EPR Assemblages as Common-Cause Resources of Nonclassicality

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Abstract

The correlations generated by measuring quantum systems challenge classical paradigms and drive research in quantum information theory to identify and quantify nonclassical properties across various resources. This dissertation focuses on resources generated in Einstein-Podolsky-Rosen (EPR) scenarios and their generalizations. In a standard bipartite EPR scenario, nonclassical correlations are produced through local measurements on half of a system prepared in an entangled state. The nonclassical resource that arises in this scenario, called an *assemblage*, is an example of a common-cause resource where the correlations between parties arise through the shared system (without communication). In this dissertation, we study the nonclassicality of EPR assemblages through a series of three research papers [1, 2, 3].

In the first paper [1], we develop a resource theory of assemblages with local operations and shared randomness as the set of free operations. Within this framework, we investigate conversions between resources using semidefinite programming and resource monotones. The second paper [2] focuses on generalizations of the bipartite EPR scenario involving post-quantum resources. We introduce a resource theory for this case and analyze the pre-order among quantum and post-quantum resources both analytically and numerically. Finally, in the third paper [3], we examine the relationship between post-quantumness in EPR scenarios and Bell scenarios. We propose a protocol that transforms post-quantum EPR assemblages into a multipartite Bell-type scenario, resulting in post-quantum device-independent correlations.

Streszczenie

Korelacje wytwarzane przez pomiary systemów kwantowych odbiegają od zasad fizyki klasycznej, co motywuje badania w teorii informacji kwantowej do zidentyfikowania i ilościowego określenia nieklasycznych własności różnych zasobów. Głównym obiektem badań niniejszej rozprawy są zasoby wytwarzane w scenariuszach Einsteina-Podolsky'ego-Rosena (EPR) i ich uogólnieniach. W standardowym scenariuszu dwupodukładowym EPR, nieklasyczne korelacje powstają wskutek lokalnych pomiarów jednego z podukładów, podczas gdy stan łączny jest stanem splątanym. Nieklasyczny zasób wytworzony w ten sposób, nazywany *asamblażem*, stanowi przykład korelacji wywołanej przez wspólną przyczynę, a nie bezpośrednie relacje przyczynowe między podukładami. Wyniki zawarte w niniejszej rozprawie opierają się na serii trzech artykułów naukowych [1, 2, 3] przedstawiających badania nieklasyczności asamblaży EPR.

Pierwszy artykuł [1] przedstawia teorię zasobów dla asamblaży, w której zbiór operacji darmowych stanowią operacje lokalne skorelowane przez dzieloną zmienną losową. W formaliźmie tym zbadane zostały przekształcenia między zasobami za pomocą programowania półokreślonego oraz monotonów zasobów. Drugi artykuł [2] skupia się na uogólnieniach dwupodukładowego scenariusza EPR, w których występują zasoby postkwantowe. Artykuł przedstawia teorię zasobów dla tych scenariuszy oraz analityczną i numeryczną analize praporządku zasobów kwantowych i postkwantowych. W trzecim artykule [3] zbadana została zależność między postkwantowością w scenariuszach EPR i scenariuszach Bell'a. Artykuł przedstawia procedure przekształcania postkwantowych asamblaży EPR w wieloukładowe scenariusze typu Bell'a, w których wytwarzane są korelacje postkwantowe.

Publications included in the dissertation

[1] Beata Zjawin, David Schmid, Matty J Hoban, and Ana Belén Sainz. “Quantifying EPR: The Resource Theory of Nonclassicality of Common-Cause Assemblages", Quantum 7, 926 (2023).

[2] Beata Zjawin, David Schmid, Matty J Hoban, and Ana Belén Sainz. “The Resource Theory of Nonclassicality of Channel Assemblages", Quantum 7, 1134 (2023).

[3] Beata Zjawin, Matty J Hoban, Paul Skrzypczyk and Ana Belén Sainz. “Activation of post-quantumness in bipartite generalised EPR scenarios", arXiv:2406.10697.

Contents

1 Introduction

One of the most fundamental questions in quantum theory pertains to identifying and quantifying resources that provide evidence of nonclassicality. Bell's theorem [4] famously highlights the need to move beyond classical paradigms when interpreting experiments involving quantum systems. This theorem has led to the development of a research field focused on exploring correlations that cannot be generated in networks consisting solely of classical systems. An effective approach for investigating nonclassical correlations, including those beyond quantum theory, is through the framework of resource theories [5, 6]. This framework focuses on studying resources relative to a restricted set of operations that are considered to be freely available. The underlying principle of resource theories is very intuitive: a resource holds greater value if it is not easily attainable. In the context of studying nonclassical correlations, the resource of interest is the nonclassicality itself. Correlations achievable using classical systems are considered invaluable within this framework.

There are different types of correlations that can arise in networks involving physical systems. Imagine two distant parties that share a common resource and perform space-like separated actions on it. We will refer to this scenario as a common-cause scenario. As the parties can be correlated through the shared resource, the results of their manipulations may exhibit some correlation. For example, if the shared resource is a classical source of randomness, and the two parties simply output the number generated by it, their outputs will be perfectly correlated. A more complex form of correlation is entanglement. When the parties share a quantum system prepared in an entangled state, they exhibit nonclassical correlations. If one party measures their share of the system, the conditional state of the subsystem of the other party after the measurement will be correlated with the measurement outcome. This correlation, which we refer to as the *Einstein-Podolsky-Rosen (EPR) correlation*, is the central focus of this dissertation. One can go a step further by having both parties measure their subsystems, which results in the

Bell scenario, where the correlation of interest is the set of conditional probability distributions describing the observed outcomes given the measurement settings. All of these considerations can be extended to cases where the shared system is not quantum but described by a more general theory. In all cases, the object of interest is some form of correlation between the two parties induced by the common cause.

To study the nonclassicality of different forms of correlations in a resource-theoretic framework, one needs to first define what are the free operations of the theory. The set of freely available operations in a theory reflects the physical constraints in the studied scenario. If the resource of interest is the nonclassicality of the correlations, the physical structure of the common-cause scenario imposes that the free operations should consist of classical common-causes (i.e., shared randomness) and arbitrary local operations. Then, the only correlations that can be generated between the parties with these free operations are the classical ones, which are deemed invaluable in this framework. Although local operations and shared randomness (LOSR) are the set of operations that best reflect the structure of the common-cause scenarios, it was only recently that they gained attention in the field of quantum foundations. This is because there is no unique choice of free operations in a resource theory, and previous research has focused on resource theories that reflect the experimental capabilities of resource manipulation rather than the fundamental causal structure of the correlations. Currently, there are several developed resource theories that investigate the transformations of quantum and post-quantum resources relative to LOSR operations [1, 2, 7, 8, 9, 10].

A general resource theory of quantum common-cause processes was introduced in Refs. [7, 8]. In these works, resources are categorized based on the type of input and output systems of the local operations. The authors consider bipartite common-cause scenarios where both parties can locally manipulate their subsystems using processes that have either trivial, classical, or quantum inputs and outputs. Refs. [7, 8] focus on important resources studied in quantum information theory like entanglement [11], EPR scenarios [12, 13, 14] or Bell scenarios [4, 15, 16]. The authors study the nonclassicality of these resources relative to LOSR transformations. They introduce type-independent methods to witnesses and quantify nonclassicality. Moreover, they adopt a resource-theoretic approach to traditional studies of quantum resources by reframing quantum games within this framework. This is feasible due to the extensive and versatile nature of the resource theory, which enables transformations between different types of resources under free operations.

