpha}} \otimes \mathbb{1}_{\mathcal{H}_{\mathcal{S}}^{\alpha}}, \qquad P^{\alpha} P^{\alpha'} = \delta^{\alpha\alpha'} P^{\alpha}, \qquad \operatorname{Tr} P^{\alpha} = m_\alpha d_\alpha. \tag{9}$$

For a given irrep α of S_n operators E_{ij}^{α} from (7) and P^α from (9) can be also written in terms of the permutation operators V_π defined through (1) that permute subsystems in $(\mathbb{C}^d)^{\otimes n}$:

$$E_{ij}^{\alpha} = \frac{d_\alpha}{n!} \sum_{\pi \in S_n} \phi_{ji}^{\alpha}(\pi^{-1}) V_\pi, \qquad P^{\alpha} = \frac{d_\alpha}{n!} \sum_{\pi \in S_n} \chi^{\alpha}(\pi^{-1}) V_\pi, \tag{10}$$

where the numbers $\phi_{ji}^{\alpha}(\pi^{-1})$ are the matrix elements of the irreducible representation of $\pi \in S_n$, and $\chi^{\lambda}(\pi^{-1}) = \sum_i \phi_{ii}^{\alpha}(\pi^{-1})$ are respective irreducible characters. Of course, the operators from the above equation satisfy the same composition rules given in (8). Indeed, the operators E_{ij}^{α} span a basis for $\mathcal{H}_{\mathcal{S}}^{\alpha}$ in the space $(\mathbb{C}^d)^{\otimes n}$, since for every element $V_\pi \in \mathcal{A}_n(d)$, we have

$$V_\pi = \sum_{\alpha} \sum_{i,j=1}^{d_\alpha} \phi_{ij}^{\alpha}(\pi) E_{ij}^{\alpha}. \tag{11}$$

Finally, using relations (10) and (11) one can find left (respective right) action of V_π on the basis elements:

$$V_\pi E_{ij}^{\alpha} = \sum_{k=1}^{d_\alpha} \phi_{ki}^{\alpha}(\pi) E_{kj}^{\alpha}, \qquad E_{ij}^{\alpha} V_\pi = \sum_{l=1}^{d_\alpha} \phi_{jl}^{\alpha}(\pi) E_{il}^{\alpha}. \tag{12}$$

The last expressions prove also that the operators E_{ij}^{α} span irreducible basis since the action is closed with respect to distinct α.

5.3 Irreducibly covariant linear maps

In the series of two papers [H1] and [H5] we characterize complete positivity and positivity of linear maps which are covariant with respect to the action of an irreducible representation of a finite group. Such maps are called *irreducibly covariant linear maps (ICLM)*. We say that a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is ICLM with respect to irreducible representations $U(g), V(g)$ for a finite group G if for any input operator $X \in \mathcal{B}(\mathcal{H})$ and for any $g \in G$, we have:

$$\Phi\left(U(g)XU(g)^\dagger\right) = V(g)\Phi(X)V(g)^\dagger. \tag{13}$$

The seminal results in the characterization of such a class of maps come from Scutaru [E11]. He proved a Stinespring-type theorem in the C^*-algebraic framework, for any completely positive linear map which is covariant with respect to a unitary representation of a locally compact group. However, his result does not give an explicit way of constructing such maps or more detailed insight into their interior structure. Nevertheless, ICLM maps forming irreducibly covariant quantum channels (ICQC) have a wide range of applications ranging from solid-state physics to quantum information science. For example $SU(2)$-covariant channels were used to describe entanglement in spin systems [E12] and allow to prove an extended version of the Lieb-Mattis-Schultz theorem by use of Matrix Product States [E13]. By increasing the dimension and considering the $SU(d)$ group one can investigate aspects of dimerization of quantum spin chains [E14]. In quantum information, covariant channels help to analyze the additivity property of the Holevo capacity [E15] and minimal output entropy [E16, E17, E18, E19, E20]. The covariance property also allows to prove strong converse properties for the classical capacity [E21] and entanglement-assisted classical capacity [E22]. They admit also interesting results in the context of Birkhoff's theorem [E23, E24] and allow for constructing new matrix inequalities from the positive cone [E25]. The above discussion and examples illustrate the relevance of (irreducibly) covariant channels and highlight the importance of understanding the structure and properties of such channels in more detail and on a more general level - without referring ourselves to a particular group, so a more systematic approach to the problem.

In paper [H1] we obtain a detailed mathematical description of quantum channels–completely positive, trace preserving (CPTP) linear maps–which are irreducibly covariant with respect to a finite group G in the case in which:

1. the input and the output Hilbert spaces of the channel are the same, i.e. $\mathcal{H} = \mathcal{K}$,
2. a particular unitary irreducible representation U is considered,
3. the tensor product $U \otimes U^c$ is simply reducible (or multiplicity free), where U^c denotes the contragradient representation, i.e. for every $g \in G$, $U^c(g) = U\left(g^{-1}\right)^T \equiv \overline{U}(g)$.

As it was observed in Corollary 15 in [H1] a map Φ which is ICLM must satisfy the following decomposition:

$$\Phi = l_{\mathrm{id}}\Pi^{\mathrm{id}} + \sum_{\alpha \neq \mathrm{id}} l_\alpha \Pi^\alpha : \quad l_{\mathrm{id}}, l_\alpha \in \mathbb{C}, \tag{14}$$

where Π^α are orthogonal projectors on irreducible components labelled by α in $U \otimes U^c$, and id labels the identity (trivial) representation. For example, by taking G to be the permutation group S_n the operators Π^α are the Young projectors from (9). From the above decomposition, it is clear that to ensure an ICLM map is ICQC some additional conditions on coefficients $l_{\mathrm{id}}, \{l_\alpha\}_{\alpha \neq \mathrm{id}}$ must be established.

To establish such conditions, in the first step, we derive the spectral decomposition of the Choi-Jamiołkowski image $J(\Phi)$ of any linear map Φ which is ICLM. The eigenvalues and orthogonal projections arising in this decomposition are expressed entirely in terms of representation characteristics of the considered group G. Ensuring $J(\Phi) \geq 0$ yields necessary and sufficient conditions on the eigenvalues of the linear map Φ, for which it is an ICQC (Theorem 40 in [H1]). In particular, we show that a map is trace-preserving ICLM if and only if $l_{\mathrm{id}} = 1$ (Proposition 25 in [H1]). We also obtain explicit expressions for the Kraus operators of such channels (Theorem 41 in [H1]). Our derived results are operative conditions when one fixes the group G and considered representation U. Namely, to ensure that a given map is ICQC we must satisfy a set of linear inequalities with respect to the coefficients in (14) - this, however, can be done easily at least numerically. This particular result gives a geometrical interpretation of the set of solutions for which the given irreducibly covariant linear map (ICLM) is an irreducibly covariant quantum channel (ICQC). Being more formal we show, that all eigenvalues of the Choi-Jamiołkowski image $J(\Phi)$ of an ICQC necessarily lie in the intersection of contracted simplex and certain subspace generated by the matrix obtained by spectral analysis of the projectors appearing in the decomposition of an ICLM (Proposition 43, Proposition 47 in [H1]).

Developed tools and characterization, which work for *any* finite, multiplicity-free group G, allow us to obtain a wide class of quantum channels that are irreducibly covariant by construction. These results are contained in Theorem 50 in Section 8.1 of the discussed paper. The construction is based on particular form of the coefficients $\{l_\alpha\}_{\alpha\neq \mathrm{id}}$ in (14):

$$l_\alpha = \frac{1}{|G|}\frac{1}{|\varphi^\alpha|}\sum_{g\in G}\chi^\alpha(g)f(g), \tag{15}$$

for some function $f : G \to \mathbb{C}$ with certain properties. In addition, we provide explicit examples of ICQCs for certain multiplicity-free groups: the symmetric groups, $S(3)$ and $S(4)$, and the quaternion group Q. In each case, we present the matrix representation and corresponding Kraus representation of the ICQC. Additionally, apart from the analytical solution, we illustrate graphically allowed space of parameters form (14) for which considered maps are ICQC. Next, for the case of $S(3)$ and Q, using the Peres-Horodecki or positive partial transpose criterion [E26, E27], we also obtain the condition under which the ICQC is an entanglement breaking channel [E28].

In paper [H5] we investigate linear maps Φ from (13) satisfying conditions 1,2 and 3, but we relax the CPTP condition of a given map Φ to just its positivity (P). The problem of classification and constructing new examples of positive maps, even with certain properties, is still an open and very complex one despite many attempts and important results in the field [E29, E30, E31, E32, E33, E34, E35, E36, E37]. One of the main obstacles to progress in this field is the non-existence of a universal operational criterion for positivity. Namely, to prove that a certain map is positive one has to check for example its block positivity, while to check the complete positivity of the given map it is enough to compute only all eigenvalues of the corresponding Choi-Jamiołkowski image, which in principle can be done effectively. Therefore, it is important to provide new examples of positive maps and methods of their construction that give insight into the internal structure of the set of maps under consideration.

We start our considerations from the straightforward (from the definition) construction of positive ICLMs for finite groups. However, such a method obviously can work effectively only for low dimensions - in our case qubit case. We constructed positive maps for two-dimensional irreps of the symmetric group $S(3)$ and the quaternion group Q. In addition, we show that any qubit ICLM positive map can be decomposed as a sum of CP and CoP (co-positive) map, which are also ICLM.

To rid of the necessity of checking the block positivity of the resulting Choi-Jamiołkowski image we developed a novel method of construction. This method is based on results contained in [P1] and exploits the concept of the inverse reduction map $R^{-1} \in \mathrm{End}[\mathrm{M}(d,\mathbb{C})]$ [E38, E39]:

$$\forall X \in \mathrm{M}(d,\mathbb{C}) \quad R^{-1}(X) = \frac{\mathrm{Tr}(X)}{d-1}\mathbf{1} - X. \tag{16}$$

To construct new families of ICLMs we exploit Theorem 15 from [H5] which is an adaptation of Theorem 1 from [P1]. In the first step, we define an operator $W := (\mathbf{1}\otimes\Phi)P_d^+$, for some linear map ICLM Φ, and we demand it is a non-positive operator. This ensures that the corresponding map for sure is not a CPTP map. In fact, the operator W is nothing else but the

Choi-Jamiołkowski image of Φ (we keep here the original notation for simplicity). Next, we demand that operator $\widetilde{W} = (\mathbf{1} \otimes R^{-1})W$ is a positive operator - this ensures the block positivity of the map Φ under investigation, according to mentioned Theorem 15. Since, the map Φ is ICLM map it must admit decomposition (14), so due to the results contained in [H1], conditions on positivity (negativity) of the corresponding Choi-Jamiołkowski images of an ICLM can be written as a set of linear inequalities with respect to the unknown coefficients $l_{\mathrm{id}}, \{l_\alpha\}_{\alpha \neq \mathrm{id}}$ from (14) - see Corollary 17 from [H5]. Using discussed here approach we construct positive maps induced by three-dimensional irreps of the symmetric group $S(4)$ and representations of the group $MU(d,n)$, which is a subgroup of the monomial unitary group $MU(d)$, where n is a natural parameter. In particular, we show that the region of parameters for an ICLM map generated by $MU(3,n)$, for which the map is positive, contains a positivity region for the generalized Choi map [E40]. What is more, the positivity region for $MU(3,n)$ is substantially larger. This result allows us to treat the generalized Choi map as a positive ICLM for $MU(3)$ group. For $MU(d,n)$ we were able also to calculate the region of positivity directly by checking positivity of the corresponding Choi-Jamiołkowski image on product vectors. This of course extends significantly range of parameters $l_{\mathrm{id}}, \{l_\alpha\}_{\alpha \neq \mathrm{id}}$ for which the considered map is positive with respect to the method based on the inverse reduction map. The results regarding the $MU(d,n)$ group can be of separate interest, since maps generated by them have been used to generalize randomised benchmarking protocols [E41], in the particular case where considered gate set are elements of a finite group, but they do not form 2-design. Moreover, the group $MU(d,n)$ contains as an element $T-$gate [E42], which together with Clifford gates forms the universal set for quantum computations, and the considered group is used in certain aspects have been used in many-body state theory [E43]. The connection between randomised benchmarking protocols and ICLMs motivated us for the further studies in this are which are discussed in the next section of this summary.

Our last contribution is dedicated for studies between celebrated Fujiwara-Algolet conditions [E44] and irreducible unital quantum channels with respect to quaternion group Q. We have proven that every unital qubit quantum channel can be expressed as a quantum channel irreducibly covariant with respect to the 2-dimensional irrep of the quaternion group. This final result is contained in Proposition 35 of [H5]. In fact our result works also for unital positive maps, giving somehow unification of all unital maps in this regime in a form of geometric constraints.

5.4 Efficient benchmarking protocol and classical simulation of quantum processes in the Weyl basis

Noise in experimental and practical implementations of quantum technologies is unavoidable due to interaction with the environment. Identifying its sources and efficient diagnostic of caused errors in the process of quantum evolution is one of the crucial steps in building a scalable quantum computer. Different implementations come with multiple hardware-dependent sources of noise and decoherence making the problem very complex. So far many different noise diagnostic methods have been developed: randomised benchmarking [E45, E46, E47, E48], state and channel tomography [E49, E50], gate set tomography [E51] or direct fidelity estimation [E52]. Generally, different tools for noise diagnostic in quantum circuits, so also in potential quantum computers, have different regions of applicability and require different costs.

In paper [H6] we introduce an approach to randomised benchmarking and classical simulation of quantum circuits that relies on Weyl unitaries. For the first time, our model enables us to identify a number of error models (including their mixtures) which works both for qubit and higher dimensional systems. In particular, we focus on experimental motivated models of noise: depolarising, dephasing channel, and noise models including over-rotations affecting the implementation of a given gate. The protocol is robust to the so-called state preparation and measurement (SPAM) errors and scalable with the system size under a natural assumption that the noise is local, without no further assumptions about the circuit structure. Presented results significantly extend works [E53, E54].

In the same paper, we applied the developed randomised benchmarking algorithm to simulate the outputs of quantum circuits on a classical computer. Having a quantum circuit with a known noise profile, we derive an analytic bound on the sufficient number of samples required to classically estimate the circuit output up to a given precision. In other words, we can establish a non-trivial computable bound on the gate noise that needs to be added to each gate in the circuit in order to render it classically and efficiently simulatable. The developed in [H6] tools do not depend on the geometry of the circuit and the particular structure of the gate set. These features differ us from preexisting methods [E55] and can be used to bound the classical simulation complexity of a wide range of quantum devices used for example in the VQE regime and near-term devices in general [E56, E57, E58].

Mathematically, when implementing a known unitary U acting on n qudits, the resulting transformation, due to noise (error) effects described by channel $\mathcal{N}$, is modeled by the quantum channel $\mathcal{N} \circ \mathcal{U}$, where $\mathcal{U}$ is the channel which corresponds to conjugation with U followed by $\mathcal{N}$. One of the main goal is to learn (estimate) parameters describing given noise channel $\mathcal{N}$ and it can be achieved by studying randomised benchmarking protocols.

Our benchmark protocol makes use of *Weyl-Heisenberg* unitaries $\{W_{(a,b)}\}_{a,b=0}^{d-1}$, which present the generalization of the Pauli matrices in higher dimensions. They are defined as $W_{(a,b)} = Z^a X^b$, where $X \in U(d)$ is the shift unitary, and $Z \in U(d)$ is the phase unitary mapping. We work with systems consisting of n qudits, then $W_{(\mathbf{a},\mathbf{b})} := W_{(a_1,b_1)} \otimes W_{(a_2,b_2)} \otimes \cdots \otimes W_{(a_n,b_n)}$ is a basis of the space $M(d^n, \mathbb{C})$, where $(\mathbf{a},\mathbf{b}) \in (\mathbb{Z}_d)^{2n}$.

As we pointed out, many practically relevant noise models, such as (local) dephasing or (local) depolarising channels, are diagonal in the Weyl operator basis. They are elements of a wider set of *Weyl diagonal channels* and denote them as $\mathcal{T}$. For any $d \geq 2$ they can be represented as convex combinations of conjugations with the Weyl operators [E59], [E60, Chapter 4]:

$$\mathcal{T}(A) = \sum_{(\mathbf{a},\mathbf{b}) \in (\mathbb{Z}_d)^{2n}} p(\mathbf{a},\mathbf{b}) W_{(\mathbf{a},\mathbf{b})} A W_{(\mathbf{a},\mathbf{b})}^{\dagger}, \tag{17}$$

where $p(\mathbf{a},\mathbf{b})$ is a probability distribution on $(\mathbb{Z}_d \times \mathbb{Z}_d)^n$, and A is an arbitrary operator. It is clear that channels from (17) belong to the class of irreducibly covariant quantum channels discussed in Section 5.3 of this summary, and we present a novel application of the discussed previously class of maps. Since the considered models of noise are Weyl diagonal channels to fully describe them is enough to learn their diagonal elements $\langle \mathcal{T} \rangle_{(\mathbf{a},\mathbf{b})}^{(\mathbf{a},\mathbf{b})}$ in the Weyl basis. We achieve this by introducing and implementing the protocol consisting the following steps:

Weyl randomized benchmarking (WRB) protocol

Input: $(\mathbf{a},\mathbf{b}) \in (\mathbb{Z}_d)^{2n}$ corresponding to the diagonal element we wish to learn and a sequence length m. Initial state ρ and POVM element E on n qudits.
Output: complex number y.