A resource theory focused only on Bell scenarios was introduced in Ref. [9]. The authors are guided by a causal modeling perspective on this resource [17], which departs from some traditional approaches and leads to LOSR as the natural set of free operations. The framework of Ref. [9] goes beyond quantum theory and considers all common-cause boxes, i.e., probabilistic processes from the classical setting variables to the classical outcome variables, where the parties have access to a common cause of arbitrary type (it can be classical, quantum, or even post-quantum). Focusing on this single type of resource, rather than the whole type-independent resource theory like in Refs. [7, 8], the authors derive multiple technical results specific to the Bell scenario. Ref. [9] includes an efficient algorithm for determining resource conversion, requiring two instances of a linear program. Moreover, the authors introduce new resource quantifiers and analyze the relative ordering of resources, with specific results derived for the subset of quantum resources.

Given that Bell scenarios are well-captured by a resource theory with LOSR as the set of free operations, a detailed study of entanglement relative to LOSR would allow one to investigate the interplay between these two resources. This is exactly the intention of Ref. [10], where LOSR-entanglement is studied. Capturing entanglement and boxes with the same set of free operations allows one to study state-to-box conversions. The authors of Ref. [10] address some common concerns about the interplay of entanglement and Bell-nonclassicality (so-called anomalies of nonlocality) and derive fundamental results regarding the pre-order of states under LOSR operations. In particular, they provide necessary and sufficient conditions for convertibility between pure entangled states.

One piece that was missing in the family of resource theories with LOSR as the set of free operations is a detailed resource theory of EPR scenarios. Although EPR correlations are formally captured by the type-independent framework of Refs. [7, 8], the work of Refs. [9, 10] have proven that the investigation of a single resource in this framework can give valuable insights into the relative nonclassicality of the specific correlations. The main goal of this dissertation is to study the consequences of framing the EPR scenario as a common-cause resource. This task naturally leads to defining a resource theory of EPR scenarios and generalizations thereof, with the set of free operations being LOSR. Moreover, it prompts the question of how EPR scenarios interplay with correlations generated in Bell-like set-ups, which was previously studied only for the multipartite EPR scenario [18]. We address these problems in three research papers that constitute the main body of this dissertation. Two of the papers are published in a peer-reviewed journal [1, 2] and one is published in the arXiv repository [3]. Our main objects of study are the resources associated to the EPR scenario, called assemblages.

The first paper, *Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages* [1], is focused on quantum EPR correlations in bipartite and multipartite scenarios. In this paper, we look at the EPR scenario through the lens of causal modeling and develop a new perspective on this resource. This leads us to define a resource theory of assemblages under LOSR and explore the conceptual and technical consequences of this approach. We show that resource conversion under LOSR operations can be evaluated with a single instance of a semidefinite program, making the problem numerically tractable. Moreover, we derive new quantifiers of nonclassicality and use them to study the relative nonclassicality of assemblages.

In the second paper, *The resource theory of nonclassicality of channel assemblages* [2], we focus on bipartite generalizations of the EPR scenario, that include post-quantum resources. This is the first study that analyzes Bob-with-input, measurement-device-independent, and channel EPR scenarios as common-cause resources. Introducing a resource theory of generalized assemblages leads us to derive semi-definite programs for assessing resource conversions relative to LOSR operations. Moreover, studying post-quantum resources with this set of free operations allows us to highlight the power of quantum entanglement in post-quantum theories by identifying a conversion between post-quantum resources that is impossible with LOSR, but is possible when parties share an entangled state.

The third paper, *Activation of post-quantumness in bipartite generalized EPR scenarios* [3], addresses the question of the interplay between assemblages and Bell-type correlations. In this paper, we introduce a protocol for certification of post-quantumness in EPR scenarios by embedding them in a larger network that generates Bell-like correlations. For a specific class of post-quantum assemblages that do not generate post-quantum correlations when converted to bipartite boxes, this protocol serves as a procedure for post-quantumness activation. We introduce and analyze protocols for generalized bipartite EPR scenarios, including Bob-with-input, measurement-device-independent, and channel EPR scenarios.

This dissertation consists of three research papers (two of which are published in a peer-reviewed journal), accompanied by an introduction and a discussion. It is structured as follows:

In Section 2, we introduce all the ideas and definitions needed to understand the three papers that constitute the main body of the dissertation. Section 2.1 introduces and motivates the framework of resource theories. Then, we briefly discuss some examples of previously studied resource theories. In Section 2.1.1 we focus on entanglement, arguably the most studied resource in quantum information theory, and in Section 2.1.2 we further motivate the studies of common-cause resources under local operations and shared randomness. This leads us to find a gap in the studies of this topic, namely a lack of a detailed resource theory of the EPR scenario, which we properly introduce in Section 2.2.1. Then, we discuss how it can be generalized in Section 2.2.2. Finally, one last technical topic that is crucial in this dissertation is semi-definite programming, which we introduce in Section 2.3. Then, we summarize the papers that constitute this dissertation. Section 3.1 is focused on the resource theory of the standard EPR scenarios (bipartite and multipartite) [1]. The resource theory of generalizations of the EPR scenario [2] is summarized in Section 3.2. Finally, in Section 3.3, we introduce the activation protocol [3]. Section 4 presents the conclusions of the dissertation.

2 Preliminaries

2.1 Resource theories

Resource theories constitute a comprehensive framework for exploring properties of physical systems in quantum theory and beyond [5, 6]. The fundamental idea of this formalism relies on classifying all possible operations on the resources of the theory into two sets: free and non-free. Then, resources are deemed valuable if they cannot be 'easily generated'. To define a resource theory, one needs to identify the *enveloping theory*, which encompasses all possible resources and operations in the considered setup, and the *free subtheory*, consisting of operations on the resources that are regarded as freely available and can generate the free resources[1]. The set of free operations partitions all resources into free and resourceful ones. One of the main problems studied within the framework of resource theories is establishing the relative value of pairs of resources. If a resource R_1 can be converted to a resource R_2 with free operations, we denote it by $R_1 \longrightarrow R_2$. If the conversion is not possible, one writes $R_1 \not\longrightarrow R_2$. Given a pair of resources R_1 and R_2, one of the following four possibilities must hold

$$\begin{aligned} R_1 \text{ is } \textit{strictly above } R_2 \text{ if } R_1 \longrightarrow R_2 \text{ and } R_2 \not\longrightarrow R_1, \\ R_1 \text{ is } \textit{strictly below } R_2 \text{ if } R_2 \longrightarrow R_1 \text{ and } R_1 \not\longrightarrow R_2, \\ R_1 \text{ is } \textit{equivalent} \text{ to } R_2 \text{ if } R_1 \longrightarrow R_2 \text{ and } R_2 \longrightarrow R_1, \\ R_1 \text{ is } \textit{incomparable} \text{ to } R_2 \text{ if } R_1 \not\longrightarrow R_2 \text{ and } R_2 \not\longrightarrow R_1. \end{aligned}$$

[1]The choice of the set of free operations for a given enveloping theory is not unique. Sometimes, it is reasonable to define the set of free operations based on the experimental capabilities, i.e., the ways in which it is physically possible to manipulate the resources in a laboratory. A different, more fundamental motivation is to choose the set of free operations solely based on the physical structure of the studied resource. In this dissertation, we endorse the latter approach and study resources from the perspective dictated by their causal structure.