1. Draw a random $(\mathbf{a}_0,\mathbf{b}_0) \in (\mathbb{Z}_d)^{2n}$, apply $W_{(\mathbf{a}_0,\mathbf{b}_0)}$ followed by a sequence $\bar{W} = (W_{(\mathbf{a}_1,\mathbf{b}_1)}, \ldots, W_{(\mathbf{a}_m,\mathbf{b}_m)})$ of uniformly random local Weyl unitaries on the n qudits interspersed with the (noisy) unitary U.
2. Apply $\bar{W}^{\dagger}$.
3. Measure the state with a POVM $\{E, \mathbf{1} - E\}$.
4. When E is measured, output $y = \chi_{(\mathbf{a},\mathbf{b})}(\mathbf{a}_0,\mathbf{b}_0) = \exp(i\frac{2\pi}{d}\langle(\mathbf{b},-\mathbf{a}),(\mathbf{a}_0,\mathbf{b}_0)\rangle)$ Else, output $y = 0$.

By selecting different sequence lengths and performing an exponential fitting one gets an estimate of the diagonal $\mu(\mathbf{a},\mathbf{b}) = \langle \mathcal{T} \circ \mathcal{U} \rangle^{(\mathbf{a},\mathbf{b})}_{(\mathbf{a},\mathbf{b})}$ in the Weyl basis. The knowledge about $\mu(\mathbf{a},\mathbf{b})$ and the fact that $\mathcal{T}$ is Weyl diagonal makes it sufficient to estimate the noise parameters because in this case: $\mu(\mathbf{a},\mathbf{b}) = \langle \mathcal{U} \rangle^{(\mathbf{a},\mathbf{b})}_{(\mathbf{a},\mathbf{b})} \langle \mathcal{T} \rangle^{(\mathbf{a},\mathbf{b})}_{(\mathbf{a},\mathbf{b})}$. This completely characterizes $\mathcal{T}$ whenever the diagonal of the unitary is nonzero. The maximal sequence length is determined by the spectral gap λ of the considered quantum channel. In fact, the parameter λ^{-1} is a measure of the depth at which the noise clearly manifests itself. Theorem 2 of paper [H6] states that for arbitrary $\epsilon > 0$ noise parameters $\mu(\mathbf{a},\mathbf{b})$ can be estimated with probability δ by performing $M = \mathcal{O}(\epsilon^{-2} \log(\delta^{-1} \log(1-\lambda)^{-1}))$ Weyl randomized benchmarking experiments each containing at most $M_{\max} = \mathcal{O}(\lambda^{-1})$ gates in the sequence. The parameter λ^{-1} can be determined using the initial knowledge for the range for noise parameters reported by the manufacturer by combining this information with a standard technique called hyper-parameter grid search. However, our protocol is not limited only to diagonal channels in the Weyl basis. Namely, with access to a suitable noiseless Clifford gate, our protocol can be applied also for any off-diagonal elements $\langle \mathcal{N} \rangle^{(\mathbf{a}_1,\mathbf{b}_1)}_{(\mathbf{a}_2,\mathbf{b}_2)}$ of any channel $\mathcal{N}$ represented in the Weyl basis. In this case we also prove the analogue of the mentioned theorem (Theorem 2 in the supplementary materials of [H6]). In particular, these can identify the parameters of other noise models including over-rotations.

In the same paper [H6] we make use of the information about noise in the circuit to bound its classical simulation complexity. Being more strict, we work in the regime of the weak simulation. It means that for a given observable E, one can classically simulate (noisy) circuit $\mathcal{C}_{\mathcal{B}} = \mathcal{N}^{(N)} \circ \cdots \circ \mathcal{N}^{(1)}$ acting on a product input state ρ, if we can estimate $\mathrm{Tr}(\sigma E)$ classically up to an additive error $\epsilon > 0$, where $\sigma = \mathcal{C}_B(\rho)$. In our work we constructed such circuit sampling algorithm, based on ℓ_1 sampling of matrices and vectors, that exploits the results of the randomised benchmarking test, and works for noisy quantum circuits in continuous and discrete time.

Circuit sampling algorithm

Input: noisy quantum circuit specified by quantum channels $\mathcal{N}^{(1)}, \ldots, \mathcal{N}^{(N)}$, initial quantum state ρ and observable E.
Output: number x s.t. $\mathbb{E}(x) = \mathrm{Tr}(E\sigma)$.

1. Sample $(\mathbf{a}_0, \mathbf{b}_0)$ from the distribution p_0.
2. For $k = 1, \ldots, n$: Sample $(\mathbf{a}_k, \mathbf{b}_k)$ from $p_k(\mathbf{a}_{k+1}, \mathbf{b}_{k+1} | \mathbf{a}_k, \mathbf{b}_k)$
3. Output x given by

$$x = \mathrm{sign}(\rho(\mathbf{a}_0, \mathbf{b}_0)) \|\rho\|_{\ell_1} E(\mathbf{a}_n, \mathbf{b}_n) \times \prod_{k=1}^{N} \|\mathcal{N}^{(k)}(\mathbf{a}_k, \mathbf{b}_k)\|_{\ell_1} \mathrm{sign}(\langle \mathcal{N}^{(k)} \rangle^{(\mathbf{a}_k,\mathbf{b}_k)}_{(\mathbf{a}_{k+1},\mathbf{b}_{k+1})})$$

In Theorem 1 of [H6] we prove that indeed the algorithm samples from the true distribution, and we give the number of samples required to be $\epsilon-$close to the empirical expectation value. We illustrate how to apply our results to simulating local quantum circuits and VQE ansatze. When the quantum noise channels in the circuit are local and the initial state and observable are product, then the complexity of our sampling algorithm scales polynomially. Curiously, whenever we encounter Clifford gates in the circuit, they do not increase the sample complexity of the algorithm because they act as signed permutations in the Weyl basis. We also extend our sampling algorithm to the case of quantum circuits that made up of quantum channels of the form $e^{t\mathcal{L}}$, where $\mathcal{L}$ is a Lindbladian (Section VII of the supplementary materials in [H6]).

Summarising, our results offer tools for both practitioners – by giving them the ability to benchmark and learn a range of complex noise models, and for the theorists – by providing a computable upper bound on the computational power of the noisy quantum devices with a clear operational interpretation.

5.5 Contribution to representation theory

Another main ingredient of the habilitation series is developments in the representation theory of the group algebra $\mathbb{C}[S_n] \equiv \mathcal{A}_n(d)$ and the *algebra of partially transposed permutation operators* $\mathcal{A}_n^{(k)}(d)$ over last k systems:

$$\mathcal{A}_n^{(k)}(d) := \operatorname{span}_{\mathbb{C}}\{V_\pi^{(k)} : \pi \in S_n\} \subset \operatorname{Hom}((\mathbb{C}^d)^{\otimes n}), \tag{18}$$

where (k) is short notation for composition of partial transpositions $t_n \circ t_{n-1} \circ \cdots \circ t_{n-k+1}$ with respect to systems $n, n-1, \ldots, n-k+1$. Sometimes, in the literature the algebra $\mathcal{A}_n^{(k)}(d)$ is referred as *permutation's partial transpose algebra* [E61]. It turns out that the elements of the algebra $\mathcal{A}_n^{(k)}(d)$ are matrix representations on the space $(\mathbb{C}^d)^{\otimes n}$ of defined elsewhere abstract Walled Brauer Algebra (WBA) [E62, E63, E64]. Every element from the algebra $\mathcal{A}_n^{(k)}(d)$ satisfies a similar relation to (5), namely we have

$$[X, U^{\otimes(n-k)} \otimes \cdots \otimes \overline{U}^{\otimes k}] = 0, \quad \forall X \in \mathcal{A}_n^{(k)}(d), \tag{19}$$

where the bar denotes complex conjugation. We say that an operator X satisfying relation (19) possesses *partial symmetry*, contrary to operators commuting with the diagonal action of $n-$fold tensor product of unitaries in (5). We see that every operator possessing partial symmetry can be written as a linear combination of partially transposed permutation operators. Conversely, every operator written in terms of operators $V_\pi^{(k)}$ satisfies the relation (19). This means we have a situation similar to the Schur-Weyl duality, and we expect an analogous decomposition of the space $(\mathbb{C}^d)^{\otimes n}$ to this from (6):

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\xi} \mathcal{H}_{\mathcal{U}}^{\xi} \otimes \mathcal{H}_{\mathcal{WBA}}^{\xi}, \tag{20}$$

where the direct sum runs over all nonequivalent irreps of the considered algebra. The labeling of the irreps is known due to the earlier papers, see for example [E65, E64] and references within. In fact, it was shown that every ξ labelling irreps is a double index $\xi = (\alpha, \mu)$, where $\alpha \vdash n-k$, and $\mu \in \alpha$ is obtained by adding k boxes to α. Our goal is to find irreducible matrix elements of objects arising from the matrix algebra $\mathcal{A}_n^{(k)}(d)$, i.e. find irreducible matrix basis, allowing us for representing operators on $\mathcal{H}_{\mathcal{WBA}}^{\xi}$ and effective computations, for an arbitrary number of systems n, number of partial transpositions k, and dimension d.

In the case of $d = 2$, we have the relation $\overline{U} = \sigma_y U \sigma_y$, where $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ is a Pauli matrix. This relation defines an isomorphism between two discussed here algebras $\mathcal{A}_n^{(k)}(2)$ and $\mathcal{A}_n(2)$. For any other $d > 2$, we do not have the mentioned isomorphism and the new algebra $\mathcal{A}_n^{(k)}(d)$ is no longer a group algebra as it is for $\mathcal{A}_n(d)$. It is very easy to see in the simplest case when $k = 1$, i.e. when partial transposition acts on the last $n-$th system. Namely, let us consider permutation operators $V_\pi \in \mathcal{A}_n(d)$ for $\pi = (a, n)$, then it is clear that $V_\pi V_\pi^\dagger = \mathbf{1}$. However, taking $V_\pi^{t_n} \in \mathcal{A}_n^{(1)}(d) \equiv \mathcal{A}_n'(d)$, we have $V_\pi^{t_n}(V_\pi^{t_n})^\dagger = dV_\pi^{t_n} = d^2 P_{an}^+$, where the operator P_{an}^+ is bipartite maximally entangled state. This property makes the considered algebra structurally very different from the algebra $\mathcal{A}_n(d)$ and makes our further analysis more complex. In fact, the direct application of the standard methods from the representation theory of the group algebra of the symmetric group S_n is insufficient here, and a new approach must be developed. However, from the general theory, we know that every finite-dimensional C^*-algebra is a direct sum of matrix algebras and consequently is a semi-simple algebra. Of course, the algebra

$\mathcal{A}_n^{(k)}(d)$ with the hermitian conjugation $\dagger$ forms a C^*-algebra and it is semi-simple algebra for any value of n and d. This means one can decompose it into direct sums of minimal (irreducible) matrix ideals M_ξ:

$$\mathcal{A}_n^{(k)}(d) \cong \bigoplus_{\xi} M_\xi \qquad \text{with} \qquad M_\xi = \operatorname{span}_{\mathbb{C}}\{F_{ij}^{\xi} : \ F_{ij}^{\xi} F_{kl}^{\theta} = \delta^{\xi\theta}\delta_{jk}F_{il}^{\xi}\}. \tag{21}$$

Our primary goal is to identify irreducible ideals M_ξ and construct irreducible basis operators F_{ij}^{ξ}.

Chronologically, a practical tool kit for irreducible representation formalism for the algebra $\mathcal{A}_n^{(k)}(d)$ has been developed for $k=1$ first, and then developed for an arbitrary k. What is more, all the results in this area have been obtained over many years in a series of papers [H3, H9, H2, H10], not always fully dedicated to purely mathematical considerations - most of the papers are dedicated to quantum teleportation protocols. Here, however, to make this summary more transparent we resign from presenting the results in this area in chronological order. Instead of this, we discuss an arbitrary $k \geq 1$, giving respective comments for the special case of $k=1$. It is also important to stress here that part of the work regarding the case $k=1$ has been done by the applicant during his PhD studies [P3, P4]. However, the approach developed there is different and of small practical use – this is well explained in Section 2.3 and Section 3 of [H2].

For clarity of further presentation, we start with a short summary of the main contribution contained in the papers from the habilitation series:

1. In paper [H10] we present effective tools for computing partial traces over an arbitrary number of systems from the irreducible operator basis in every irrep of S_n emerging from the Schur-Weyl duality. Additionally, we prove novel orthogonality rules for S_n motivated by the celebrated Schur orthogonality relations [E66]. In particular, we are talking here about Proposition 6, Lemma 9, and Corollary 10 from the mentioned paper. These results on this level of generality extend the results from [E10, E67] for P_μ and they are genuinely new for the operators E_{kl}^{μ}. We write more about these result in Section 5.5.1.

2. In papers [H3, H10, H2] we use the connection that the algebra $\mathcal{A}_n^{(k)}(d)$ of partially transposed permutation operators is in fact the matrix representation of the Walled Brauer Algebra on the space $(\mathbb{C}^d)^{\otimes n}$. This connection, due to [E65], gives us all the ideals of the considered algebra (but not minimal), shows how they are nested, and presents the connection with the irreps of the symmetric groups S_{n-k} and S_{n-2k}. These results give us a tool for constructing an irreducible matrix basis of the Walled Brauer Algebra on the space $(\mathbb{C}^d)^{\otimes n}$, which is the first result of such kind in the literature. In particular, we identify the ideal, which is the main object for further understanding (multi) port-based teleportation schemes. This identification is implied by the symmetries exhibited in our new teleportation protocols discussed in Section 5.6.

3. Using constructed basis we find *irreducible matrix elements* of the basic objects for our study - namely, the permutation operators partially transposed on last k systems, as well permutation operators belonging to S_{n-k}, when the partial transposition does not change the initial operator. This part is also summarised shortly from the technical point of view in Section 5.5.1.

Obtained results are a non-trivial extension of the tools used in the Schur-Weyl duality to the case when one has to deal with symmetry of a different type (partial symmetry) – $U^{\otimes(n-k)} \otimes \overline{U}^{\otimes k}$, where the existing tools cannot be applied straightforwardly. We deliver tools for studying and understanding the Walled Brauer Algebras on the most friendly level for potential applications - irreducible matrix representation. These tools allow us for effective calculations of compositions and partial traces of operators exhibiting partial symmetries which we exploit in the description of quantum teleportation protocols in the next sections, but not only.

It turns out that the considered commutant and the general theory of the Walled Brauer Algebra play a non-trivial role in many other aspects of physics. In antiferromagnetic systems, the representation theoretic approach based on WBA reduces the complexity of the numerical diagonalization of the considered Hamiltonians [E68]. Our tools, in principle, can allow for more analytical treatment of this problem, possibly leading for further simplifications. Elements of the WBA theory can be also successfully applied to certain aspects of particle physics [E69] and gravity theories [E70, E71], where again representation approach leads to simplifications in analytical considerations. The analysis presented here can be also applied in the field of quantum information theory - besides considered here quantum teleportation protocols. In fact, one could use the theory of the matrix representation of WBA for studying and characterizing PPT property of multipartite quantum states, in the spirit of the seminal paper by Eggeling and Werner [E72]. Next, one could try to analyze ICLM, and their $k-$positivity, which are generated by the considered here algebra [E73, E74, E25]. In particular, restricting ourselves to ICQC we could construct new examples of new quantum channels for which the minimum output Rényi entropy is not additive [E75]. The first step in this direction, of constructing ICQC has been done in [E76], where the authors consider Temperley-Lieb quantum channels. Having a solid mathematical toolkit we are in a position for the full algebraic description of the universal $M \to N$ quantum cloning machines. For the case, $1 \to N$ it was done partially in [P2] and recently extended in [E77]. In principle, one could also try to apply our tools to obtain tighter bounds on the worst-case entanglement fidelity for approximate covariant error correction codes [E78, E79, E80], which could have implications for fault-tolerant quantum computing or some aspects of the AdS-CFT duality [E81]. Finally, using developed methods we could analyze higher-order quantum operations. In particular, motivated by [E82, E83], we could construct quantum combs producing transposition of unknown unitary operation when one has access to its k uses. This wide range of selected topics shows a huge potential for developed by us mathematical tools for applications in other problems of modern physics and mathematics and clearly proves that they were not tailored only for one particular application.