These relations provide information about the value of a resource. If $R_1 \longrightarrow R_2$, then R_1 can be used for the same tasks as R_2; hence, R_1 is at least as valuable as R_2 in the resource theory. In this dissertation, we focus on single-copy deterministic conversions. However, the framework of resource theories is well suited to study other types of conversions, such as single-copy stochastic conversion (where the conversion must not happen with probability 1) or multi-copy conversion (where more than one copy of the resource is available).

Determining whether conversions under free operations are possible for any pair of resources determines the pre-order over resources in the theory. A pre-order is a transitive and reflexive binary relation between resources, i.e., it requires that every resource can be mapped to itself $R \longrightarrow R$ and if $R_1 \longrightarrow R_2$ and $R_2 \longrightarrow R_3$, then $R_1 \longrightarrow R_3$. Equivalent resources in the theory form equivalence classes. One can also consider conversion relations between these equivalence classes, which are described by a partial order (the conversion relations between equivalence classes are antisymmetric).

There are two primary approaches to testing if a conversion is possible. The first approach relies on designing an algorithm that, when provided with two resources, simply verifies whether one can be converted into the other. The complexity of this algorithm depends on the resources under study and the set of free operations of the theory. In some cases, the algorithm may be fast and straightforward. There also exist resource theories in which formalizing the set of free operations is notably complex, making it impossible to design an effective general algorithm for this purpose. The second approach involves employing resource monotones. Formally, a resource monotone is an order-preserving map from the pre-order of resources to the total order of real numbers. The monotone function is non-increasing under conversions with free operations. Let $M(R)$ denote the real number that the monotone M assigns to a resource R. If for a pair of resources R_1 and R_2 the following relation holds: $M(R_1) < M(R_2)$, then the monotone M witnesses that $R_1 \not\longrightarrow R_2$. Typically, a single resource monotone provides only partial information about the relative order of resources. A collection of resource monotones is often required to fully characterize the pre-order. Nonetheless, resource monotones serve as useful tools for studying the resources. For example, the real number assigned to a resource by the monotone can be viewed as a quantification of its resourcefulness.

In addition to analyzing individual resources, it's valuable to study the composition of them. An interesting research direction in the framework of resource theories is resource activation. In the context of quantum resource theories, this concept was first studied concerning states and Bell nonclassicality. In Ref. [19], it was shown that there exist bipartite quantum states ρ and σ such that neither of them can be converted into a nonclassical box, however, the joint state $\rho \otimes \sigma$ can. This phenomenon is observed across various resource theories.

2.1.1 Resource theory of entanglement

The framework of resource theories is often used to study the properties of quantum systems. There exist multiple quantum resource theories, including those for coherence [20, 21, 22], quantum reference frames and asymmetry [23], athermality [24, 25, 26, 27, 28, 29], and so on. One of the most celebrated quantum resource theory is that of entanglement relative to local operations and classical communication (LOCC) [11, 30, 31, 32]. The set of LOCC operations was introduced to understand entanglement as a resource for information processing. In information processing tasks (e.g., teleportation protocol), it's common for the parties to have access to classical communication channels. Therefore, it's reasonable to consider LOCC as the free operations of the theory, as the parties can, for example, communicate the classical outcomes of their local measurements to each other. The most general states that can be generated with LOCC operations alone, i.e., the free states of the theory, are separable states. It is worth noticing that the popularity of LOCC in entanglement studies motivated similar sets of operations to be considered for different resources, one example being a resource theory of 'steering' under local operations and one-way classical communication (1W-LOCC) [33].

The implications of the resource theory of entanglement under LOCC operations remain an active area of research. Characterizing the set of LOCC operations is challenging, making it difficult to study resource conversion in this framework. For example, there is currently no known method for determining whether a given map belongs to LOCC (it is only known what are the necessary conditions) [6]. Another topic of active discussion is the number of classical communication rounds needed in protocols [34, 35, 36]. Due to the technical difficulties inherent in the characterization of LOCC, significant research efforts have been dedicated to investigating entanglement theory within a broader set of free operations that are easier to characterize. One example is the set of separable operations (SEP) [37], which LOCC is a subset of. The set of free states is the same in resource theories under SEP and LOCC. Considering a broader set of free operations can be valuable in proving no-go theorems, as any task that surpasses the capabilities of the resource theory with a broader set of free operations will also exceed the capabilities of the weaker one. While theories studying larger sets of free operations are beneficial for practical applications, it's important to note that the foundational significance of the chosen set of free operations is relatively limited.

In this dissertation, we advocate for defining free operations in a resource theory based on the fundamental structure of the resource. In the case of entanglement, the physical structure of the scenario (two parties that share a common system) makes the natural choice of free operations to be LOSR [10], which is a subset of LOCC. The set of LOSR operations proved to be important when considering state-to-box conversions [38], but it was only recently that a formal resource theory of entanglement under LOSR was developed [10]. In Ref. [10], a detailed comparison of the resource theories of entanglement under LOCC and LOSR is presented. One important feature of the resource theory of entanglement relative to LOSR is that it is a single instance of a larger resource theory of common-cause scenarios, since LOSR is a valid and well-motivated set of free operations for a larger class of resources (quantum and post-quantum), including Bell nonclassicality, EPR scenarios, and more.

2.1.2 Resource theory of common-cause processes

To understand the similarity between different types of common-cause resources, it is useful to introduce graphical notation [39]. Let single and double lines correspond to classical and quantum systems, respectively. Thick lines represent systems that could be classical, quantum, or even post-quantum. Let boxes represent local processes. Then, a box with two output quantum wires illustrated in Fig. 2.1(a) represents a bipartite quantum state. The two wires represent the physical systems of Alice and Bob, each of which is in a different, distant laboratory. The Bell scenario is represented in Fig. 2.1(b), where Alice and Bob each measure their quantum systems. Single-lined inputs and outputs of their local boxes represent the classical setting of the measurement and the classical outcome. Boxes can also have quantum inputs or outputs. For example, in Fig. 2.1(d), Bob has access to a quantum channel – a box with a quantum input and a quantum output that represents a completely positive and trace preserving (CPTP) map [40, 41]. A common-cause resource is one where distant parties share a common-cause (which can be classical, quantum, or even post-quantum) and perform local operations on it. The only restriction on the resource is the no-signaling principle, i.e., the parties cannot communicate with each other via local operations on the shared system. The following examples of common-cause

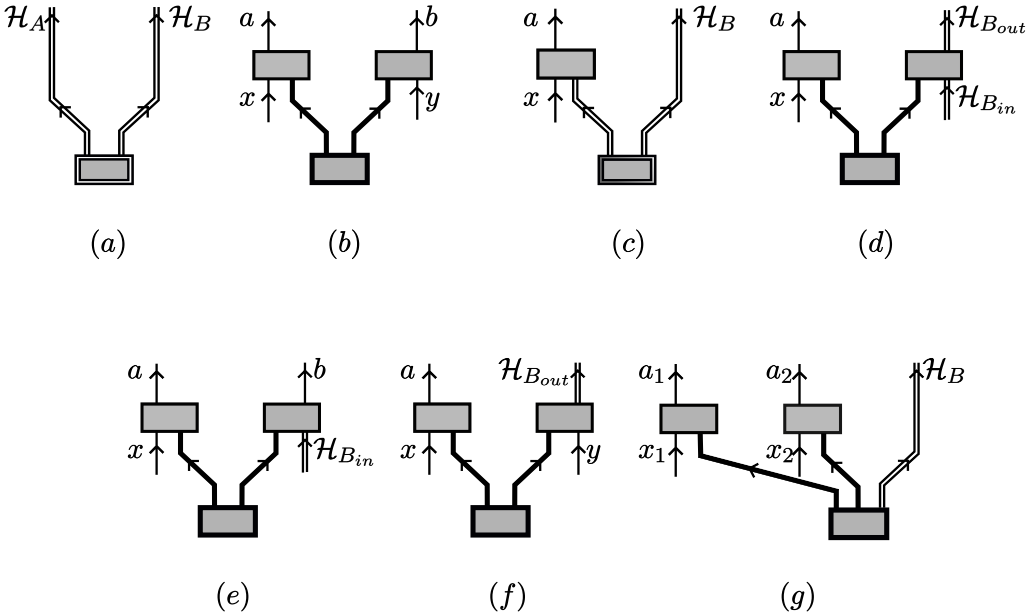

Figure 2.1: **Examples of common-cause resources.** (a) Quantum state. (b) Bell scenario. (c) Standard EPR scenario (bipartite). (d) Channel EPR scenario. (e) Measurement-device-independent EPR scenario. (f) Bob-with-input EPR scenario. (g) Standard EPR scenario (multipartite). Quantum and classical systems are depicted by double and single lines, respectively. Thick black lines depict systems that could be post-quantum.

resources are illustrated in Fig. 2.1:

(a) Quantum state [10, 11]: a bipartite system prepared in a possibly entangled state.

(b) Bell scenario [9, 42]: two parties share a system and perform measurements on it.

(c) EPR scenario [1, 12, 13, 14]: two parties share a system, and Alice performs measurements on her share of the system.

(d) Channel EPR scenario [2, 43]: a bipartite EPR scenario where Bob has access to a quantum channel.

(e) Measurement-device-independent EPR scenario [2, 44]: a bipartite EPR scenario where Bob has a measurement channel with a quantum input and a classical output.

(f) Bob-with-input EPR scenario [2, 45]: a bipartite EPR scenario, where Bob has a classical input that allows him to locally influence the state preparation of his quantum output system.

(*g*) Multipartite EPR scenario [1, 46]: three parties share a multipartite system and the Alices perform measurements on their shares of the system.

The nonclassicality (relative to LOSR operations) of all these resources depends on the type of common cause that Alice and Bob share. If the common cause is classical, the resources are free. This definition of classicality coincides with the well-known definitions of separable states, 'unsteerable' assemblages, local boxes, and so on for the states, the EPR scenario, the Bell scenario, and other common-cause resources. If the common cause is a quantum or a post-quantum system, the resource could be nonclassical. The overall goal of the resource theories of common-cause processes is to quantify the nonclassicality of resources.

Let us now specify the most general LOSR operations on a bipartite common-cause resource. The most general local operation that Alice (or Bob) can apply is given by a *comb* [47]. Additionally, Alice's and Bob's local combs can be correlated by classical shared randomness. A comb, illustrated in Fig. 2.2(a) for a general resource (in green), allows for pre- and post-processing of the local systems. In general, this operation can change the type of resource. The pre- and post-processing stages can be connected by a side channel of an arbitrary type; however, it is often the case that the type of the side channel can be simplified. The precise structure of the comb will differ depending on the enveloping theory of resources. For example, if the enveloping theory only consists of quantum states, the most general LOSR processing is given in Fig. 2.2(b). A different LOSR processing is illustrated in Fig. 2.2(c), which represents a box-to-box transformation.

One of the key advantages of LOSR resource theories is their universality – they can be formulated to study specific resources or incorporate all common-cause scenarios and conversions among them. Recently, a program of characterizing resource theories of common-cause processes has been very active. A general resource theory of quantum common-cause processes was introduced in Refs. [7, 8]. Resource theories for specific scenarios were introduced in Ref. [9] for the Bell scenario and in Ref. [10] for entanglement. In this dissertation, we focus on the nonclassicality of the EPR scenario, which we formally introduce in the next section.

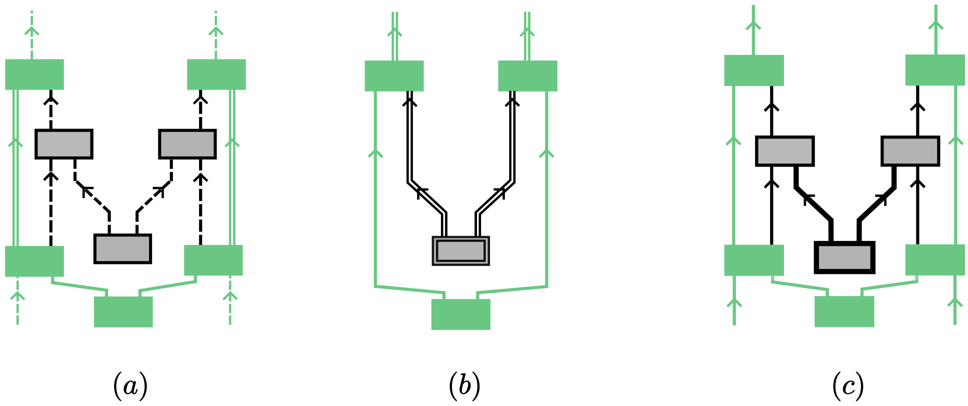

Figure 2.2: **LOSR transformations of different types of bipartite common-cause scenarios.** Resources (in black) are being transformed by local combs correlated by shared randomness (in green). (a) A generic LOSR transformation of a common-cause resource. Dashed lines represent arbitrary system types. (b) The most general LOSR transformation of a quantum state. (c) The most general LOSR transformation of a Bell scenario.

2.2 EPR scenarios

In this section, we introduce the central object of our study: the EPR scenario and various generalizations thereof. The technical and conceptual details pertaining to these scenarios are described in more detail in Refs. [1, 2].