5.5.1 The algebra of partially transposed permutation operators - technical summary

This section contains a compressed technical summary of the main results regarding the representation theory of the algebra of partially transposed permutation operators $\mathcal{A}_n^{(k)}(d)$ contained in the series of papers [H3, H2, H10]. We start by introducing the notion of *partially irreducible representations (PRIR)* for easier computations of partial traces and compositions of operators with the partial symmetries, which is a matrix approach to the Gelfand-Tsetlin construction [E9] in the case of the symmetric group. First, we have to clarify the introduced and used by us notation. Namely, let us take $\mu \vdash n$ and $\alpha \vdash n-k$, for $k < n$. By index $r_{\mu/\alpha}$ we denote a path on Young's lattice from diagram μ to α. This path is uniquely determined by choosing a chain of covered Young frames from μ to α, differencing by one box in each step:

$$r_{\mu/\alpha} = (\mu, \mu_{n-1}, \dots, \mu_{n-k+1}, \alpha), \quad \text{and} \quad \mu \ni \mu_{n-1} \ni \cdots \ni \mu_{n-k+1} \ni \alpha. \tag{22}$$

In particular, it means that every basis index i_μ, labeling matrix elements of ϕ^μ, can be written uniquely using a path on Young's lattice as

$$i_\mu \equiv (r_{\mu/\alpha}, l_\alpha), \qquad \alpha \in \mu, \tag{23}$$

where l_α denotes now index running only within the range of the irrep α. Additionally, by writing $\delta_{i_\mu j_\nu}$, where $\mu \vdash n$ and $\alpha \vdash n-k$, we understand the following

$$\delta_{i_\mu j_\nu} = \delta^{r_{\mu/\alpha} \widetilde{r}_{\nu/\beta}} \delta_{l_\alpha l'_\beta} = \delta_{\mu\nu} \delta_{\mu_{n-1}\nu_{n-1}} \cdots \delta_{\mu_{n-k+1}\nu_{n-k+1}} \delta_{\alpha\beta} \delta_{l_\alpha l'_\beta}. \tag{24}$$

Using the introduced convention, we can illustrate the matrix approach to the Gelfand-Tsetlin construction (PRIR). Namely, any irrep ϕ^{μ} of S_n can be always unitarily transformed to $PRIR$ with respect to subgroup $S_{n-k} \subset S_n$ in the following way

$$\forall \sigma \in S_{n-k} \quad \phi^{\mu}_{R}(\sigma) = \left(\delta^{r_{\mu/\alpha}\widetilde{r}_{\mu/\beta}} \varphi^{\alpha}_{i_\alpha j_\alpha}(\sigma)\right) = \bigoplus_{\alpha \in \mu} \varphi^{\alpha}(\sigma), \tag{25}$$

where indices i_α, j_α run from 1 to dimension of the irrep α. The off-diagonal blocks are zero blocks and they need not to be square. However, for an arbitrary $\sigma \in S_n$ the form of the resulting decomposition (25) with respect to subgroup S_{n-k} is more complex since we have non-zero off-diagonal blocks:

$$\forall \sigma \in S_n \quad \phi^{\mu}_{R}(\sigma) = \left((\phi^{\mu}_{R})_{i_\alpha}^{r_{\mu/\alpha},}{}_{j_\beta}^{\widetilde{r}_{\mu/\beta}}(\sigma)\right), \tag{26}$$

where again the matrices on the diagonal are of the dimension of the corresponding irrep φ^{α} of S_{n-k}. Having the notion of the PRIRs we prove the first main technical results contained in Lemma 9 and Corollary 10 of [H10] regarding the partial trace over operators E^{μ}_{ij} and P^{μ} from (10):

$$\operatorname{Tr}_{(k)} E_{i_\beta}^{r_{\mu/\beta}}{}_{j_{\beta'}}^{\widetilde{r}_{\mu/\beta'}} = \frac{m_\mu}{m_\beta} E^{\beta}_{i_\beta j_\beta} \delta_{r_{\mu/\beta}\widetilde{r}_{\mu/\beta'}}, \qquad \operatorname{Tr}_{(k)} P_\mu = \sum_{\beta \in \mu} m_{\mu/\beta} \frac{m_\mu}{m_\beta} P_\beta, \tag{27}$$

where we use simplified notation for partial trace over last k systems $\operatorname{Tr}_{(k)} = \operatorname{Tr}_{n-2k+1,\ldots,n-k}$. These results are used later in the construction of the irreducible operator basis for the considered algebra $\mathcal{A}^{(k)}_{n}(d)$. In this algebra, for the reasons motivated by (multi) port-based teleportation protocols, and described in further sections, we distinguish a particular partially transposed permutation operator:

$$V^{(k)} := V^{t_n}_{(n-2k+1,n)} V^{t_{n-1}}_{(n-2k+2,n-1)} \cdots V^{t_{n-k+1}}_{(n-k,n-k+1)}, \tag{28}$$

which is the composition of disjoint partially transposed transpositions represented on the space $(\mathbb{C}^d)^{\otimes n}$. Our primary goal was to construct an irreducible matrix operator basis for a two-sided ideal generated by the element $V^{(k)}$ from (28) and elements of the algebra $\mathcal{A}^{(k)}_{n}(d)$:

$$\mathcal{M} := \{V_\tau V^{(k)} V^{\dagger}_{\tau'} \mid \tau, \tau' \in S_{n-k}\} \subset \mathcal{A}^{(k)}_{n}(d). \tag{29}$$

Having worked out expressions for partial trace over an arbitrary number of particles from irreducible basis operators of the symmetric group (27), we have constructed an irreducible matrix basis for the ideal $\mathcal{M}$. This is one of the main result concerning mathematical tools and is contained in Theorem 11 of [H10]:

$$F_{i_\mu}^{r_{\mu/\alpha}}{}_{j_\nu}^{r_{\nu/\alpha}} = \frac{m_\alpha}{\sqrt{m_\mu m_\nu}} E_{i_\mu}{}^{r_{\mu/\alpha}}{}_{1_\alpha} V^{(k)} E_{1_\alpha}^{r_{\nu/\alpha}}{}_{j_\nu}, \tag{30}$$

where m_μ, m_ν and m_α are multiplicities of respective irreps of S_{n-k} and S_{n-2k} in the Schur-Weyl duality. Additionally, the above-defined operators satisfy the following composition rule

$$F_{i_\mu}^{r_{\mu/\alpha}}{}_{j_\nu}^{r_{\nu/\alpha}} F_{k_{\mu'}}^{r_{\mu'/\beta}}{}_{l_{\nu'}}^{r_{\nu'/\beta}} = \delta^{r_{\nu/\alpha} r_{\mu'/\beta}} \delta_{j_\nu k_{\mu'}} F_{i_\mu}^{r_{\mu/\alpha}}{}_{l_{\nu'}}^{r_{\nu'/\alpha}}. \tag{31}$$

The results contained in (30) and (31) are the analogue of the irreducible matrix basis E^{α}_{ij} given in (10), with orthogonality relations (8), for the group algebra $\mathcal{A}_n(d)$ on the space $(\mathbb{C}^d)^{\otimes n}$. The first attempt for the construction of the irreducible matrix basis operators has been done for $k = 1$ in paper [H2] in Theorem 38. In this paper, we have made a crucial in understanding the general theory of $\mathcal{A}'_n(d)$ presented by the Author earlier in his scientific career in [P3, P4]. Namely, we rid of certain technical problems which make direct applications of the developed tools, even for $k=1$ very complex. In particular, we can mention here the necessity of choosing non-zero elements of the operators spanning the algebra $\mathcal{A}_n(d)$, which can be done in principle for fixed n and d, but it fails in the general considerations - see Theorem 12 and Remark 13 in [H2]. Additionally, all very complex expressions for objects describing the algebra $\mathcal{A}'_n(d)$ by using the concept of the reduced representation can be simplified significantly - Section 4 of [H2]. Despite the progress which has been made, for higher k further analysis was necessary which is presented in paper [H10].

Going further, in the same Theorem 11 from [H10], we prove that the generating element $V^{(k)}$ of the ideal $\mathcal{M}$ can be written in terms of the operators from (30), which is relation similar to (11):

$$V^{(k)} = \sum_{\mu,\nu} \sum_{r_{\mu/\alpha}, \widetilde{r}_{\nu/\alpha}} \sum_{l_\alpha} \frac{\sqrt{m_\mu m_\nu}}{m_\alpha} F_{l_\alpha \ r_{\mu/\alpha} \ \widetilde{r}_{\nu/\alpha} \ l_\alpha}^{\quad r_{\mu/\alpha} \ \widetilde{r}_{\nu/\alpha}}. \tag{32}$$

This particular result is a generalization of Theorem 38 in [H2] for an arbitrary number of partial transpositions k. This relation, together with properties of the irreducible matrix basis for $\mathcal{A}_n(d)$ allows us to prove Lemma 12 from the discussed paper, in which we evaluate respective left actions of the basis operators:

$$F_{k_\beta \ r_{\mu/\beta} \ r_{\nu/\gamma} \ l_\gamma}^{\quad r_{\mu/\alpha} \ r_{\nu/\alpha}} V^{(k)} = \sum_{\mu'} \sum_{r_{\mu'/\gamma}} \frac{\sqrt{m_\nu m_{\mu'}}}{m_\gamma} F_{k_\beta \ r_{\mu/\beta} \ r_{\mu'/\gamma} \ l_\gamma}^{\quad r_{\mu/\gamma} \ r_{\mu'/\gamma}} \delta^{r_{\nu/\alpha} r_{\nu/\gamma}} \tag{33}$$

and

$$F_{k_\beta \ r_{\mu/\beta} \ r_{\nu/\gamma} \ l_\gamma}^{\quad r_{\mu/\alpha} \ r_{\nu/\alpha}} V_\tau = \sum_{k_\nu} \phi^\nu_{j_\nu k_\nu}(\tau) F_{i_\mu \quad k_\nu}^{r_{\mu/\alpha} r_{\nu/\alpha}} \tag{34}$$

where $\phi^\nu_{j_\nu k_\nu}(\tau)$ are the matrix elements of V_τ for $\tau \in S_{n-k}$, which can be evaluated using relation (11). These results allow us to derive expressions for irreducible matrix elements of the objects in interest (Lemma 13 in [H10]):

$$\left(V^{(k)}\right)_{k_\beta \ r_{\mu/\beta} \ r_{\nu/\gamma} \ l_\gamma}^{\quad r_{\mu/\alpha} \ r_{\nu/\alpha}} = \delta_{k_\beta l_\gamma} \delta^{r_{\mu/\alpha} r_{\mu/\beta}} \delta^{r_{\nu/\alpha} r_{\nu/\gamma}} \frac{\sqrt{m_\mu m_\nu}}{m_\alpha}, \tag{35}$$

and

$$(V_\tau)_{i_\mu \quad j_\nu}^{r_{\mu/\alpha} r_{\nu/\alpha}} = \delta^{r_{\mu/\alpha} r_{\nu/\alpha}} \delta_{i_\mu j_\nu} \sqrt{\frac{m_\mu}{m_\nu}} \sum_{k_\mu} \phi^\mu_{k_\mu i_\mu}(\tau), \tag{36}$$

As we can see the elements from (35) and (36) are connected with the parameters describing irreps of the symmetric groups S_n and S_{n-2k}. We also construct a set of projectors, which are the analog of the Young projectors given in (10). Every such projector in its action produces restriction to irreducible blocks of the algebra $\mathcal{A}_n^{(k)}(d)$ in the ideal $\mathcal{M}$ (see Definition 15 in [H10])):

$$\forall \alpha \ \forall \mu \in \alpha \quad F_\mu(\alpha) := \sum_{r_{\mu/\alpha}} \sum_{k_\mu} F_{k_\mu \quad k_\mu}^{r_{\mu/\alpha} r_{\mu/\alpha}}. \tag{37}$$

This kind of operators have been firstly defined for $k=1$ in Theorem 1 of paper [H3] in the context of port-based teleportation schemes. Moreover, the operators satisfy orthogonality rules, similar to these in (9), in indices α, μ, see Lemma 15 in [H10]. Namely, we have

$$F_\mu(\alpha) F_\nu(\beta) = \delta^{\mu\nu} \delta^{\alpha\beta} F_\mu(\alpha). \tag{38}$$

Further, we will see that these projectors play a central role in the description of the measurements in the multi port-based teleportation schemes. Motivated by their applications we prove a series of more technical lemmas and facts in the spirit of the relations given through (27). In particular, the main results are contained in Lemma 18, Lemma 19, and Lemma 21 of paper [H10]. Since we do not describe here in detail all the proofs concerning teleportation schemes, so we decided not to list these expressions here explicitly.

5.6 Variants of port-based teleportation schemes

The port-based teleportation (PBT) protocol was introduced in 2008 by Hiroshima and Ishizaka in a series of two papers [E84, E85]. Its breakthrough relies on the fact that the receiver does not need to apply a correction on his/her side to recover the transmitted quantum state, contrary to the teleportation protocol proposed in 1993 by Bennett et al in [E86]. The lack of the last step led to novel applications of PBT in the areas where the ordinary teleportation scheme is not applicable.

First important result is providing a new architecture for the universal programmable quantum processor, performing computation by teleportation [E7, E87, E84, E85]. PBT schemes were used effectively in position-based cryptography to engineering protocols for instantaneous implementation of measurement and computation. This application resulted in new attacks on the cryptographic schemes, reducing the amount of resource in the form of consumable entanglement from doubly exponential to exponential [E88]. The qubit PBT schemes have found a connection with the field of communication complexity and a Bell inequality violation [E89]. This completely new approach allowed to show that for any quantum advantage in communication complexity, there exists a way of obtaining measurements statistics that violate some Bell inequality, so there is a deep and general connection between non-locality and quantum advantage in communication complexity. The qubit PBT can be also interpreted as a universal simulator for qubit channels [E90] improving simulations of the amplitude damping channel. Extending PBT protocols to so-called PBT stretching allows to obtain limitations of the fundamental nature for quantum channels discrimination [E91]. Recently, aspects of PBT play a role in the construction of universal quantum circuits for inverting general unitary operations [E92, E82], theory of storage and retrieval of unitary quantum channels [E93], and even they are used in the context of AdS/CFT duality [E94]. This wide range of applications and the possibility of further developments in many problems of the modern quantum information theory motivates us to study the PBT protocol in all its variants in more detail, especially in the higher dimensional case.

In all versions of PBT sender (Alice) and receiver (Bob) share N pairs of maximally entangled states, each called a *port*, see the right-hand side graphic of Figure 3. The collection of all shared entangled states (ports) we will call here as the *resource state* and its of the form

$$|\Psi\rangle_{AB} = (O_A \otimes \mathbf{1}_B)|\Psi^+\rangle_{AB} = (O_A \otimes \mathbf{1}_B)|\psi^+\rangle_{A_1B_1} \otimes |\psi^+\rangle_{A_2B_2} \otimes \cdots \otimes |\psi^+\rangle_{A_NB_N}, \tag{39}$$

where $A = A_1A_2 \cdots A_N$, $B = B_1B_2 \cdots B_N$, and O_A, with normalization constraint $\mathrm{Tr}(O_A^\dagger O_A) = d^N$, is a global operation applied by Alice to increase the efficiency of the protocol [E85, E95], [H3, H4]. In non-optimal PBT $O_A = \mathbf{1}_A$, while for the optimal scheme, its explicit form is known and discussed in [E85], [H4]. Alice to send to Bob an unknown state ψ_C of a particle applies a joint measurement Π_i^{AC} on her halves of the ports and on the state to be teleported, getting a classical outcome $1 \leq i \leq N$, transmitted further by a classical channel to Bob. This classical message indicates the port on which the teleported state arrives and no further correction is needed. We can distinguish two variants of the protocol:

- *Deterministic protocol (dPBT):* An unknown quantum state ψ_C is always transmitted to the receiver but the transmission is imperfect. The teleportation channel $\mathcal{N}$ in general situation can be expressed as:

$$\mathcal{N}(\Psi_C) = \sum_{i=1}^{N} \mathrm{Tr}_{AC}\left[\Pi_i^{AC}\left(\left(O_A \otimes \mathbf{1}_{\widetilde{B}}\right)\sigma_{A_i\widetilde{B}}\left(O_A^\dagger \otimes \mathbf{1}_{\widetilde{B}}\right) \otimes \psi_C\right)\right], \tag{40}$$

  where by Tr_{AC} denotes partial trace over all systems AC but $\widetilde{B}$. The states $\sigma_{A_a\widetilde{B}}$ are called *signal states* and have the following explicit form

$$\sigma_{A_i\widetilde{B}} := \sigma_i := \frac{1}{d^{N-1}}\mathbf{1}_{\overline{A}_i} \otimes P^+_{A_i\widetilde{B}}, \tag{41}$$

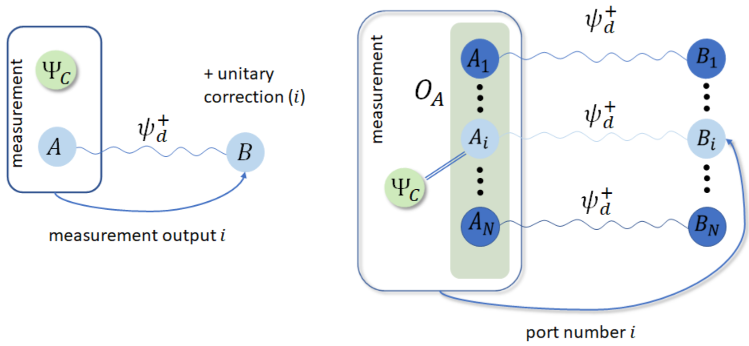

Figure 3: The left panel presents a schematic description of the standard quantum teleportation procedure introduced in [E86]. In this scheme, the receiver to recover the transmitted state must apply unitary correction which depends on the classical message send by the sender. In the right panel, we present a schematic description the port-based teleportation protocol [E84, E85]. Here, on the contrary to the standard teleportation scheme, parties share N maximally entangled pairs (ports), and the receiver to recover the transmitted state must just pick up the right port according to the classical message sent by the sender. No correction is needed here, however, due to no-programming theorem [E96] transmission is not perfect, resulting with the fidelity of teleportation smaller than 1. The perfect transmission is possible only in the asymptotic scenario when $N \to \infty$.

where $P^+_{A_i\widetilde{B}}$ is projector on maximally entangled state between systems A_i and $\widetilde{B}$. To learn about the quality of the protocol we check how well one can transmit quantum correlations. It means we compute the entanglement fidelity $F(\mathcal{N})$ between the output of the teleportation channel $\mathcal{N}$ when teleporting a subsystem C from a maximally entangled state P^+_{CD}, and the state after perfect transmission $P^+_{\widetilde{B}D}$:

$$F(\mathcal{N}) = \operatorname{Tr}\left[P^+_{\widetilde{B}D}(\mathcal{N}_C \otimes \mathbf{1}_D)(P^+_{CD})\right] = \frac{1}{d^2}\sum_{i=1}^{N} \operatorname{Tr}\left[\left(O^\dagger_A \otimes \mathbf{1}_{\widetilde{B}}\right)\Pi_i^{A\widetilde{B}}\left(O_A \otimes \mathbf{1}_{\widetilde{B}}\right)\sigma_{A_i\widetilde{B}}\right]. \quad (42)$$

In particular, in paper [E85] it was shown that the teleportation channel $\mathcal{N}$ is in fact quantum depolarising channel, and as its output, we obtain an isotropic state with mixing parameter $p = p(N,d)$ connected with the entanglement fidelity $F(\mathcal{N})$ by the following expression [E91]:

$$F(\mathcal{N}) = 1 - p + \frac{p}{d^2}. \quad (43)$$

Due to the recent result presented in [E97], we know that measurements in the form of *square-root measurements* (SRM) are optimal in both PBT versions (non- and optimal PBT). The optimal measurements in the non-optimal case are:

$$\forall 1 \leq i \leq N \qquad \Pi_i^{AC} = \frac{1}{\sqrt{\rho}}\sigma_{A_iC}\frac{1}{\sqrt{\rho}}, \quad \text{where} \quad \rho = \sum_{i=1}^{N}\sigma_{A_iC}. \quad (44)$$

The operator ρ^{-1} is restricted to the support of ρ, so to ensure summation of all POVMs to identity $\mathbf{1}_{AC}$ on the whole space $(\mathbb{C}^d)^{\otimes N+1}$, we add to every Π_i^{AC} an excess term $\Delta = \frac{1}{N}\left(\mathbf{1}_{AC} - \sum_{i=1}^N \Pi_i^{AC}\right)$. However, this extra term does not change the entanglement fidelity $F(\mathcal{N})$ of the channel $\mathcal{N}$, since the complement Δ is orthogonal to the space on which all operators describing PBT are supported, see [E85, E84], and [H3].

- *Probabilistic protocol (pPBT):* The transmission is always perfect with $F(\mathcal{N}) = 1$, but there is a non-zero probability of failure of the whole process (no teleportation). In this scheme Alice has access to $N+1$ measurements $\{\Pi_0^{AC}, \Pi_1^{AC}, \ldots, \Pi_N^{AC}\}$ with measurement Π_0^{AC}

corresponding to the failure of teleportation procedure. The efficiency of the protocol is described by the average probability of success p_{succ} of the scheme equals to [E85], [H3]:

$$p_{succ} = \frac{1}{d^{N+1}} \sum_{i=1}^{N} \operatorname{Tr}\left[O_A^{\dagger} \Pi_i^{AC} O_A \right]. \tag{45}$$

The requirement of perfect transmission constraints form of allowed measurements accessible for Alice, namely can be of the following form [E85], [H3]:

$$\forall 1 \leq i \leq N \qquad \Pi_i^{AC} = P^{+}_{A_i C} \otimes \Theta_{\overline{A}_i}, \tag{46}$$

where $\overline{A}_i$ denotes all states $A_1 A_2 \cdots A_N$ but A_i. The optimal form of the operators $\Theta_{\overline{A}_i}$ in both versions of the pPBT protocols (both optimal and non-optimal cases) was computed for qubits in [E85] and for higher dimensions in [H3].

It is important here to say a few words about used notational convention. Namely, in both cases, deterministic and probabilistic, we denote the optimization operation by O_A. Of course, we have to remember that they are different for different versions of the considered protocol. The same holds for the measurements used by Alice. It will be always clear from the context about which O_A or measurements we are talking about.

In every case, the perfect transmission, with unit fidelity in deterministic or unit probability of success in probabilistic PBT, is possible only in the asymptotic limit $N \to \infty$. This is a consequence of the no-go theorem for the universal processor with finite system size (no-programing theorem) [E96]. This limitation of PBT makes the question of how well parties can send quantum states via PBT schemes with the finite number N of ports of arbitrary dimension d fundamental one, especially in the light of the listed at the beginning applications.

Clearly, the straightforward application of numerical methods do not give good results, since the dimension of all objects describing PBT protocols grows like d^{N+1}, which causes real computational problems. Up to now the efficiency analysis of all variants of PBT was known for qubits and it has been done in [E84, E85]. The analysis was possible by exploiting the representation theory of $SU(2)^{\otimes N}$, in particular properties of Clebsch-Gordan coefficients, together with methods from the semidefinite programming. Unfortunately, such methods do not work effectively in higher dimensions. This is because in the case of $SU(d)^{\otimes N}$ there is no closed-form of the Clebsch-Gordan coefficients and to compute them we need an exponential overhead in N and d. In the case of $d \geq 2$ only partial results have been known, mostly by lower/upper bounds on the entanglement fidelity and the probability of success for the non-optimal PBT [E88, E85, E98]. For the deterministic case bounds rely on straightforward analysis spectral properties of the operator ρ [E85] or connection with the state discrimination problem [E88], while for the probabilistic PBT come from the no-cloning theorem and the no-signalling principle [E98]. However, in all cases, it was not known how close they are to the real values for a finite number of ports N and in any dimension d. The first analytical attempt to describe the exact efficiency of PBT in higher dimensions has been done in [E99] by exploiting elements of Temperley-Lieb algebra theory, mostly in its graphical representation. The authors presented closed expressions for entanglement fidelity as well as the probability of success for an arbitrary d and $N = 2, 3, 4$. Graphical methods, however, also are not satisfactory, mostly due to the fact of a large number of cases that must be considered separately, which grows exponentially with the number of ports N. Therefore, there was a need to develop new, more effective methods for the analysis of PBT protocols. For this purpose, we decided to identify and then use the existing symmetries in the problem, which allows us to apply elements of group representation theory and finite algebras. Below we will list and discuss the most important results obtained regarding the study of PBT efficiency in higher dimensions.

5.6.1 Port-based teleportation in arbitrary dimension

As we pointed out, an outstanding open problem was to find the description of PBT in higher dimensions and their optimal versions in the most general setting – with the optimal measurement that works for an arbitrary dimension and number of ports, as well as establish fundamental limits of their performance, by delivering reasonably computational easy expressions for entanglement fidelity and probability of success. In papers [H3, H4, H2], together with co-authors, I fully characterize the performance of all existing PBT protocols in arbitrary dimensions d as well as an arbitrary number of ports N. The possibility of the full description of all variants of PBT relies on two key innovations described in Section 5.5:

1) A novel connection between the PBT protocols and algebra of partially transposed permutation operators $\mathcal{A}'_d(n)$.

2) New and effective tools for computing products of operators with partial symmetries combined with partial traces and partial transpositions, both in analytical and numerical approaches.

As it was stated above the first step was to identify the symmetries in PBT protocols and show their connection with irreducible representations of the algebra $\mathcal{A}'_d(n)$. Recall that a bipartite maximally entangled state is $U \otimes \overline{U}$ invariant [E100], where the bar denotes complex conjugation of an element U of the unitary group $\mathcal{U}(d)$. This implies the following symmetries for the all signal states $\sigma_i^{A\widetilde{B}}$ from (41):

$$[U^{\otimes N} \otimes \overline{U}, \sigma_i^{A\widetilde{B}}] = 0, \quad \forall\, U \in \mathcal{U}(d), \tag{47}$$

where $\overline{U}$ acts on $\widetilde{B}$, and $U^{\otimes N}$ acts on systems $A = A_1 \cdots A_N$. Construction $\sigma_i^{A\widetilde{B}}$ allows us to identify the additional symmetry – covariance with respect to elements from the group S_N, acting on first N systems:

$$V_\pi \sigma_i^{A\widetilde{B}} V_\pi^\dagger = \sigma_{\pi(i)}^{A\widetilde{B}}, \quad \forall\, \pi \in S_N. \tag{48}$$

In particular, choosing an arbitrary state from the set, say $\sigma_N^{A\widetilde{B}}$ all the others can be generated by acting on it with an element from the coset S_N/S_{N-1}, whose elements in the representation V are of the form $V[(i, N-1)]$, for $i = 1, \dots, N-1$. The same kind of covariance as in (48) also holds for all measurements $\{\Pi_i^{AC}\}_{i=1}^N$ in all variants of PBT:

$$V_\pi \Pi_i^{AC} V_\pi^\dagger = \Pi_{\pi(i)}^{AC}, \quad \forall\, \pi \in S_N. \tag{49}$$

Recall that the operator ρ from (44) is a sum over all possible states σ_i^{AC}:

$$\rho = \frac{1}{d^N} \sum_{i=1}^N V_{(A_i,C)}^{t_C} \equiv \frac{1}{d^N} \sum_{i=1}^N V'_{(i,n)}. \tag{50}$$

In the last equality, we renumbered systems according to the rule $A_1 \mapsto 1, A_2 \mapsto 2, \dots, A_N \mapsto N, C \mapsto n = N+1$. For notational simplicity by $'$ we denote the partial transposition over $n-$th system. Relation (50) implies that ρ also exhibits symmetries described in (47) and in addition, it is invariant with respect to the action of elements from S_N. We will extensively rely on this property later on in this section. Therefore, one of the main building blocks for effective description of PBT can be written in terms of *partially transposed permutation operators* form the algebra $\mathcal{A}'_d(n)$.

Exploiting the above described symmetries and mathematical toolkit described in Section 5.5.1 we proven Theorem 1 and Proposition 2 from [H3] saying that the operator ρ from (50) has the following spectral decomposition:

$$\rho = \sum_{\alpha \vdash n-2} \sum_{\mu \in \alpha} \lambda_\mu(\alpha) F_\mu(\alpha), \tag{51}$$

where $F_\mu(\alpha)$ are projectors on irreducible components of algebra $\mathcal{A}'_n(d)$ from (37) for $k=1$, with corresponding eigenvalues

$$\lambda_\mu(\alpha) = \frac{N}{d^N}\frac{m_\mu d_\alpha}{m_\alpha d_\mu}. \tag{52}$$

This result allows for solving one of the main technical obstacle, namely evaluate ρ^{-1} or $1/\sqrt{\rho}$ from (44). Thanks to the spectral decomposition (51) we can compute entanglement fidelity of the teleporting channel $F(\mathcal{N})$ from (42), when $O_A = \mathbf{1}_A$ (non-optimal deterministic PBT), in terms of purely group-theoretic parameters describing irreps of S_N in the Schur-Weyl duality (Theorem 12 in [H3]):

$$F = \frac{1}{d^{N+2}}\sum_{\alpha\vdash n-2}\left(\sum_{\mu\in\alpha}\sqrt{d_\mu m_\mu}\right)^2. \tag{53}$$

The quantities from the final expression are effectively computable. For this purpose, we have written dedicated software in Python with use of the group-theoretic package Sage [E101]. Results of the numerical simulations are depicted in Figure 4. From this figure we see that entanglement fidelity decreases when dimension d of the port increases, but it goes to 1 for fixed d and increasing port number N. Additionally, in paper [H2] we have proven that indeed expression (53) approaches 1 with $N\to\infty$ (Theorem 52) for arbitrary port dimension d.

In paper [H4] we provided analysis efficiency analysis of the optimal PBT, i.e. when Alice optimizes measurements and the resource state using operation O_A from (39). Here the situation is more complicated compared to the non-optimal deterministic PBT since we must also find an explicit form of the optimal measurements, the optimal form of O_A ensuring maximization of F. This can be done by formulating and solving primal and dual semidefinite programs (chapter 5.1 with Theorem 30 and chapter 5.2 with Theorem 31 of paper [H4]). Again exploiting internal symmetries of the problem the SDP problems can be solved analytically. What is more, solutions for the primal and the dual problem coincide giving exact value of the entanglement fidelity and optimal form of the considered operators (Proposition 32 in [H4]). The optimal entanglement fidelity is described by static objects - principal minors of so-called *teleportation matrix* M_F^d introduced in chapter 4 of [H4]:

$$F_{opt} = \frac{1}{d^2}||M_F^d||_\infty. \tag{54}$$

In Figure 4 we depict optimal values of entanglement fidelity F_{opt} from (54) with entanglement fidelity F for the non-optimal protocol given through (53) - we see increased efficiency of the optimized PBT scheme. As we can see, to evaluate fidelity F_{opt} we must know maximal eigenvalue $\lambda_{\max}(M_F^d)$ of the matrix M_F^d, so we must know more information about its interior structure. Rows and columns of the teleportation matrix are indexed by Young frames of N boxes arranged in strictly decreasing lexicographical order. Then its every matrix element encodes mutual relations between Young frames of N and $N-1$ boxes. Namely, an element of the teleportation matrix M_F^d, is of the following form (Definition 4 in [H4]):

$$M_F := (n_\mu\delta_{\mu,\nu} + \Delta_{\mu,\nu}), \tag{55}$$

where n_μ is a number of Young frames $\alpha\vdash N-1$ for which $\alpha\in\mu$, and

$$\Delta_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu/\nu = \square, \\ 0 \text{ otherwise.}\end{cases} \tag{56}$$

The symbol $\mu/\nu = \square$ denotes Young frames μ,ν which can be obtained one from another by moving a single box. In Figure 5 we present explicit forms of the teleportation matrix for various values of port dimension d and ports number N. It turns out that an analytical form of $\lambda_{\max}(M_F^d)$ is known only in two cases, when $d\geq N$ (Corollary 8 in [H4]), and when $d=2$

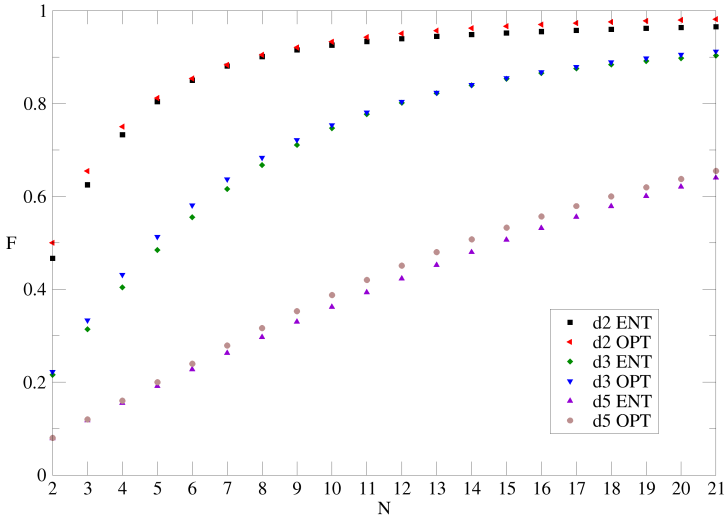

Figure 4: Graphic depicts the comparison of the non-optimal and optimal deterministic PBT scheme. The symbol dX ENT denotes the fidelity of the deterministic PBT when the resource state consists of maximally entangled pairs and measurements are of the form of the square-root measurements; X corresponds to the dimension of the teleported state. The symbol dX OPT denotes the best possible fidelity achieved by optimizing the resource state and measurement simultaneously, i.e. when one considers the optimal variant of the teleportation scheme.

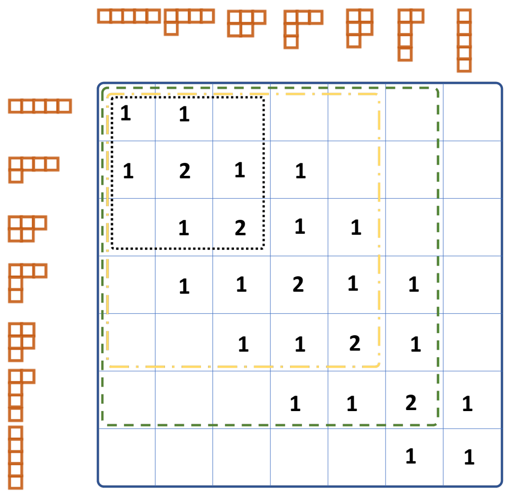

Figure 5: Graphic depicts teleportation matrices for the optimal deterministic PBT for a fixed number of ports $N = 5$ and various dimensions of ports $d = 2,3,4$ and $d \geq 5$. In the graphic, we depict subsequent principal minors which are defined by Young frames of a height no greater than d. The whole matrix corresponds to the case when the port dimension is no smaller than 5. The dashed green square corresponds to $d = 4$. Next, we have respectively $d = 3$ (yellow) and $d = 2$ (black). The empty cells contain zeros.

(Section 5.3 of [H4]). In the first case, the maximal eigenvalue is just equal to N giving optimal entanglement fidelity $F_{opt} = N/d^2$. In the latter case, the teleportation matrix is a tridiagonal matrix for which eigen-spectrum is known due to Losonczi result [E102]. Applying this we reproduce the result for optimal fidelity obtained in [E85] for qubits, which is

$$F_{opt} = \cos^2\left(\frac{\pi}{N+2}\right). \tag{57}$$

In the other cases one has to use efficient numerical methods described in Section 5.4 of [H4]). Finally, as we mentioned, the solution of the optimization problems allows us not only to derive the connection between optimal fidelity and teleportation matrix but also derive the optimal form of Alice's measurements, which is the element of the algebra $\mathcal{A}_n'(d)$

$$\forall 1 \leq i \leq N \quad \Pi_i = \Pi\sigma_i\Pi, \quad \text{with} \quad \Pi = \frac{d^N}{\sqrt{N}}\sum_{\alpha}\sum_{\mu\in\alpha}\sqrt{\frac{m_\alpha}{d_\alpha}\frac{v_\mu}{m_\mu}}F_\mu(\alpha), \tag{58}$$

where $F_\mu(\alpha)$ are the eigen-projectors from expression 37. The optimal operation O_A from (39) is element of the algebra $\mathcal{A}_n(d)$

$$O_A = \sqrt{d^N}\sum_{\mu}\frac{v_\mu}{\sqrt{d_\mu m_\mu}}P_\mu. \tag{59}$$

In the two above expressions, numbers v_μ are entries of the eigenvector corresponding to the maximal eigenvalue of the teleportation matrix. These numbers can be evaluated analytically for $d \geq N$ and $d = 2$, in the other cases we must support our findings by numerics.

In both cases of the probabilistic PBT, tools relying on the representation theory of the symmetric group S_n, the algebra of partially transposed permutation operators $\mathcal{A}_n'(d)$, and methods coming from semidefinite programming play a substantial role. However, contrary to the non-optimal deterministic case, here we do not know the form of the optimal measurements - they are not in the form of the square-root measurements (44), and they must obey the extra rule, given by expression (46) to ensure teleportation with unit fidelity. Solving analytically primal and dual problems we have shown that these solutions match each other, giving the optimal value of the average success probability and optimal form of measurements.