2.2.1 Standard EPR scenario

The **bipartite EPR scenario** is illustrated in Fig. 2.1(c). It consists of two distant parties, Alice and Bob, that share a physical system. Alice measures her share of the system: with probability $p(a|x)$ she observes an outcome a after selecting a measurement setting x. Bob, contrary to the Bell scenario, does not measure his subsystem, which is now described by a conditional marginal state $\rho_{a|x}$. This definition of an EPR scenario is a generalization of the original thought experiment considered by Einstein, Podolsky and Rosen [12, 13]. The correlations between Alice and Bob in this set-up are sometimes taken to be evidence for spooky action-at-a-distance, where Alice 'steers' Bob's state. Hence, this scenario is referred to as 'quantum steering' in the literature [14, 48]. In this dissertation, we embrace an alternative perspective on this phenomenon in which Alice has no direct causal influence on the physical state of Bob's system; rather, her knowledge about the state of Bob's system is updated upon measurements performed on a system correlated with his. For this reason, which is explained in more detail in Ref. [1], we use the term EPR scenario rather than EPR 'steering' throughout the dissertation.

Formally, the relevant object of study in the EPR scenario is an *assemblage* [49] defined as $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\sigma_{a|x}\}_{a,x}$, where each unnormalized state $\sigma_{a|x}$ is given by $\sigma_{a|x} := p(a|x)\rho_{a|x}$ [2]. Due to general relativity, a condition called no-signaling principle imposes that Bob's state is independent of Alice's choice of measurement x without the knowledge of her output a, i.e., $\sum_a \sigma_{a|x} = \rho_B$ for all values of x, where ρ_B is Bob's reduced state.

In quantum theory, the bipartite system shared between Alice and Bob can be described by a density matrix ρ_{AB} and Alice's measurements are given by positive operator-valued measures (POVMs), which we denote by $\{M_{a|x}\}_{a,x}$. Then, the unnormalized states admit a quantum realization of the form

$$\sigma_{a|x} = \mathrm{tr}_A\{(M_{a|x} \otimes \mathbb{I})\rho\}. \tag{2.1}$$

In the case of bipartite EPR scenarios, the GHJW theorem, proven by Gisin [50] and Hughston, Jozsa, and Wootters [51], shows that all possible assemblages that emerge from Alice and Bob sharing a common cause can be described in this form. In other words, it means that *all* bipartite common-cause assemblages admit a quantum realization. It is worth noticing that the assemblages generated in this scenario are by definition non-signaling. Since POVMs satisfy $\sum_a M_{a|x} = \mathbb{I}$, it is indeed the case that $\sum_a \sigma_{a|x} = \rho_B$.

A bipartite assemblage that can be generated using only classical resources can be written as

$$\sigma_{a|x} = \sum_{\lambda} p(\lambda)p(a|x,\lambda)\rho_{\lambda}, \tag{2.2}$$

with $p(a|x,\lambda)$ being valid conditional probability distributions for all values of λ. Here, ρ_λ is Bob's local state that is sampled according to $p(\lambda)$. From the causal perspective, these are the

[2] In this dissertation, we use the following simplified notation. Each EPR scenario is defined with respect to the set of possible classical inputs and outputs. In the standard EPR scenario, the sets to be specified are the set of possible measurement outputs $\mathbb{A}$ and the set of measurement settings $\mathbb{X}$. Here, we only consider measurements that have the same number of outcomes. Then, the assemblage is formally defined as an ensemble of ensembles, i.e., $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\{\sigma_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$. Instead of this, we write $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\sigma_{a|x}\}_{a,x}$. The same simplification of notation is used for the generalizations of the EPR scenario in the later sections of this dissertation.

assemblages generated by Alice and Bob through a classical common cause, which corresponds to the variable λ. Traditionally, these assemblages have been referred to as 'unsteerable' assemblages.

One possible generalization of the standard EPR scenario is to the **multipartite** set-up. Here we consider a case with $k+1$ separated parties, k Alices and one Bob. For the sake of simplicity, let us now focus on the case of $k=2$ (two Alices, one Bob). This EPR scenario is illustrated in Fig. 2.1(g). Each Alice chooses an input x_i and observes an outcome a_i with probability $p^i(a_i|x_i)$; here $i \in \{1,2\}$. As Alices perform measurements on their respective subsystems, Bob's subsystem is characterized by a conditional marginal state $\rho_{a_1a_2|x_1x_2}$. In this scenario, the assemblage is defined as $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2} = \{\sigma_{a_1a_2|x_1x_2}\}_{a_1,a_2,x_1,x_2}$ and its elements are given by $\sigma_{a_1a_2|x_1x_2} := p(a_1a_2|x_1x_2)\rho_{a_1a_2|x_1x_2}$. Similarly to the bipartite scenario, the no-signalling condition is given by $\sum_{a_1}\sigma_{a_1a_2|x_1x_2} = \sum_{a_1}\sigma_{a_1a_2|x_1'x_2}$ and $\sum_{a_2}\sigma_{a_1a_2|x_1x_2} = \sum_{a_2}\sigma_{a_1a_2|x_1x_2'}$. Bob's reduced state in this case is given by $\sum_{a_1,a_2}\sigma_{a_1a_2|x_1x_2} = \rho_B$.

If the physical state that the parties share is quantum, which we denote by $\rho_{A_1A_2B}$, the elements of the assemblage can be written as

$$\sigma_{a_1a_2|x_1x_2} = \text{tr}_{A_1A_2}\{(M_{a_1|x_1} \otimes M_{a_2|x_2} \otimes \mathbb{I})\rho_{A_1A_2B}\}. \quad (2.3)$$

The elements of the assemblages which can be generated classically in this set-up can be written as

$$\sigma_{a_1a_2|x_1x_2} = \sum_\lambda p(\lambda)p^1(a_1|x_1,\lambda)p^2(a_2|x_2,\lambda)\,\rho_\lambda, \quad (2.4)$$

where $p^i(a_i|x_i,\lambda)$ is a conditional probability distribution for the i-th Alice and ρ_λ is Bob's local state for every λ. An interesting feature of the multipartite EPR scenario is that there exist non-signaling assemblages that are not compatible with a quantum common-cause [52]. However, in this dissertation, we focus on a subset of multipartite assemblages that admit a quantum realization.

2.2.2 Bipartite generalizations of the EPR scenario

In the standard EPR scenario, Alice performs a measurement on her subsystem, and Bob does not perform any local operation. One possible generalization of the EPR scenario relies on allowing Bob to process his system in various ways. In this work, we focus of three different bipartite generalizations of the standard EPR scenario: the channel EPR scenario [2, 43], the Bob-with-input EPR scenario [2, 45] and the measurement device-independent EPR scenario [2, 44].