In the non-optimal case, when parties share N copies of maximally entangled state optimal average success probability p_{succ} from (45) reads (Theorem 3 in [H3]):

$$p_{succ} = \frac{1}{d^N}\sum_{\alpha} m_\alpha^2 \min_{\mu\in\alpha}\frac{d_\mu}{m_\mu}, \tag{60}$$

where the minimization is taken over all $\mu \vdash N$ which can be obtained from a given $\alpha \vdash N-1$ by adding a single box. Numerical values obtained from expression (60) are plotted in Figure 6. Optimal form of the operators $\{\Theta_{\bar{i}}\}_{i=1}^N$, and it what follows optimal measurements $\{\Pi_i\}_{i=1}^N$ from (46) is the following (Proposition 7 in [H3]):

$$\Pi_i^{AC} = P_{i,n}^+ \otimes \sum_{\alpha\vdash N-1}\frac{d}{\gamma_{\mu^*}(\alpha)}P_\alpha, \tag{61}$$

where the quantity $\gamma_{\mu^*}(\alpha)$ is a maximal eigenvalue of the operator $d^N\rho$ for a given α, and P_α is a Young projector from (9) acting on all systems but i and n. Measurement Π_0^{AC} corresponding to the failure of the protocol is obtained from the normalization condition $\sum_{i=0}^N \Pi_i^{AC} = \mathbf{1}_{AC}$.

For the optimal case, where Alice optimizes the resource state and measurements simultaneously, the general approach is similar to the above ones - however, is more involved from the technical point of view. Please compare respective SDP problems given in equations (24),(25) in [H3]), and equations (19), (20) of the same paper. Nevertheless, still we can solve optimization problems analytically, and have the strong duality property, getting the optimal value of the average success probability p_{succ} and corresponding measurements $\{\Pi_i\}_{i=1}^N$ (Theorem 8 in [H3]):

$$p_{succ} = 1 - \frac{d^2-1}{N+d^2-1}, \qquad \Pi_i^{AC} = P_{i,n}^+ \otimes \frac{d^{N+1}}{N\sum_{\nu\vdash N} m_\nu^2} \sum_{\alpha\vdash N-1} \frac{m_\alpha}{d_\alpha} P_\alpha. \tag{62}$$

This time the expression for the optimal average success probability p_{succ} depends only on the global parameters describing protocol like the number of ports N and their dimension d. Numerical values obtained from expression (62) are plotted in Figure 6. From the above expression, we get straightforwardly asymptotic behaviour of the considered protocol. For example, for $d=2$ we see that $p_{succ} = 1 - \mathcal{O}(1/N)$, and we reproduce result from [E85], while for $d>2$ we give novel results and analysis. It is important to stress here that the remarkably compact expression for p_{succ} from (62) was possible to obtain thanks to technical lemmas on purely group-theoretic character proven in Appendix E of paper [H3].

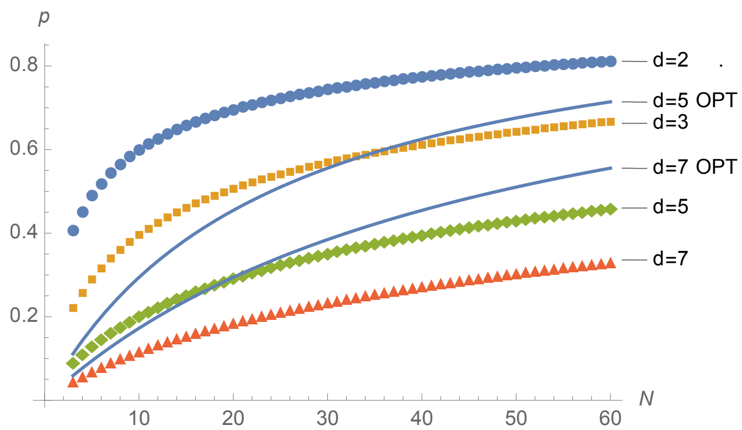

Figure 6: Dotted lines depict the average success probability when the resource state is fixed to N copies of maximally entangled pairs. Solid lines depict the average success probability for the optimal protocol with simultaneous optimization over resource state and measurements. We see that the optimal protocol clearly outperforms the non-optimal one for fixed dimension d.

5.6.2 Structure of the measurements and the signal states in port-based teleportation

As we described earlier, signal states and measurements in the PBT scheme belong to the algebra $\mathcal{A}_n'(d)$. This means we can apply the developed mathematical tool kit for their detailed description. For the separate interest deserve the results contained in papers [H2] and [H9], which allow us for further studies of the recycling protocols for PBT discussed in the next section. In this section, we cover results strictly related to the title of this section, since purely representation theoretic results have been summarised in Section 5.5.1.

Let us start by summarising the results contained in paper [H2]. The main achievement in this area was to find an analytical form of eigenvectors for operators $V'_{(a,n)}$, where $1 \leq a \leq n-1$ - this has been done in Proposition 54. These operators are in fact unnormalized signal states σ_a from equation (41). Using the result of the proposition we were able to prove that the entanglement fidelity F from (53) approaches 1 when $N \to \infty$ for any $d \geq 2$. This is contained in Theorem 52, in which as a by-product we obtained a lower bound on F depending only on global parameters, which are the number of ports N and port dimension d:

$$F \geq \frac{N}{d^2+N-1}. \tag{63}$$

This recovers the result of Köning and Beigi [E88], however, we obtain it by completely different, purely group-theoretic methods, without referring ourselves to the state discrimination problem.

In the paper [H9] we moved one step further with our analysis of the measurements $\{\Pi_i\}_{i=1}^N$ from (44). In the first step, in Proposition 4, by exploiting Lemma 35 from [H2], we evaluated their matrix elements in terms of irreducible matrix elements of $V'_{(a,n)}$. This result led us to composition law for two arbitrary measurements and is contained in Proposition 5 giving us a very interesting conclusion. Namely, whenever local dimension is high enough, i.e. $h(\alpha) < d$, where α labels irreps of S_{N-1}, the resulting POVMS are projective measurements - they satisfy a rule $\Pi_i\Pi_j = \delta_{ij}\Pi_i$. In all other cases, considered POVMs are pseudo-projectors on every irrep of the algebra $\mathcal{A}'_n(d)$ with a different constant. Having these findings we reached our main objective, which is calculating irreducible matrix elements of the square-root of any POVM (Proposition 6), and its trace with the operator $V'_{(a,n)}$ (Theorem 8). These results are very technical and there is no reason to present all of them in detail in this summary. We refer the Reader to Section IV in [H9] for all the details in this matter. Here, only for the self-consistence of this summary, we cite the final result concerning evaluating the following traces (Theorem 8 in [H9]):

$$
\begin{aligned}
\operatorname{Tr}\left(\sqrt{\Pi_a} V'_{(a,n)}\right) = & \sum_{\alpha:h(\alpha)<d} \frac{1}{n-1} \left(\sum_{\nu\in\alpha} \sqrt{m_\nu d_\nu} \right)^2 + \\
& + \sum_{\alpha:h(\alpha)=d} \frac{1}{\sqrt{(n-1)d_\alpha - d_\theta}} \frac{\sqrt{d_\alpha}}{\sqrt{(n-1)}} \left(\sum_{\nu\neq\theta} \sqrt{m_\nu d_\nu} \right)^2,
\end{aligned}
\tag{64}
$$

where $n = N + 1$. We see that the final expression depends only on group theoretic quantities such as multiplicities and dimensions of irreducible representations of the symmetric groups S_N and S_{N-1} in the Schur-Weyl duality, and does not depend on the index $1 \leq a \leq N$. The symbol d_θ denotes dimensions of the irreps θ of S_N which do not occur in decomposition when the port dimension is too small. More formally, the irreps θ belong to the following set:

$$
\Theta := \{\theta \vdash N \mid \theta \in \alpha \vdash N-1 \text{ with } h(\alpha) = d \text{ and } h(\theta) = d+1\}. \tag{65}
$$

When one considers irreps of S_{N-1} for which $h(\alpha) < d$, then Θ is an empty set. Notice that for a given Young frame α with $h(\alpha) = d$ there is only one θ with $h(\theta) = d + 1$. Additionally, we present closed expressions for the above-described objects in the qubit case, depending only on the number of ports N, see Lemma 9. It was possible because in this regime all Young frames labeling respective irreps must have up to two rows.

5.6.3 Recycling for deterministic port-based teleportation scheme

From Section 5.6.1 we have learned that no matter what kind of PBT protocol we wish to implement, to achieve reasonable efficiency measured in entanglement fidelity (dPBT) or probability of success (pPBT) one must use a substantial number of ports N. This however leads to the question of how port-based teleportation, in any known variant, is costly in terms of maximally entangled pairs which parties must prepare before they run the protocol. If the resulting resource state after one round of teleportation is heavily damaged then it can happen it is no longer a valid resource for port-based teleportation. This however could lead to the conclusion that after even a single round of the transmission process parties must have to prepare and send among them a new resource state - this is obviously a very serious constraint. However, the general situation is not so dramatic - in paper [E103] authors show that there exists a variant of the PBT scheme called the recycling protocol for PBT, in which parties can re-use all the ports with which they have left, still getting asymptotically ideal teleportation. The recycling protocol in any variant of PBT can be summarised as follows:

1. Alice performs a measurement $\{\Pi_i^{AC}\}_{i=1}^N$ obtaining an outcome $1 \leq i \leq N$.
2. Alice sends outcome i to Bob by a classical channel.

3. Parties apply a transposition (SWAP) between $i-$th and 1st port
4. Parties do not use port 1 in next rounds of the protocol - they only use remaining $N-1$ ports.
5. Parties repeat steps 1-4 using remaining ports to complete transmission of k states.

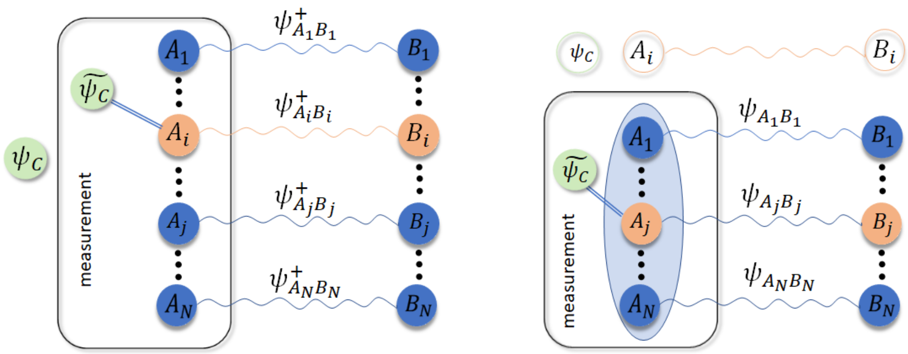

Figure 7: Schematic description of the recycling scheme for teleporting two unknown quantum states $\psi_C, \widetilde{\psi}_C$. On the left, we see the usual port-based teleportation procedure, when transmission occurred through $i-$th port. After this, parties are left with $N-1$ ports, we do not have port $\psi^+_{A_iB_i}$, since it has been consumed for teleporting state ψ_C – figure on the right-hand side. After the measurement in the first round, each port is no longer in the form of a maximally entangled state, and there are some correlations between all the ports (light blue ellipse). In the second round, when parties wish to transmit the state $\widetilde{\psi}_C$ must use this distorted resource state.

As it was explained in [E103] the recycling protocol $\mathcal{P}_{rec}(N,2,k)$ is indeed efficient if the fidelity $F(\mathcal{P}_{rec}(N,2,1))$ between states in the idealized situation, where the state is teleported and the remaining resource state is untouched, and the real state of the resource after application of a joint measurement in PBT if high enough. For the qubit case and dPBT the Authors concluded with the following lower bound

$$F(\mathcal{P}_{rec}(N,2,1)) \geq 1 - \frac{11}{4N} + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{66}$$

Next, having a lower bound on fidelity $F(\mathcal{P}_{rec})$ after one round of the recycling protocol, one can establish a similar lower bound after k rounds of the protocol (Lemma 2 in [E103]):

$$F(\mathcal{P}_{rec}(N,2,k)) \geq 1 - \frac{11k}{2N}. \tag{67}$$

We easily see that the error after each round is at most additive in the number of rounds k. These results imply that in every round of teleportation Alice one can apply the square-root measurement obtaining high efficiency of teleportation when parties just re-use the remaining ports.

In paper [H9] we focus on extensions of the results on the recycling protocol for dPBT beyond the qubit case and for the first time we discuss recycling for the optimal version of dPBT. What is more, we derived exact expressions for $F(\mathcal{P}_{rec}(N,d,1))$ in any variant of dPBT. Let us summarise our findings in more detail.

The first and the most crucial step toward the final results was the analysis of the interior structure of these objects from the point of view of representation theory. As we wrote in Section 5.6.1 the SRMs belong to the algebra $\mathcal{A}'_d(n)$ and tools developed in papers [H3, H2] can be effectively applied here. In this matter, we prove several propositions among which the most important are: composition law which shows when the considered measurements become projective, we compute matrix elements of the measurements in irreducible blocks, and finally

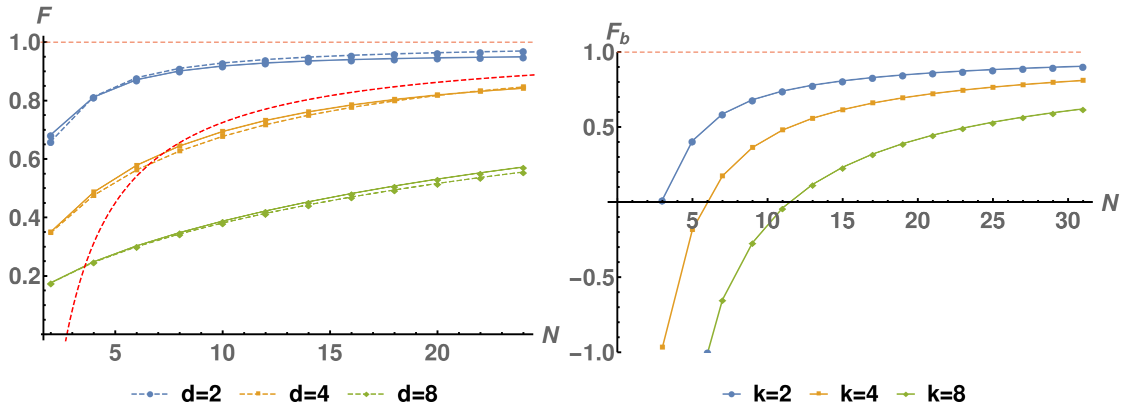

Figure 8: The left panel presents values of fidelity $F(P_{rec}(N,d,1))$ evaluated for non- (dashed lines) and optimal (solid lines) dPBT. From these plots, we see that the resource state for optimal PBT is not necessarily better for recycling protocol than the resource state for its non-optimal counterpart. In fact, for $d=2$ values of $F(P_{rec}(N,2,1))$ for the optimal version are even worse than for the non-optimal one. The red dashed line shows the lower bound on fidelity in the qubit case given by Eq. (66) up to $\mathcal{O}(1/N^2)$ part. The right panel presents the lower bound for $F(P_{rec}(N,2,k))$ (the qubit case) and various numbers of teleportation rounds k. From the plot, we see that $F(P_{rec}(N,2,k))$ is relatively high even for not a too large number of ports.

we present how to effectively calculate square-roots from the measurements occurring in dPBT. This has been summarised in the previous section of the summary.

Applying these findings we present explicit expressions for the quantity $F(\mathcal{P}_{rec}(N,d,1))$ in the case of non-optimal and optimal dPBT for andy $d\geq 2$ in the case of dPBT (see Theorem 12 in [H9]):

$$F(\mathcal{P}_{rec}(N,d,1))=\frac{\sqrt{N}}{d^{N+1}}\left[\sum_{\alpha:h(\alpha)<d}\frac{1}{N}\left(\sum_{\nu\in\alpha}\sqrt{m_\nu d_\nu}\right)^2+\sum_{\alpha:h(\alpha)=d}\frac{1}{\sqrt{Nd_\alpha-d_\theta}}\frac{\sqrt{d_\alpha}}{\sqrt{N}}\left(\sum_{\nu\neq\theta}\sqrt{m_\nu d_\nu}\right)^2\right], \tag{68}$$

while for optimal dPBT we obtained (see Theorem 14 in [H9]):

$$F(\mathcal{P}_{rec}(N,d,1))=\frac{1}{d^{1/2}}\sum_{\alpha\vdash N-1}\sum_{\mu\in\alpha}\frac{v_\alpha v_\mu}{m_\alpha^{1/2}}\frac{\sum_{\substack{\nu\neq\theta\\ \nu\in\alpha}}\sqrt{m_\nu d_\nu}}{\sqrt{Nd_\alpha-d_\theta}}. \tag{69}$$

The numbers v_μ, v_{nu} are elements of the eigienvector corresponding to the maximal eigenvalue of the teleportation matrix described briefly in Section 5.6.1 and in details in paper [H4]. The group theoretic parameters have been evaluated numerically by using the SAGE package [E101]. In particular case, when $d=2$, we present effectively computable expressions depending *only* on the number of ports N, without any need to refer to specialized computer simulations – please see Lemma 13 and Lemma 14 in [H9]. Having direct formulas for $F(P_{rec}(N,d,1))$ in both variants of dPBT we could give a lower bound on $F(P_{rec}(N,d,k))$, after k rounds of teleportation:

$$F(P_{rec}(N,d,k))\geq 1-2k(1-F(P_{rec}(N,d,1))), \tag{70}$$

which obviously tends to 1 when $F(P_{rec}(N,d,1))\to 1$. We summarize our findings on Figure 8 taken from [H9].

Finally, from the obtained results we conclude that there is no particular connection between the type of the dPBT (so resources states) and values of the recycled fidelity $F(P_{rec}(N,d,1))$. Namely, one can see that the entanglement fidelity $F(|\Psi^+\rangle_{AB},|\Psi\rangle_{AB})$ between the resource states for non-optimal and optimal dPBT is low when we increase the number of ports N (see Lemma 16 in [H9]), but the resulting recycled fidelity for them does not differ much.