In the **channel EPR scenario**, Bob has access to a quantum channel, as illustrated in Fig. 2.1(d). It is possible that this channel is affected by subsystem B of the shared system AB. Then, Bob applies a process $\Gamma^{BB_{in}\to B_{out}}$ on the systems in his laboratory. Let $\mathcal{I}_{a|x}(\cdot)$ denote the instrument applied on Bob's quantum system B_{in} to produce a quantum system B_{out}, given that Alice has performed measurement labeled by x and obtained the outcome a. The central object of study is then the *channel assemblage* of instruments $\mathbf{I}_{\mathbb{A}|\mathbb{X}} = \{\mathcal{I}_{a|x}(\cdot)\}_{a,x}$. The no-signaling condition in the channel EPR scenario is given by

$$\text{tr}\left\{\mathcal{I}_{a|x}(\rho)\right\} = p(a|x) \quad \forall\, a, x, \rho\,, \tag{2.5}$$

$$\sum_a \mathcal{I}_{a|x}(\cdot) = \Lambda^{B_{in}\to B_{out}}(\cdot) \quad \forall\, x, \tag{2.6}$$

where ρ represents any normalized state of quantum system B_{in} and $\Lambda^{B_{in}\to B_{out}}(\cdot)$ is a quantum channel. If the system that Alice and Bob share is quantum and denoted by ρ_{AB}, the elements of the channel assemblage are the following:

$$\mathcal{I}_{a|x}(\cdot) = \text{tr}_A\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes\Gamma^{BB_{in}\to B_{out}})[\rho_{AB}\otimes(\cdot)_{B_{in}}]\right\}. \tag{2.7}$$

If the system that Alice and Bob share is classical, then the elements of the channel assemblage can be written as

$$\mathcal{I}_{a|x}(\cdot) = \sum_\lambda p(a|x,\lambda)p(\lambda)\mathcal{I}_\lambda(\cdot), \tag{2.8}$$

where $\mathcal{I}_\lambda$ is a CPTP map (from system B_{in} to B_{out}) for each λ, and $p(a|x,\lambda)$ is a valid conditional probability distributions for all values of λ.

Let us now introduce the **measurement-device-independent (MDI) EPR scenario** that is shown in Fig. 2.1(e). When Bob's input remains quantum but his output is classical, Bob's local process is a measurement channel. Then, Bob has access to a processing $\Theta_b^{BB_{in}\to B_{out}}$. The input of this processing is the input state B_{in} together with the half of the system AB Bob shares with Alice (denoted by B). The output of this process is a classical variable b which lives in the space B_{out}. Formally, the relevant *MDI assemblages* in this scenario are given by $\mathbf{N}_{\mathbb{AB}|\mathbb{X}} = \{\mathcal{N}_{ab|x}(\cdot)\}_{a,b,x}$. Here, $\mathcal{N}_{ab|x}(\cdot)$ is a channel corresponding to the POVM element with outcome b given that Alice observed an outcome a upon a setting x. The elements of a valid MDI assemblage must satisfy the no-signaling conditions

$$\sum_b \mathcal{N}_{ab|x}(\rho) = p(a|x) \quad \forall\, a, x, \rho\,, \tag{2.9}$$

$$\sum_a \mathcal{N}_{ab|x}(\cdot) = \Omega_b^{B_{in}\to B_{out}}(\cdot) \quad \forall\, b, x, \tag{2.10}$$

where $\{\Omega_b^{B_{in}\to B_{out}}(\cdot)\}_b$ is a collection of measurement channels and ρ represents any normalized state of quantum system B'. A quantumly-realisable MDI assemblages are specified by elements of the form

$$\mathcal{N}_{ab|x}(\cdot) = \mathrm{tr}\left\{(M_{a|x} \otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A \otimes \Theta_b^{BB_{in}\to B_{out}})[\rho_{AB} \otimes (\cdot)]\right\}. \tag{2.11}$$

When the common cause that Alice and Bob share is classical, the elements of the MDI assemblage are given by

$$\mathcal{N}_{ab|x}(\cdot) = \sum_\lambda p(\lambda) p(a|x,\lambda) \mathcal{N}_{b,\lambda}(\cdot), \tag{2.12}$$

where the measurement channels $\mathcal{N}_{b,\lambda}$ depend on the value of the classical common cause λ.

The **Bob-with-input EPR scenario** is illustrated in Fig. 2.1(f). In this scenario, Bob has a state preparation channel: he chooses the value of a classical variable y, referred to as 'Bob's input', which influences the state preparation of a quantum system in his laboratory. The *Bob-with-input assemblage* is given by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{XY}} = \{\sigma_{a|xy}\}_{a,x,y}$. To ensure that the assemblage is non-signaling, it must satisfy the following conditions:

$$\mathrm{tr}\left\{\sum_a \sigma_{a|xy}\right\} = 1 \quad \forall\, x, y\,, \tag{2.13}$$

$$\mathrm{tr}\left\{\sigma_{a|xy}\right\} = p(a|x) \quad \forall\, a, x, y\,, \tag{2.14}$$

$$\sum_a \sigma_{a|xy} = \sum_a \sigma_{a|x'y} \quad \forall\, x, y, x'. \tag{2.15}$$

If the common-cause Alice and Bob share is a quantum state ρ_{AB} and the most general local operation Bob's device can implement is a collection of CPTP maps $\{\xi_y\}_y$, the elements of the assemblage are the following:

$$\sigma_{a|xy} = \mathrm{tr}_A\{(M_{a|x} \otimes \xi_y)\rho_{AB}\}. \tag{2.16}$$

If the common cause shared by Alice and Bob is classical, the elements of the assemblage can be expressed as

$$\sigma_{a|xy} = \sum_\lambda p(\lambda)p(a|x\lambda)\rho_{\lambda,y}. \tag{2.17}$$

Here, λ is the shared classical variable that is sampled according to $p(\lambda)$, $p(a|x\lambda)$ is a well-defined conditional probability distribution for all values of λ, and the quantum states $\rho_{\lambda,y}$ are locally generated by Bob, depending on the values of the classical variables y and λ.

For the purpose of this work, it is important to notice that all the generalizations of the EPR scenario have a *common-cause* causal structure: Alice and Bob share a common cause and perform local operations on it. The differentiation among these scenarios is made by specifying the type of Bob's input and output systems. In general, we will not take the common cause AB to be a quantum system. This common cause may be a classical, quantum, or even post-quantum system.

2.3 Semidefinite programming

In this section, we give an overview of the basics of semidefinite programming (SDP). This knowledge is useful for understanding the programs we introduce in Refs. [1, 2]. We follow the presentation of Ref. [53], which is well suited for considering the applications of semidefinite programming in quantum information. Although here we focus on the simplest problems with a single optimization variable, all the problems presented in this section can be reformulated to accommodate multiple optimization variables and constraints.

Semidefinite programs belong to the the class of convex optimization problems. They have a nice characterization, useful properties, and can often be efficiently solved [54]. They also have multiple applications in quantum information [55], including studying the properties of EPR scenarios [53]. Let A and B be Hermitian operators, and $\Phi(\cdot)$ be a hermiticity-preserving linear map. The *primal problem* in semidefinite programming is formulated as follows:

$$\begin{aligned} \text{given} \quad & A, \Phi(\cdot), B \\ \max_{X} \quad & \text{tr}\{AX\} \\ \text{s.t.} \quad & \Phi(X) = B \\ & X \geq 0. \end{aligned} \tag{2.18}$$

The linear function $\text{tr}\{AX\}$ is referred to as the *primal objective function.* The maximum value of the primal objective function, which we denote by α, is referred to as the *primal optimal value.* The *dual problem* can be derived by considering the best possible upper bound on α. Let $\Phi^\dagger(\cdot)$ be the conjugate map that satisfies $\text{tr}\{\Phi(X)Y\} = \text{tr}\left\{X\Phi^\dagger(Y)\right\}$ with X and Y being arbitrary hermitian operators. The dual problem can be formulated as

$$\begin{aligned} \text{given} \quad & A, \Phi(\cdot), B \\ \min_{Y,Z} \quad & \text{tr}\{YB\} \\ \text{s.t.} \quad & Z = \Phi^\dagger(Y) - A \\ & Z \geq 0. \end{aligned} \tag{2.19}$$

Here, $\text{tr}\{YB\}$ is the *dual objective function.* The result of this minimization problem, which we denote by β, is the *dual optimal value.* The primal and dual problems are closely related to each other. It is always the case that β upper bounds α; this is known as *weak duality.* Moreover, it is often the case that the optimal values coincide, i.e., $\alpha = \beta$. This condition is called *strong duality.*

In Refs. [1, 2] we focus on a class of SDPs known as *feasibility problems*, in which the objective function vanishes. A feasibility problem can be written as

$$\begin{aligned} \text{given} \quad & \Phi(\cdot), B \\ \text{find} \quad & X \\ \text{s.t.} \quad & \Phi(X) = B \\ & X \geq 0. \end{aligned} \tag{2.20}$$

This program checks if the feasible set is equal to the empty set. If this is the case, which means that there exists no X that satisfies the relevant constraints, the optimal value is equal to $-\infty$.