5.6.4 Teleporting a large amount of quantum information

In the previous sections dedicated to port-based teleportation protocols, we have discussed the transmission of a single quantum system by using N ports, each of dimension d. Natural is to ask how to effectively transmit a composed quantum system or many independent quantum states exploiting variants of PBT protocols. We come up with five different scenarios:

1. Application of a separate PBT protocol to every single system separately. The disadvantage of this approach is the necessity of using a large number of separate resource states and sets of measurements - each resource state and set of measurements for a single system. Our goal however is to work in the scenario of N ports, without allowing for other, new resource states for parties. Because of these reasons such a kind of teleportation is not interesting for us for further analysis.

2. Application of the recycling protocol for the deterministic PBT [E103], [H9]. As it was described in Section 5.6.3, in this protocol parties re-use the resource state for the next rounds of teleportation, in each round dealing with the distorted resource state with one port less. Results in this regime have been obtained recently, for the time being only for the (non-)optimal deterministic case [H9], and explicit efficiency in this regime has not been yet evaluated for the teleporting channel. In particular, no analysis is known for the probabilistic variant of the recycling scheme. Additionally, in every step, one has to apply a different set of measurements, but that is not the case which in we are interested.

3. Application of a standard PBT scheme with a large enough port dimension, which is equal to $d = D^k$, where k is a number of teleported systems, and D stands for their local dimension. Such a scenario is very inefficient because entanglement fidelity and average success probability drastically decrease when port dimensions grow. This causes a need of exploiting a large number of shared entangled pairs in the resource state, see Table 1.

Teleportation protocol	**Entanglement fidelity F**
Non-optimised dPBT	$F = 1 - \frac{d^2-1}{4N} + O\left(N^{-3/2+\delta}\right)$
Optimised dPBT	$F \geq 1 - \frac{d^5+O\left(d^{9/2}\right)}{4\sqrt{2}N^2} + O\left(N^{-3}\right)$
Teleportation protocol	**Averaged probability of success p_{succ}**
Non-optimised pPBT	$p_{succ} = 1 - \sqrt{\frac{d}{N-1}}\mathbb{E}[\lambda_{\max}(\mathbf{G})] + o\left(N^{-1/2}\right)$
Optimised pPBT	$p_{succ} = 1 - \frac{d^2-1}{d^2-1+N}$

Table 1: Asymptotic behaviour of dPBT and pPBT with arbitrary port dimension d and port number N. All the results are taken from [E95] and [H3].

4. Application of so-called *packaged PBT*, when N ports we divide into k groups (each package has N/k ports) and we perform on every group separate PBT protocol, see the left graphic of Figure 9. The total entanglement fidelity $F_{pack}(N,k)$ is equal to the product of entanglement fidelities $F(N/k,1)$ of every package, so in total we have $F_{pack}(N,k) := F(N/k,1)^k$. Such protocol has been firstly suggested in [E103] and developed in [H7].

5. Application of so called *multi port-based teleportation scheme (MPBT)*. The existence of such kind of protocol has been suggested in [E103] in non-optimal deterministic variant, however, its rigorous description along with performance analysis was not known until the papers [H10, H8, H7] appeared. In the vanilla scheme sender and receiver share N maximally entangled pairs, but his time parties want to transmit a composite $k-$partite state $\Psi_C = \Psi_{C_1C_2\cdots C_k}$ or k independent states $\Psi_C = \psi_{C_1} \otimes \psi_{C_2} \otimes \cdots \otimes \psi_{C_k}$, see the right graphic in Figure 9. To do so, the sender applies a joint measurement on the state Ψ_C and his/her part

of the resource state, getting as an outcome multi-index $\mathbf{i} = \{i_1, i_2, \dots, i_k\}$, which is send to the receiver by a classical channel. To recover the state, the receiver must just pick up k ports in the right order, according to obtained multi-index. It is easy to see that in this variant the sender must have access to $k!\binom{N}{k}$ measurements, this number is of course equal to the number of possible multi-indices $\mathbf{i}$. In the probabilistic version of the MPBT, we must add one additional measurement corresponding to the failure of the teleportation process. We denote the set of all possible values of the multi-index $\mathbf{i}$ by $\mathcal{I}$.

As we stressed above, due to the lack of technical tools, efficiency analysis of the MPBT protocol, its generalization to optimal deterministic case, as well as, formulation of its probabilistic counterpart has been missing. We fully addressed these issues in the series of papers [H7, H10, H8] and we will summarise our findings in the next two paragraphs.

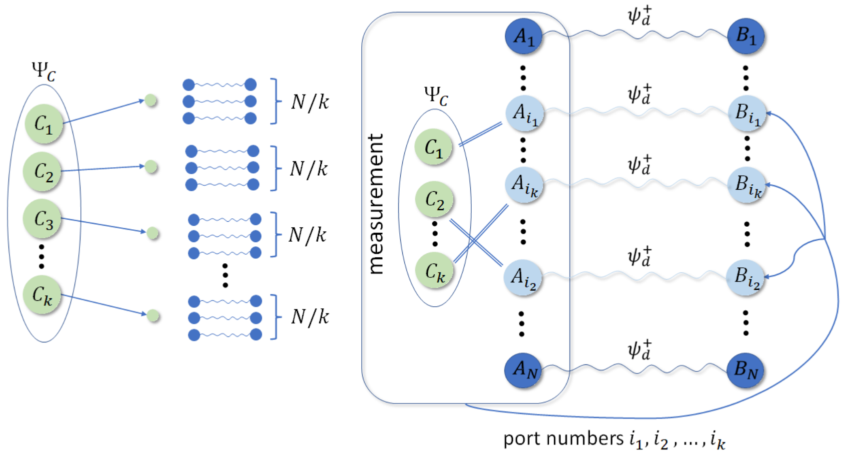

Figure 9: The left graphic presents the configuration for the packaged PBT protocol with k packages, each with N/k ports. The right-hand side graphic depicts a schematic description for multi-port based teleportation protocol (MPBT) with N ports. In both cases, parties transmit $k-$partite unknown quantum state Ψ_C with the highest possible efficiency. Both protocols are described in the main text of this summary.

Representation theoretic approach to MPBT protocols As it was for the PBT protocols described in Section 5.6.1 our goal was to describe deterministic and probabilistic MPBT protocol in (non-)optimal regimes by evaluating their efficiency and optimal measurements and resource state. This has been done in two papers [H10, H8]. The general idea behind the MPBT protocols is similar to the ordinary PBT when $k = 1$. However, from the technical point of view, the problem is much more complicated since one needs tools coming from the representation theory of the algebra of partially transposed operators $\mathcal{A}_n^{(k)}(d)$ for more than one partial transposition - this has been summarised in Section 5.5. Why we need this particular type of representation is easy to see from the symmetries satisfied by all the objects describing MPBT protocols. Namely, all the signal states $\{\sigma_{\mathbf{i}}^{AB}\}_{\mathbf{i}\in\mathcal{I}}$ satisfy the following commutation relations:

$$\begin{aligned} &\left[U^{\otimes(n-k)} \otimes \overline{U}^{\otimes k}, \sigma_{\mathbf{i}}^{AB}\right] = 0, \qquad \forall\, U \in \mathcal{U}(d), \\ &\left[V_\pi, \sigma_{\mathbf{i}}^{AB}\right] = 0, \qquad \forall\, \pi \in S_{n-2k}, \end{aligned} \tag{71}$$

and they are covariant with respect to the symmetric group $S(n-k)$:

$$V_\pi \sigma_{\mathbf{i}}^{AB} V_\pi^\dagger = \sigma_{\pi(\mathbf{i})}^{AB}, \qquad \forall\, \pi \in S_{n-k}, \tag{72}$$

where $n = N + k$. The same type of symmetries are satisfied for respective measurements $\{\Pi_{\mathbf{i}}^{AB}\}_{\mathbf{i}\in\mathcal{I}}$ applied by Alice. From the definition of the set $\mathcal{I}$, we see that for $k = 1$ we reduce the above relations to these described in Section 5.6.1. Having developed necessary representation theoretic tools, mostly in paper [H10], we could write down respective optimization SDP problems in the algebra $\mathcal{A}_n^{(k)}(d)$ and solve them analytically. The solutions of the primal and dual problems match each other, giving us optimal values of entanglement fidelity, average success probability, as well as the optimal form of measurements.

In the deterministic non-optimal case Alice chooses the square-root measurements of the form

$$\forall\, \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = \frac{1}{\sqrt{\rho}}\sigma_{\mathbf{i}}^{AC}\frac{1}{\sqrt{\rho}} + \Delta, \qquad \rho = \sum_{\mathbf{i}\in\mathcal{I}} \sigma_{\mathbf{i}}^{AC}, \tag{73}$$

where Δ is an additional term ensuring summation to the identity operator on the whole space $(\mathbb{C}^d)^{\otimes n}$. We see that the primary technical obstacle is to find the inverse of $\rho = \sum_{\mathbf{i}} \sigma_{\mathbf{i}}$. However, using discussed symmetries we can prove the following eigendecomposition of ρ (Theorem 17 in [H10]):

$$\rho = \sum_{\alpha\vdash N-k}\sum_{\substack{\mu\vdash N\\ \mu\in\alpha}} \lambda_\mu(\alpha)F_\mu(\alpha), \qquad \lambda_\mu(\alpha) = \frac{k!\binom{N}{k}}{d^N}\frac{m_\mu}{m_\alpha}\frac{d_\alpha}{d_\mu}, \tag{74}$$

where $F_\mu(\alpha)$ are the eigen-projectors from expression (37). This time Young frames μ of N boxes are obtained from Young frames α of $N-k$ boxes by adding k boxes. Having the above, we prove Theorem 22 from [H10] giving group-theoretic expression for the entanglement fidelity:

$$F = \frac{1}{d^{N+2k}}\sum_{\alpha\vdash N-k}\left(\sum_{\mu\in\alpha} m_{\mu/\alpha}\sqrt{m_\mu d_\mu}\right)^2, \tag{75}$$

where $m_{\mu/\alpha}$ denotes number of ways in which μ can be obtained from α by adding k boxes. For the special case of $k=1$, we always have $m_{\mu/\alpha}=1$, and the above expression reduces to entanglement fidelity for the non-optimal dPBT given by (52). This number can be evaluated numerically using our software and for the qubit case, there exists a compact analytical expression for it [E104, E105]. In the left panel of Figure 10 we present numerical values of entanglement fidelity for this scheme compared with the ordinary PBT discussed in Section 5.6.1, but with respectively higher port dimension. We see that the MPBT protocol is more efficient even than the optimal PBT. For the non-optimal probabilistic scheme studied in [H10], to ensure the unit entanglement fidelity we must demand from the measurement analogous relation to (46), as it was for the standard pPBT:

$$\forall \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = P^+_{A_{\mathbf{i}}C} \otimes \Theta_{\overline{A}_{\mathbf{i}}}, \tag{76}$$

where the form of $\{\Theta_{\overline{A}_{\mathbf{i}}}\}_{\mathbf{i}\in\mathcal{I}}$ is derived from SDP programs (grey boxes on page 7 in [H10]). The solution for these optimization programs, together with the expression for the averaged success probability is presented in Theorem 23 in [H10]. Namely, measurements are elements of the algebra $\mathcal{A}_n^{(k)}(d)$ from (18):

$$\forall\, \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = \frac{k!\binom{N}{k}}{d^{2N}} P^+_{A_{\mathbf{i}}C} \otimes \sum_{\alpha\vdash N-k} P_\alpha \min_{\mu\in\alpha}\frac{1}{\lambda_\mu(\alpha)}, \tag{77}$$

and the average probability of success is of the form:

$$p_{succ} = \frac{k!\binom{N}{k}}{d^{2N}}\sum_{\alpha\vdash N-k}\min_{\mu\in\alpha}\frac{m_\alpha d_\alpha}{\lambda_\mu(\alpha)}, \tag{78}$$

where the minimisation is taken over all Young frames μ which can be obtained from a given Young frame α of $N-k$ boxes by adding k boxes. Numerical values of p_{succ}, compared with

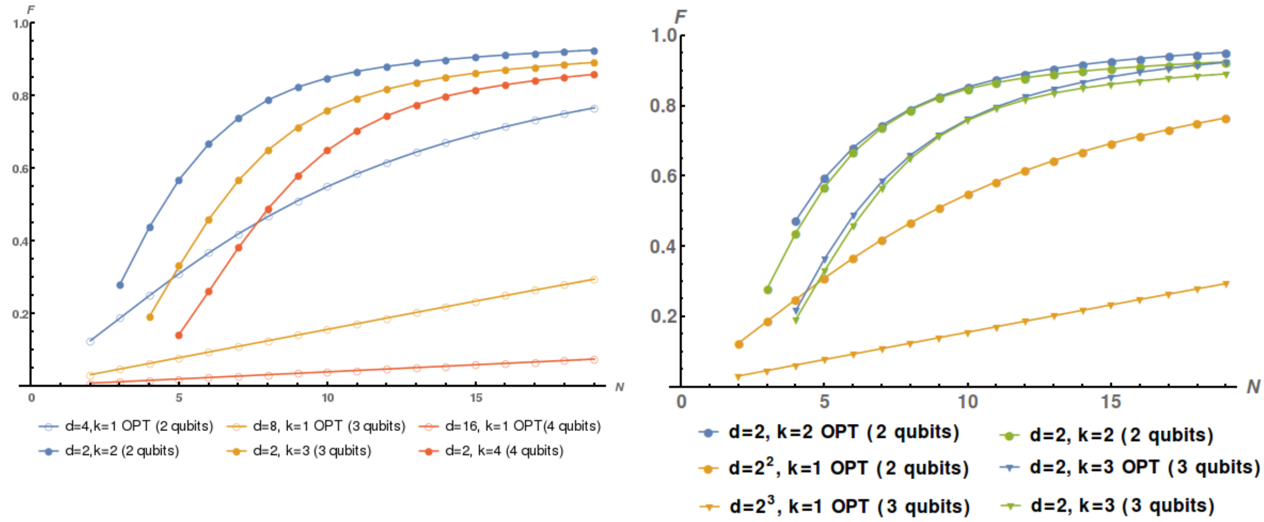

Figure 10: In the left panel we show the performance of the non-optimal deterministic MPBT protocol measured in entanglement fidelity F, for various choices of initial parameters which are local dimension d, number of ports N and number of teleported particles k. We achieve better performance than the standard optimal PBT (OPT) with respective high port dimension for teleporting a state of two qubits ($d = 2, k = 2$). In the right panel, we present a comparison of the optimal deterministic MPBT scheme ($k > 1$, OPT) with its non-optimal version. In every case, the efficiency in the optimal case is higher. Additionally, to illustrate the efficiency jump, we plot entanglement fidelities for the standard optimal PBT ($k = 1$).

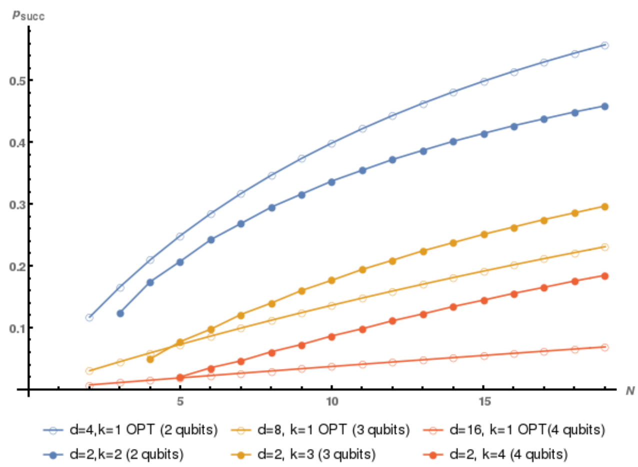

Figure 11: The performance of the probabilistic version of non-optimal MPBT protocol, measured in average success probability p_{succ}, for various choices of initial parameters which are local dimension d, number of ports N, and number of teleporting particles k. One can see that we start achieving better performance than the corresponding optimal PBT scheme with appropriate port dimension for a state of three qubits ($d = 2, k = 3$).

the standard optimal probabilistic PBT scheme with enough large port dimension are depicted in Figure 11.

The optimal deterministic and probabilistic case has been studied mainly in paper [H8]. Solving analytically primal and dual optimization problem we came up to the conclusion that the optimal entanglement fidelity is described by the *generalized teleportation matrix* $M_F^{d,k}$. This matrix has been defined through Definition 6 in [H8] and studied in chapter 6.2 in the same paper. According to Theorem 7 from [H8], we have the following expression for the entanglement fidelity

$$F = \frac{1}{d^{2k}}\lambda_{\max}(M_F^{d,k}), \tag{79}$$

where k is the number of teleported systems and $\lambda_{\max}(M_F^{d,k})$ is maximal eigenvalue of generalised teleportation matrix $M_F^{d,k}$. Finally, in Theorem 8 we present the optimal form of Alice's measurement

$$\Pi_{\mathbf{i}} = \Pi\sigma_{\mathbf{i}}\Pi \quad \text{with} \quad \Pi = \sum_{\alpha\vdash N-k}\sum_{\mu\in\alpha} \frac{d_\mu}{\sqrt{k!\binom{N}{k}}}\sqrt{\frac{m_\alpha}{d_\alpha}\frac{v_\mu}{m_\mu}}F_\mu(\alpha), \tag{80}$$

where $F_\mu(\alpha)$ are the eigen-projectors from expression 37. We see that the measurements belong to the algebra $\mathcal{A}_n^{(k)}(d)$ from (18), while the optimal operation O_A is an element of the group algebra $\mathcal{A}_n(d)$ from (2):

$$O_A = \sqrt{d^N}\sum_\mu \frac{v_\mu}{d_\mu m_\mu}P_\mu, \tag{81}$$

where P_μ denotes Young projector on irreps of S_N labelled by $\mu \vdash N$. In both equations, the numbers $v_\mu > 0$ are the components of an eigenvector corresponding to the greatest eigenvalue of the generalized teleportation matrix $M_F^{d,k}$ which can be evaluated numerically. In the case of optimal probabilistic protocol, by proving Theorem 9 in [H8] of purely group-theoretic taste, we have derived closed expression for the averaged success probability p_{succ} (Theorem 3 [H8]):

$$p_{succ} = \frac{N!}{(N-k)!}\frac{(d^2+N-k-1)!}{(d^2+N-1)!}. \tag{82}$$

Using the above result, in Theorem 4 in [H8] we show that whenever the number of teleported states $k = k(N)$ changes as $o(N)$, then the averaged success probability approaches 1 with $N \to \infty$ for arbitrary port dimension. Moreover, in the special case of qubits, we show that in the optimal MPBT averaged success probability goes to 1 with $N \to \infty$ whenever $k = o(N)$. This result improves the non-optimal MPBT protocol for which we have the same behaviour for $k = o(\sqrt{N})$. Figure 12 presents the comparison of numerical values of the averaged success probability p_{succ} for the optimal MPBT with the standard probabilistic PBT with respectively large port dimension. Finally, in Theorem 5 in [H8] we derived optimal form of measurements and operation O_A as elements of the group algebras $\mathcal{A}_n(d)$ and $\mathcal{A}_n(d)$ respectively:

$$\Pi_{\mathbf{i}} = P^+_{A_{\mathbf{i}}B}\otimes\sum_{\alpha\vdash N-k}\frac{d^{N+k}\frac{m_\alpha}{\sum_{\nu\vdash N}m_\nu^2}}{k!\binom{N}{k}d_\alpha}P_\alpha, \quad O_A = \sqrt{d^N}\sum_\mu\sqrt{\frac{m_\mu}{d_\mu\sum_{\nu\vdash N}m_\nu^2}}P_\mu. \tag{83}$$

By P_α, P_μ we denote Young projectors on irreps of S_{N-k} and S_k respectively.

MPBT protocols - asymptotic analysis In this section, we summarise the main results contained in paper [H7]. The main purpose of this paper is to evaluate and compare the transmission capability of all variants of MPBT protocols and compare it with other effective methods for transmission of a large amount of quantum information described at the beginning of Section 5.6.4. We have been interested in how the efficiencies vary when the number of transmitted states is $k \sim N^\alpha$, where $\alpha \in (0,1)$, and the number of ports N to infinity. This is somehow the most natural way of considering the asymptotic behaviour of MPBT protocols since only in this scenario number of teleported states k changes together with the number of ports, showing us the real capabilities of a given scheme. In the traditional approach to channel capacity, the central quantity is the asymptotic rate k/N - number of sent qubits (qudits) per number of channel (or entangled pair) uses. Here the main objective becomes to identify asymptotic exponents α for which the transmission is possible since the mentioned rate vanishes for the PBT. Such considerations, where we ask how much quantum information we can reliably send

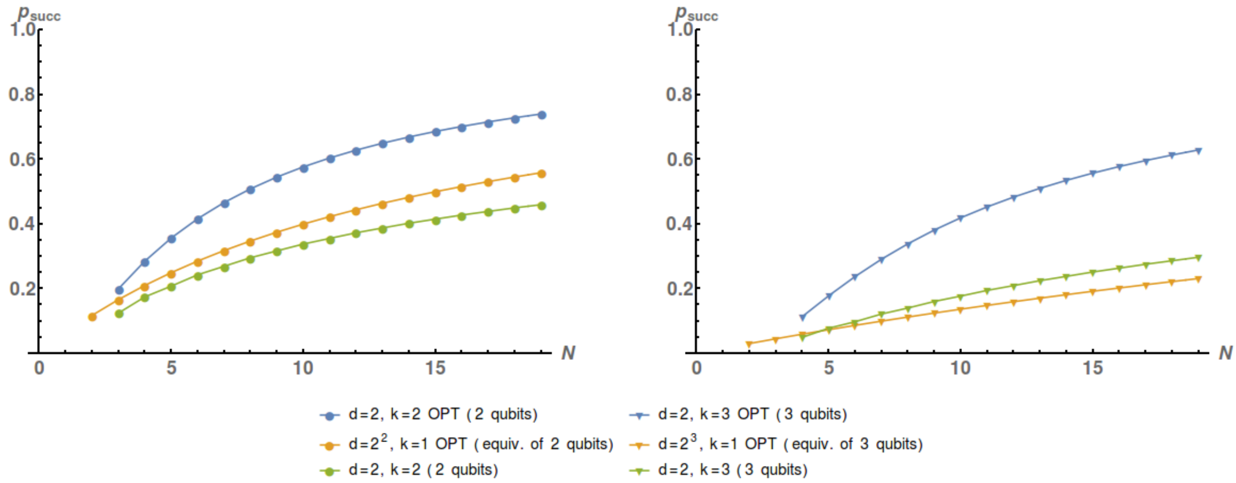

Figure 12: The right-hand side figure presents the comparison of the optimal MPBT scheme ($k > 1$, OPT) with its non-optimal version and standard PBT scheme with high enough port dimension. In all variants, the efficiency is substantially higher. Additionally, on the left-hand side plot, we present the probability of success for the standard optimal PBT, for port dimensions $d = 4$ and $d = 8$. We see that our scheme for $k = 2, d = 2$ performs better, which was not the case in the non-optimal MPBT procedure for the same parameters. This is the only such case.

via teleportation protocols have been considered for the first time and led us to new qualitative and quantitative results.

We have shown that in general the number of systems to be teleported can be changed dynamically by the sender with the growing number of ports, still ensuring high efficiency in the deterministic and probabilistic scheme. In any variant of either PBT or MPBT, we obtain critical-like behaviour of the quality of transmission. We identify the critical values α_{cr} of exponents α for several variants, both for exact asymptotic values of a figure of merit and their lower bounds. Whenever the value of α is below the critical value α_{cr} the values of fidelity (or probability of success, depending on the scheme) describing transmission are 1, and 0 otherwise.

Now, let us now briefly summarise the main results focusing independently on deterministic and probabilistic MPBT protocol:

1. **Deterministic protocols** In all deterministic (M)PBT protocols we can relate resulting entanglement fidelity F of transmission with the probability of success p_{dist} of discriminating set of signal states $\{\sigma_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}}$, each given with equal probability $1/|\mathcal{I}| = 1/k!\binom{N}{k}$. This idea was first suggested by Beigi and Köning in [E88] for $k = 1$ and further developed by us in paper [H7]. For MPBT protocols and any feasible set of measurements $\{\Pi_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}}$ the mentioned relation is of the form:

$$F = \frac{k!\binom{N}{k}}{d^{2k}} p_{dist}, \qquad p_{dist} = \frac{1}{k!\binom{N}{k}} \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}(\Pi_{\mathbf{i}}\sigma_{\mathbf{i}}). \tag{84}$$

   To find a lower bound on F as a measurements $\{\Pi_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}}$ we have chosen square-root measurements given by expression (73). For such a choice we proved the generalization of Lemma A.3 from [E88], where we derived lower bound p_{dist} in terms of trace from the normalized operator $\overline{\rho} = \rho/\,\mathrm{Tr}(\rho)$, where ρ is in (74):

$$p_{dist} \geq \frac{1}{d^{N-k}\,\mathrm{Tr}\,\overline{\rho}^2}. \tag{85}$$

   Next, we have proven one of the central technical result contained in Lemma 1 in [H7], where using purely combinatorial reasoning we evaluated $\operatorname{Tr}\overline{\rho}^2$, which is:

$$\operatorname{Tr}(\overline{\rho}^2) = d^{-N-k}\binom{N}{k}^{-1}\binom{d^2+N-1}{k}. \tag{86}$$

This however, led us to two lower bounds on F derived in Theorem 2 in [H7], depending only on global parameters - number of ports N, port dimension d, number of teleported systems k:

$$F \geq \binom{N}{k}\binom{d^2+N-1}{k}^{-1} \geq \left(1-\frac{d^2-1}{d^2+N-k}\right)^k. \tag{87}$$

Having the above result we are in the position for investigating behaviour of MPBT protocol for $N \to \infty$, when changing k adaptively with N, so in interesting for us regime. It turns out that in the deterministic scheme, even the optimal packaged PBT scheme is outperformed by the lower bound evaluated for non-optimal MPBT. In particular, we argue that asymptotically in non-optimal deterministic MPBT, one can teleport a much larger amount of quantum information, i.e. with $\alpha_{\mathrm{cr}} = 1$, in comparison to optimal packaged port-based teleportation, where $\alpha_{\mathrm{cr}} = 2/3$, and one can teleport faithfully only up to $o(N^{2/3})$ qubits. The summary of our findings is presented in Table 2 and the left graphic in Figure 13.

	$F=0$	F_{cr}	$F=1$
Pack.PBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2\,,\ F_{\mathrm{cr}} = e^{-3a^2/4}$	$\alpha < 1/2$
Pack.OPBT	$\alpha > 2/3$	$\alpha_{\mathrm{cr}} = 2/3\,,\ F_{\mathrm{cr}} = e^{-\pi a^3}$	$\alpha < 2/3$
MPBT	$-$	$\alpha_{\mathrm{cr}} = 1, \quad F_{\mathrm{cr}} \geq e^{-\frac{3a}{1-a}}$	$\alpha < 1$

Table 2: Comparison of the asymptotic behaviour of two variants of packaged PBT with MPBT in deterministic version, where $k = aN^\alpha$. By "cr" we denote the critical values of parameter α for which the asymptotic value of F exhibits a jump.

2. **Probabilistic protocols** Results regarding asymptotic behaviour of non-optimal probabilistic case, have been obtained by exploiting group-theoretic expression for the probability of success derived in [H10], see also (78) of this summary. In this case, we have restricted ourselves to the case of qubits, where the considered expression turns into elegant form, expressed in terms of angular momentum parameters, where we can rid of the minimization in (78):

$$p_{succ} = \frac{1}{2^N}\frac{1}{N+1}\sum_{s=0(\frac{1}{2})}^{\frac{N-k}{2}}(2s+1)^2\binom{N+1}{\frac{N-k}{2}-s}. \tag{88}$$

Unlike in the deterministic variant in probabilistic non-optimal MPBT the scaling is the same as for probabilistic optimal packaged PBT, allowing for teleportation asymptotically $o(N^\alpha)$ qubits with $\alpha_{\mathrm{cr}} = 1/2$. This shows the qualitative difference between a deterministic scheme and a probabilistic one. Additionally, for a finite number of ports N we derived lower and upper bound for p_{succ} in our protocol (Proposition 18 in [H7]). We achieved this by combining advanced tools from statistical analysis, in particular non-straightforwardly the celebrated Berry-Essen theorem [E106, E107] with direct estimates of binomial expressions by their Gaussian approximations. For the optimal probabilistic MPBT, we work with exact values of success probability for an arbitrary dimension of the port d. It was possible since the exact formula for the probability of success has been derived in [H8] (see equation (82) of this summary) and transformed to useful for us form:

$$p_{succ} = \prod_{m=2}^{d^2} \left(1 - \frac{k}{N-1+m} \right). \tag{89}$$

Considering the above expression we show that one can transmit asymptotically $o(N^\alpha)$ with a critical exponent equal to 1, clearly outperforming PBT variants. That was illus-

	$p_{succ} = 0$	$p_{succ,\mathrm{cr}}$	$p_{succ} = 1$
Pack.PBT	$\alpha > 1/3$	$\alpha_{\mathrm{cr}} = 1/3\,,\ p_{succ,\mathrm{cr}} = e^{-ca^{3/2}}$	$\alpha < 1/3$
Pack.OPBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2,\ p_{succ,\mathrm{cr}} = e^{-3a^2}$	$\alpha < 1/2$
MPBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2$, eq. (15) in [H7]	$\alpha < 1/2$
OMPBT	$-$	$\alpha_{\mathrm{cr}} = 1,\quad p_{succ,\mathrm{cr}} = (1-a)^3$	$\alpha < 1$

Table 3: Comparison of the asymptotic behavior of probability of success p_{succ} of all variants of packaged PBT with MPBT in probabilistic version when $k = aN^\alpha$. We have $c = \sqrt{8/\pi}$ and by "cr" we denote the critical values of parameter α for which the asymptotic value of p_{succ} exhibits a jump.

trated in Table 3 and by the right graphic in Figure 13 to see the discussed comparison.

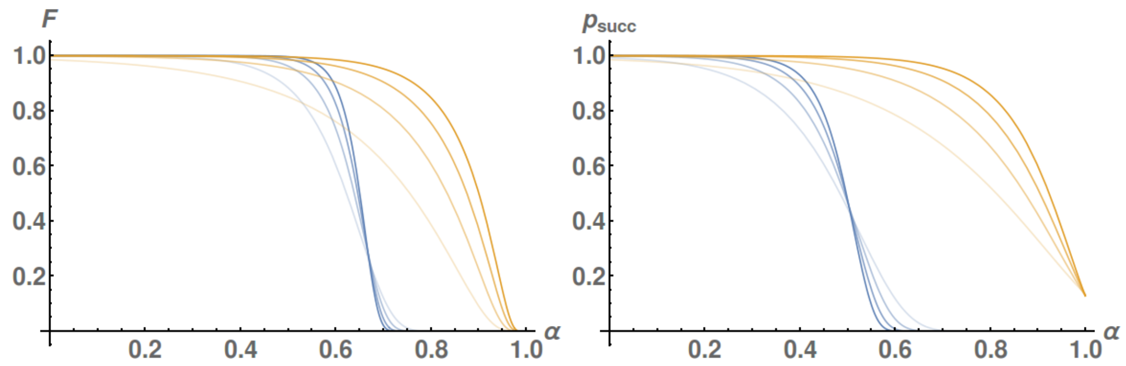

Figure 13: Left graphic: Asymptotic values of the lower bound on entanglement fidelity in MPBT (orange lines) compared to the fidelity of Pack. OPBT (blue lines), where $k = aN^\alpha, a = 1,\ \alpha \in (0,1)$. N runs through $10^2, 10^3, 10^4$ and 10^5 as the lines become thicker. Right graphic: Probability of success, compared for Pack. OPBT (blue lines), and OMPBT (orange lines) protocols, where $k = aN^\alpha, a = \frac{1}{2}$. N runs through $10^2, 10^3, 10^4$ and 10^5 as the lines become thicker.

6 Presentation of teaching, organizational, and popularisation of science achievements

6.1 Teaching achievements

Academic teaching:

1. Lecture in *Classical Mechanics* (45 hours, the academic year 2022/2023, winter semester, for the 2nd year of physics students)
2. Tutorials in *Classical Mechanics* (45 hours, the academic year 2022/2023, winter semester, for the 2nd year of physics students)
3. Lecture in *Classical Information Theory* (in English, 30 hours, the academic year 2021/2022, summer semester, for the 1st year of the second-degree studies for quantum information technology students)
4. Laboratories in *Python with the basics of algorithmic* (45 hours, the academic year 2021/2022, summer semester, for the 1st year of bio-informatics students)
5. Lecture in *Classical Mechanics - elements of special relativity* (15 hours, the academic year 2021/2022, summer semester, for the 2nd year of physics students)
6. Tutorials in *Classical Mechanics* (30 hours, the academic year 2021/2022, summer semester, for the 2nd year of physics students)
7. Tutorials in *Stochastic processes: basics and applications* (30 hours, the academic year 2021/2022, summer semester, for the 2nd year of mathematical modelling and data analysis students)
8. Tutorials in *Linear Algebra with Geometry* (30 hours, the academic year 2013/2014, winter semester, for the 1st year of physics students)
9. Tutorials in *Introduction to Mathematics* (30 hours, the academic year 2012/2013, winter semester, for the 1st year of bio-informatics students)

PhD student supervising:

1. Co-supervisor of Piotr Adam Kopszak, 2019-present, Faculty of Physics and Mathematics, University of Wrocław
2. Supervisor of Tomasz Patryk Młynik in research grant Sonata 16 (UMO-2020/39/D/ST2/01234), 2022-present, Faculty of Mathematics, Physics and Informatics, University of Gdańsk

Non-academic teaching

1. Lecturing at the summer school *"Next generation of quantum information scientists. Gdańsk series of international school for students"*, 11.07.2022-22.07.2022, Faculty of Mathematics, Physics and Informatics, the University of Gdańsk. I delivered 8 hours of lectures entitled *"Introduction to quantum computing"* (quantum circuit model, the universality of the quantum circuit model, fundamental quantum algorithms, etc.).