If the feasible set is not equal to the empty set, the optimal value is equal to zero, and the problem is feasible. Any feasibility problem can be reformulated as a feasible one by relaxing the semidefinite constraint as follows:

$$
\begin{aligned}
\text{given} \quad & \Phi(\cdot), B \\
\max_{X} \quad & \mu \\
\text{s.t.} \quad & \Phi(X) = B \\
& X \geq \mu \mathbb{I}.
\end{aligned}
\tag{2.21}
$$

In this case, a non-negative optimal value implies that the problem was feasible. If the optimal value is negative, the problem was infeasible.

3 Summary of the dissertation

3.1 Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages

In Ref. [1], we develop a resource theory of nonclassicality of standard bipartite and multipartite EPR scenarios. We take the enveloping theory to be quantum resources, i.e., we only focus on assemblages that admit a quantum realization. The main motivation behind this resource theory is a causal approach to the correlations generated in EPR scenarios. Considering assemblages as common-cause resources implies that the free operations that best reflect the physical structure of the scenario are local operations and shared randomness. Therefore, we derive the most general LOSR operation on a standard EPR assemblage and study the nonclassicality of resources relative to these free operations. In this summary of Ref. [1], our primary focus is on bipartite scenarios. The results regarding the multipartite case are based on very similar techniques; hence, we will not recall all the technical details here.

3.1.1 The resource theory

The enveloping theory of this resource theory consists of all bipartite assemblages generated in standard EPR scenarios. The free operations are LOSR. Consequently, the free subtheory consists of assemblages that can be generated with LOSR operations, as illustrated in Fig. 3.1(a). In this figure, Alice's outcome a depends on the measurement setting x and the value of the common cause λ. Bob's quantum state ρ_λ is generated locally depending on the value of λ. Free assemblages admit a decomposition of the form specified in Eq. (2.2), as they correspond to the set of 'unsteerable' assemblages.

The most general LOSR transformation of a standard assemblage consists of a local comb on Alice's side and a local comb on Bob's side [47], both correlated by shared randomness. Alice's

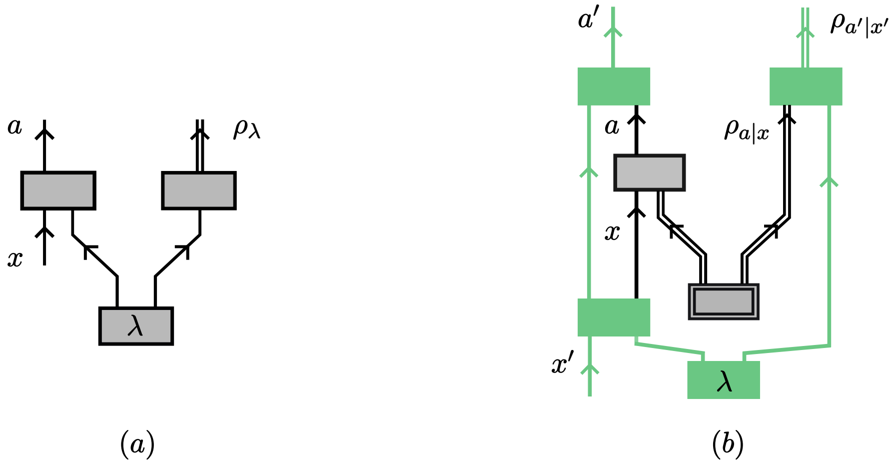

Figure 3.1: (a) LOSR-free standard bipartite assemblage. (b) The most general LOSR operation on a standard bipartite assemblage.

comb consists of a classical pre- and post-processing of her input and output variables. As Bob has no input, his comb can be simplified to only include a post-processing of his quantum state that can be described by a CPTP map. These operations are illustrated in Fig. 3.1(b), where the classical variable λ represents the shared randomness. In Ref. [1, Section 2.2], we show that this operation transforms the elements of one assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ into a new assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ with elements:

$$\sigma'_{a'|x'} = \sum_{\lambda} \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda) \mathcal{E}_{\lambda}(\sigma_{a|x}). \tag{3.1}$$

Here, $\mathcal{E}_{\lambda}$ is a CPTP map and $D(\cdot)$ represents deterministic probability distributions, i.e., $D(a'|a,x',\lambda) = \delta_{a',f_{\lambda}(a,x')}$ and $D(x|x',\lambda) = \delta_{x,g_{\lambda}(x')}$.

To study resource conversions under LOSR, it is useful to rewrite Eq. (3.1) using the Choi-Jamiołkowski isomorphism [56, 57]. Let $\mathcal{E} : \mathcal{H}_B \to \mathcal{H}_{B'}$ be a CPTP map. Define the operator W on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ as $W = (\mathcal{E} \otimes \mathbb{I}_{B'}) \left|\Omega\right\rangle \left\langle\Omega\right|$, with $\left|\Omega\right\rangle = \frac{1}{\sqrt{d_B}} \sum_{i=1}^{d_B} \left|ii\right\rangle$. Then, the map applied on an arbitrary quantum system ρ_B with dimension d_B can be written as

$$\mathcal{E}(\rho_B) = d_B \, \mathrm{tr}_B \left\{ W \left(\mathbb{I}_{B'} \otimes \rho_B^T \right) \right\}. \tag{3.2}$$

We will refer to W as a Choi matrix (or a Choi state) of the map $\mathcal{E}$. Recall that the following two conditions are equivalent: *(i)* $\mathcal{E}$ is a CPTP map, *(ii)* $W \geq 0$ (W is a positive semidefinite matrix) and $\mathrm{tr}_{B'} \{W\} = \frac{1}{d_B} \mathbb{I}_B$.

Using the Choi-Jamiołkowski isomorphism, we can rewrite Eq. (3.1) as

$$\sigma'_{a'|x'} = \sum_{\lambda} \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, d_B\, \mathrm{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a|x} \right) \right\} . \tag{3.3}$$

Here, the Choi matrix W_λ is a $(d_B \times d_{B'})$ by $(d_B \times d_{B'})$ matrix.