6.2 Organisational achievements

1. Organisation of the summer school *"Next generation of quantum information scientists. Gdańsk series of international school for students"*, 11.07.2022-22.07.2022, Faculty of Mathematics, Physics and Informatics, University of Gdańsk. Together with Dr. Sergii Strelchuk (The University of Cambridge, United Kingdom), I prepared the whole teaching schedule, lecture notes, list of problems with solutions for 40 hours of lectures + 20 hours of tutorials. `https://gqi.ug.edu.pl/`

2. Scientific committee of the planned conference *Mathematical Structures in Quantum Mechanics*, 19-22.06.2023, Faculty of Mathematics, Physics and Informatics, University of Gdańsk. `https://gwmp.ug.edu.pl/`

6.3 Popularisation of science achievements

1. 16.03.2023, Open days of the University of Gdańsk, the title of the talk: *"Na problemy jutra FIZYKA!"*

2. 09-11.09.2019, Zdolni z Pomorza (eng. Talented from Pomerania), Mathematics of board games (workshop)

3. 19.11.2017, Polish Science Cafe, Trinity Hall, Cambridge, the title of the talk *"A few steps towards everyday quantum cryptography"* (invited)

7 Other scientific achievements

7.1 Bibliometric data

Source: Google Scholar (14.03.2023)

- Number of peer-reviewed publications: 25 (9 before PhD)
- Number of unpublished arXiv preprints: 7 (5 of them are under review process in scientific journals, please see `https://arxiv.org/search/?searchtype=author&query=Studzi%C5%84ski%2C+M&order=-announced_date_first&size=50&abstracts=hide`)
- Total number of citations: 560
- H-index: 11
- I10-index: 12

Source: Web of Science (14.03.2023)

- Number of peer-reviewed publications: 23 (9 before PhD)
- Total number of citations: 345 (total), 309 (without self-citations)
- H-index: 9

7.2 Awards

1. 08.2021-07.2024, The Minister scholarship for outstanding young researchers, Polish Ministry of Education and Science, UMO-SMN/16/0938/2020

2. 05.10.2021, University of Gdańsk Rector's prize of 3rd degree

7.3 Track record before PhD

7.3.1 Research included in my PhD thesis:

1. **Developments in representation theory** The celebrated Peres-Horodecki criterion for the separability of quantum states is one of the main result in quantum information theory. It says that whenever $2 \otimes 2$ and $2 \otimes 3$ quantum systems stay positive after partial transposition the state is separable. In higher dimensions, the positive partial transposition (PPT) property gives a necessary condition for separability. In the general situation deciding whether a given quantum state is separable/entangled is a very complex task and we can apply numerical methods like for example Doherty's algorithm, which is not always conclusive or apply the notion of an entanglement witness. Another way of approaching the problem is restricting the set of considered quantum states, for example to the set of states that possess certain symmetries, but still, contain a wide range of objects. In the series of papers [P3],[P4] we develop ideas firstly introduced by Werner and Eggeling for the states which are $U \otimes U \otimes U$ invariant and become positive after partial transposition. The main goal was to develop a mathematical tool allowing in principle for constructing PPT states in the multiparty regime when the partial transposition acts on the last system. The main observation for our construction was the fact that objects which are $U^{\otimes n}$ invariant, after mentioned partial transposition, become $U^{\otimes(n-1)} \otimes \overline{U}$ invariant, where the bar denotes complex conjugation. In fact, such objects belong the algebra of partially transposed permutation operators, where the partial transposition acts on the last system. In our work, we identify irreducible (minimal) left ideals of the considered algebra and give a recipe for constructing an irreducible basis in each of them. We have done this twofold, in paper [P4] by constructing a set of non-orthogonal basis vectors, to which we had to apply an orthogonalization procedure by using inversion of the corresponding Gram matrix. In paper [P3], we applied methods of construction coming from purely algebra theory, obtaining an orthogonal set of irreducible basis operators. However, in this approach, we are still left with the technical problem of finding a subset of a linearly independent set of operators, which is connected with the inversion problem of the mentioned Gram matrix. Nevertheless, we were able to prove how to decompose every partially transposed permutation operator in terms of the proposed basis operators and work with the quantum systems exhibiting such symmetries. In particular, developed methods allow us for a detailed description of quantum cloning machines (see below) and allow for the construction of PPT quantum states for low number of systems n. In particular, using our tools we were able to reconstruct the previously known results for $U^{\otimes 3}$ invariant states.

2. **Description of the universal quantum cloning machines** The impossibility of producing perfect clones of an unknown quantum state is one of the main feature distinguishing quantum mechanics from the classical world. This feature follows directly from the linearity of the underlying quantum theory. However, still quantum mechanics allows for producing clones that are not perfect or speaking more formally the fidelity between clones and the state to be cloned is lower than 1. Having this property, we can ask how well one can clone quantum states, what is the limit of the resulting fidelity imposed by quantum theory? In two papers [P2],[P5] we consider the most general scenario of universal quantum cloning machines (UQCM) producing N clones from one input state. We started our considerations by investigating the problem for qubits [P5], where we were able to show that the total system admits symmetries allowing for the application of the Schur-Weyl duality. Namely, the problem of finding the allowed fidelity of the clones can be reduced to finding allowed fidelity but on every irreducible space separately. Then to obtain the full allowed region it is enough to take so-called the convex hull of the obtained subregions. In particular, we analyzed in detail the case of $1 \to 3$ UQCM and

presented graphics of the allowed region by theory region of fidelities of the clones in three-dimensional space. Reconstruction of the clones with prescribed cloning fidelities is also discussed. For higher dimensions, the Schur-Weyl duality does not work and new methods must be derived. In [P2] we argued that the total system under consideration is in fact element of the algebra studied in papers [P3],[P4]. By applying these tools we could perform a similar analysis of quantum cloning machines as it was for qubits.

3. **Contribution to entanglement distillation** One of the most important and fundamental resource in quantum information protocols is pure entanglement. Unfortunately, in practice, if we want to run some communication task we have access only to mixed entanglement, which is a limiting factor. One way to rid of this difficulty, before we apply our target protocol, is to distill pure entanglement in a form of maximally entangled pairs. To do so we apply a procedure called entanglement distillation. In paper [P6] we present a distillation procedure from two-qubit states which are mixtures of three mutually orthogonal states. By exploiting symmetries of the problem we could apply tools coming from the Schur-Weyl duality and Young symmetrizers. Beig more strict, we distill entanglement by projecting n copies of the state on permutationally invariant subspaces and applying on-way hashing protocol. Having the problem rewritten in a purely group-theoretic manner we could find analytical expressions for the rate of the protocol. Additionally, we solved this problem using an equivalent approach relying on algebraic association schemes called the Johnson scheme. We also generalize our approach to higher dimensions.

7.3.2 Research not included in my PhD thesis:

1. **Geometric properties of the private states** The main task of quantum cryptography is to obtain the classical secret key for data encryption from quantum states. This can be achieved by maximally entangled pairs or a much more broader class of states, called private bits. These entangled states are built of two parts - the bipartite key part, from which we produce the secret key, and the bipartite part protecting the key from an eavesdropper called the shield part. In paper [P7], we analyzed the geometrical properties of private states from the point of view of their distance from the set of separable states. We were interested in private states which admit PPT property (they remain positive operators after partial transposition) and are far away as possible from the separable states. It means that from such states no pure entanglement can be distilled but still we can obtain a cryptographic key. We show that there are such private states that for the fixed dimension of the key part the distance from the set of separable states increases with the growing dimension of the shield part. The novel feature in our approach is the fact we did not need to use the method called boosting, relying on tensoring many copies of private state to achieve a state arbitrarily far from the separable states. This leads us to better distance scaling than in the previous attempts. Additionally, we presented explicit construction of private states under consideration.

2. **Studies on local random quantum circuits** Random unitary matrices are an important resource in quantum computing and information theory in general. The possible applications range from quantum cryptography, quantum data-hiding, and foundations of statistical mechanics, up to quantum thermodynamics problems like equilibration of quantum states. Unfortunately, to implement a random Haar unitary we need an exponential number of quantum qubit gates and random bits - these properties make impossible practical implementations. One way to deal with this problem is to consider approximate unitary $t-$designs which mimic properties of the Haar measure for polynomials of degrees up to t. There was even a conjecture, proved later, that polynomial-sized random quantum circuits acting on n qubits form an approximate unitary poly$(n)-$ design. It was done

by presenting a ratio of convergence to a given $t-$design as a function of n. The main idea is to exploit tools coming from many body physics and translate the problem into a problem of finding bounds on the spectral gap of a certain Hamiltonian. In paper [P8] we numerically investigate previously known results in this area, especially focusing on derived earlier bounds. For various ranges of parameters $n \leq 20$ and $t \leq 5$, we have proven that previously known lower bounds, in principle give a little information about the real situation, and only prove that the gap is closed. It means that there could be a large difference in actual values of the spectral gap and corresponding lower bound. Additionally, we compare our numerical results with other lower bounding techniques and conclude their lack of tightness or even show that they can be inconclusive. Progress in numerical results was possible due to applications of the Schur-Weyl duality and representation theory for the group algebra of the symmetric group $S(n)$. Our methods combining numerics with an analytical approach for constructing irreducible bases can be useful in studying many-body systems in general.

3. **Transformations between equivalent irreducible representations** It is a well-established fact that group representation theory is a powerful tool in physics, allowing for exploiting the underlying symmetries in the system under considerations to simplify its description, finding corresponding conservation laws, and make it more elegant. From the point of view of computing certain values describing a given system, restricting calculations to every irreducible block reduces the complexity of computations, and very often allows even for analytical description. However, to perform calculations one needs to know matrix representations of the group elements on every nonequivalent component. It can happen that one representation is more useful for a certain task than another one, and there is a need to translate our description into a more friendly for us form. In paper [P9] we give an algorithm on how to construct a unitary transformation between equivalent irreducible representations for a finite group G. We show that it is possible to construct a unitary conjugation between sets of equivalent irreducible representations, using very particular properties of irreducible representations of finite groups. In particular, by using the orthogonality relations for irreps, which is something different than previously known approaches based on solving a set of linear equations. We present examples of how the proposed method works in practice for the permutation group $S(n)$. In particular, we analyze the similarity transforms for a class of equivalent pairs of permutation group irreps showing that the resulting transformation matrix has an anti-diagonal form by using the Young-Yamanouchi representation. We also analyze some generalizations of the celebrated orthogonality relations for irreducible representations of finite groups.

4. **Integrability of classical Hamiltonian systems** Many problems of classical mechanics is described by a set of linear differential equations. For example, in the Hamiltonian approach, for a system of n degrees of freedom, its evolution is described by a set of $2n$ differential equations. A natural question is to ask whether we can find an analytical solution for a given set of equations. In fact, we ask then about the integrability property of a system under consideration. In paper [P10] we study the integrability in the Liouville sense of natural Hamiltonian systems with a homogeneous rational potential. Such potentials describe a wide range of physical systems like cosmological models with a conformal scalar field, generalized Hélion-Heiles potentials describing the flat motion of stars in galaxies, Yang-Mills system with $SU(2)$ gauge fields, and many others - this makes our restriction strongly physically motivated. In the case of two degrees of freedom, we show universal relation between the eigenvalues of Hessian of the potential evaluated at all so-called proper Darboux points. It turns out, this property shows that considered potentials satisfy a finite number of necessary conditions for their integrability coming from the differential Galois group of variational equations along certain particular solutions. This gives us a tool for the classification and characterization of such potentials. In particular,

we have constructed a number of examples of potentials satisfying necessary conditions and proved their integrability by deriving corresponding functionally independent first integrals of motion.

7.4 Additional track record after PhD

1. **Limitations for private randomness repeaters** Most of the cryptography protocols rely on two main ingredients – private randomness and private key. The distributed secure key, which is a correlated string of bits, between distant and honest parties ensures the security of communication which is crucial for the safety of the quantum internet. One way of safe distribution of the secure key among parties in a quantum network is the formalism of the quantum repeaters by exploiting maximally entangled states and entanglement swapping, called the paradigm of network key swapping. This formalism has been developed for the most general scenario involving private states. In paper [P11] we focus on the network properties of the private randomness - complementary to the private key resource. Our first step was to show that any private state is a particular case of a so-called independent state, i.e. state containing perfect, directly accessible, private randomness. Using this connection we were able to prove an upper bound on the rate of repeated randomness in a similar scenario as it was for the private states. It is equal to twice the relative entropy with respect to the maximally mixed state and holds for states with positive partial transposition (PPT). This bound provides a fundamental limit on the possibility of transferring privacy among the considered network. We show the usefulness of the derived bound by applying it to separable Werner states and showing a gap between the localizable and the repeated private randomness for sufficiently large dimensions. Additionally, in a restricted scenario, we were able to derive an analogous bound for arbitrary states, not necessarily PPT ones.

2. **Leakage of private data and connection with markovianity** All proofs of the security of quantum key distribution do not take into account the everyday environment in which communicated parties have to deal with. It means their usefulness is only theoretical since we neglect the imperfections in the production of the quantum key distribution (QKD) equipment and the active attacks of the eavesdropper known as Trojan Horse Attacks (THAs). The latter leads to the leakage of the secret key and it is recently been extensively studied. In paper [P12] we consider the scenario of THA where an eavesdropper has access to the raw key of the honest parties' device. The main result contains lower bounds on the leakage of private randomness and private key. In particular, we have shown that the private randomness in distributed setting can not drop down by more than $S(a) + \log_2 |a|$, where S is the von Neumann entropy, and a system that has been leaked out to eavesdropper. In the considered scenario we also focus on so-called irreducible private states and prove that their two-way distillable key is non-lockable. This is the first result of such kind for the non-trivial class of mixed states. Additionally, we observed that for the private states under consideration the key drop is independent on the size of the shield part. It means in particular, that a larger shield does not protect the key more effectively. Finally, we proved the connection between the (non)markovianity of quantum dynamics and hacking. Namely, we show that an invertible map is non-CP-divisible if and only if there exists a state whose key witnessed by a particular privacy witness increases in time.

3. **Distilling secure key from reducible private states** As we pointed out earlier obtaining a secret key for secure communication/data encryption from quantum states is one of the most fundamental achievements of quantum information science. This was demonstrated firstly by exploiting maximally entangled states. However, the approach of extracting a secret key can be extended to private states which is a class of states containing maximally entangled ones. Nowadays, we have a rich theory describing the quantitative relation

between secrecy and this class of states but the theory is still developed. In paper [P13] we formulated the first protocol which distills the secret via measuring the shield part of reducible private states – we are not restricted only to the key part as it was in all previously known protocols. We provided an upper bound on its performance in terms of regularised relative entropy of entanglement. This bound outperforms bounds relying on the relative entropy of entanglement used earlier in the literature. Using obtained bound we have made a connection between the set of irreducible private states/strictly irreducible private states and the existence of entangled key-undistilable states (bound key states). Namely, if the latter states exist then the sets of irreducible and strictly irreducible private states are equal. The reverse reasoning also holds, giving us if and only if conditions. Assuming the bound key states exist, we consider several properties of the irreducible private states. In particular, we derived a lower bound on the trace norm distance between bound key states and private states that works in sufficiently large dimensions.

4. **Theory of positive maps and entanglement witnesses** As we pointed out earlier one way of deciding whether a given $d \otimes d$ quantum state ρ is entangled/separable is to use the notion of an entanglement witness. A non-positive operator W is an entanglement witness if its expectation value with all separable states is non-negative. This makes the approach of studying entanglement by entanglement witnesses very general, but technically very complex. To check whether a given operator W is an entanglement witness we have to find a positive, but not completely positive linear map Λ for which W is its Choi-Jamiołkowski image. Due to the lack of characterization of the positive cone such searching is in principle hard and other methods are required. In paper [P1] we deliver one additional method for entanglement witnesses construction. Namely, we prove that it is enough to use non-positive maps, but one has to restrict consideration to the subdomain when the considered map is positive. In fact, a map must be a surjection between set of rank $k \leq d$ projectors and a set of rank-one projectors. A map satisfying these demands is considered by us the inverse reduction map R^{-1} for which we have presented the explicit construction of the new class of entanglement witnesses which can be treated as a generalization of the Choi entanglement witness.

5. **Studies on quantum error correction codes** In the last years, we could observe the interest in studying the connection between quantum error correcting codes and a powerful connection between theories of quantum gravity and field theory known as AdS/CFT correspondence. This relates gravitational theories in the interior (bulk) of anti-de Sitter (AdS) spaces to conformal field theories (CFT) defined on the boundary of the space. Recently, the connection between quantum information science and a special toy model of the AdS/CFT correspondence has been proposed. In paper [P14] we investigate this connection and explore the relation between AME states and quantum error correcting codes and apply our findings to improve our understanding of the AdS/CFT correspondence. Namely, we consider the perfect tensor of the network as AME states, and we study the spreading of the encoded quantum information in the network. Since we can treat it as a quantum code, we can use tools like code concatenation and stabilizer formalism. In the first step, we derive the form of logical operators and stabilizer generators for every code emerging from $n + m$ qudit stabilizer AME state, encoding n into m qudits with $n \leq m$. Our calculations are valid for every local dimension which is a prime number, which follows from previously known properties of certain graph states. We find that the upper bound of the entanglement entropy of the boundary state is saturated for AME input states for every connected bipartition of the boundary, and we postulate its optimality. In fact, for the AME states we show that our bound correspond with the celebrated Ryu-Takayanagi formula that connects the entanglement entropy of the conformal field with the geometry of associated AdS space.

6. **Studies on quantum second laws of thermodynamics – information-theory approach** In the era of miniaturization and developing new and more precise experimental methods of constructing nanoscale devices, we must ask what are limitations restricting their functioning. In particular, we can focus on thermodynamics machines, like nano- or even quantum-engines or refrigerators, and ask about their functioning and possibly find unexpected capabilities compared to their classical counterparts. This an important theoretical goal that touches thermodynamical fundamental limits of Nature – a set of thermodynamic laws that apply on the quantum scale. In paper [P15] we have made a fundamental step in this direction. For the first time, we focus on the fully quantum regime – where we consider evolution of quantum states with coherences when the experimenter has access to a heat bath at temperature T, and perform unitary operation commuting with the total Hamiltonian (thermal operation). We ask what are the limitations on the evolution of the off-diagonal elements of a density operator between distinct energy levels under thermal operations. These are in fact genuine second laws of thermodynamics for off-diagonal elements. We show that there exist two families of bounds, one for diagonal elements of states (thermo–majorization) and the second one for coherences. By considering the so-called damping matrix we derived an upper bound on the coherences after transformation in terms of the initial coherences and probability of transitions between the energy levels. In particular, we show that obtained bounds do not mix the coherences among each other, and they do not mix with bounds on the diagonal elements. In the qubit case, we proved that obtained bounds are tight by constructing a general thermal operation saturating them. However, for larger dimensions, we constructed an explicit three-level quasi-cycle for which the bounds cannot be saturated from its construction. This motivated us to introduce and study a larger class of thermal operations called by us enhanced thermal operations.

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