3.1.2 Properties of the pre-order

The resource theory we have introduced can be used to study the relative nonclassicality of assemblages. In Ref. [1, Section 2.3], we show that deciding whether one assemblage can be converted to another with LOSR operations requires a single instance of a semidefinite program. This SDP is a feasibility problem and has the following formulation:

SDP 1. $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$**.**
The assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ *can be converted into the assemblage* $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ *under LOSR operations, denoted by* $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$*, if and only if the following SDP is feasible:*

$$\begin{aligned}
\textit{given} \quad & \{\{\sigma_{a|x}\}_a\}_x \,,\ \{\{\sigma'_{a'|x'}\}_{a'}\}_{x'} \,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'} \,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
\textit{find} \quad & \{(W_\lambda)_{BB'}\}_\lambda \\
\textit{s.t.} \quad & \begin{cases} W_\lambda \geq 0 \,, \\ \mathrm{tr}_{B'}\left\{W_\lambda\right\} \propto \frac{1}{d}\, \mathbb{I}_B \quad \forall \lambda \,, \\ \sum_\lambda \mathrm{tr}_{B'}\left\{W_\lambda\right\} = \frac{1}{d}\, \mathbb{I}_B \,, \\ \sigma'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, d_B\, \mathrm{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a|x} \right) \right\} . \end{cases}
\end{aligned} \tag{3.4}$$

When the conversion is not possible, we denote it by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$*.*

A few comments pertain to the formulation of the program. To obtain SDP 1, it is crucial to rewrite the LOSR processing using the Choi-Jamiołkowski isomorphism, as the optimization variables, $\{(W_\lambda)_{BB'}\}_\lambda$, represent the Choi matrices of Bob's local processing. The constraints of the SDP ensure that $\{(W_\lambda)_{BB'}\}_\lambda$ correspond to a CPTP map. Moreover, the formulation of this program relies on the fact that Eq. (3.3) involves only finite sums. The variable λ encodes the (finite) number of deterministic distributions $D(a'|a,x',\lambda)$ and $D(x|x',\lambda)$. Here, the number of deterministic strategies is equal to $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|}$.

SDP 1 can be used to study conversions between assemblages that are hard to evaluate analytically. In Ref. [1, Section 2.4], we use it to study conversions among assemblages from an infinite family index by two parameters. The code in Matlab [58] (which uses the software CVX [59, 60], the solver SDPT3 [61] and the toolbox QETLAB [62]) is available in an online repository [63]. In this summary, however, we will only focus on the analytical results regarding the pre-order.

The main result of Ref. [1] regarding the pre-order of assemblages is the existence of an infinite number of incomparable resources. We now define these assemblages and outline the proof of their incompatibility.

Assume that Alice and Bob share a quantum system prepared in a state $|\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, with $\theta \in (0, {}^{\pi}/4]$. Define Alice's measurements on her subsystem as $M_{a|0} = \frac{1}{2}\{\mathbb{I} + (-1)^a\sigma_z\}$ and $M_{a|1} = \frac{1}{2}\{\mathbb{I} + (-1)^a\sigma_x\}$, where σ_z and σ_x are Pauli matrices. This EPR scenario is characterized by the dimension of Bob's quantum state, $d_B = 2$, and Alice's input and output space, $\mathbb{A} = \mathbb{X} = \{0, 1\}$. We denote this infinite family of assemblages by $\mathbf{\Sigma}^{\theta}_{\mathbb{A}|\mathbb{X}}$. The elements of $\mathbf{\Sigma}^{\theta}_{\mathbb{A}|\mathbb{X}}$ are given by

$$\sigma^{\theta}_{a|x} = \mathrm{tr}_{\mathrm{A}}\left\{M_{a|x} \otimes \mathbb{I}\,|\theta\rangle\,\langle\theta|\right\}. \tag{3.5}$$

To witness that the assemblages index by the angle $\{\theta\}$ are all pairwise incomparable, we derive an infinite family of resource monotones. We start by defining an EPR functional $S_\eta[\mathbf{\Sigma}]$ that is uniquely maximized by an assemblage from the family $\mathbf{\Sigma}^{\theta}_{\mathbb{A}|\mathbb{X}}$ with a fixed parameter $\theta = \eta$, hereon denoted by $\mathbf{\Sigma}^{\eta}_{\mathbb{A}|\mathbb{X}}$.

Definition 2. ***EPR functional*** $S_\eta[\mathbf{\Sigma}]$

The EPR functional $S_\eta[\mathbf{\Sigma}]$ *is defined as*

$$S_\eta[\mathbf{\Sigma}] = \mathrm{tr}\left\{\sum_{a\in\mathbb{A},\,x\in\mathbb{X}} F_{a,x}\,\sigma_{a|x}\right\}, \tag{3.6}$$

with the operators $F_{a,x}$ *given by*

$$F_{0,0} = \alpha\mathbb{I} + B_0 + B_1 = -F_{1,0} =, \; F_{0,1} = B_0 - B_1 = -F_{1,1}, \tag{3.7}$$

with

$$\alpha = \frac{2}{\sqrt{1+2\tan^2(2\eta)}}\,, \tag{3.8}$$
$$B_0 = \cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,,$$
$$B_1 = \cos(\mu)\,\sigma_z - \sin(\mu)\,\sigma_x\,,$$
$$\tan(\mu) = \sin(2\eta)\,.$$

This construction is inspired by the family of Bell inequalities presented in Ref. [64, Eq. (1)], that are uniquely maximized by the states $|\theta\rangle$ (via a self-testing result of Ref. [64]). Then, we use a yield-based construction [65] to define the monotone M_η:

Definition 3. *The resource monotone* M_η*, for* $\eta \in (0, \pi/4]$*, is defined as*

$$\mathrm{M}_\eta[\mathbf{\Sigma}] := \max_{\widetilde{\mathbf{\Sigma}}}\{S_\eta[\widetilde{\mathbf{\Sigma}}] : \mathbf{\Sigma} \xrightarrow{\text{LOSR}} \widetilde{\mathbf{\Sigma}}\}. \tag{3.9}$$

Each of the monotones $\mathrm{M}_\eta[\mathbf{\Sigma}]$ is uniquely maximized by the assemblage $\mathbf{\Sigma}^{\eta}_{\mathbb{A}|\mathbb{X}}$. Therefore, for every pair $\theta_1 \neq \theta_2$, the monotones M_{θ_1} and M_{θ_2} given by Definition 3 certify the incompatibility of the pair $\mathbf{\Sigma}^{\theta_1}_{\mathbb{A}|\mathbb{X}}$ and $\mathbf{\Sigma}^{\theta_2}_{\mathbb{A}|\mathbb{X}}$. It follows that the assemblages in the family $\{\mathbf{\Sigma}^{\theta}_{\mathbb{A}|\mathbb{X}}\}_{\theta\in(0,\pi/4]}$ are incomparable, as summarized in the corollary below.

Corollary 4. *The infinite family of EPR monotones* $\mathsf{M} = \{\mathrm{M}_\eta \,|\, \eta \in (0, \pi/4]\}$ *certifies that the infinite family of assemblages* $\{\mathbf{\Sigma}^{\theta}_{\mathbb{A}|\mathbb{X}}\}_{\theta\in(0,\pi/4]}$ *is composed of pairwise unordered resources.*

This result is given in Ref. [1, Corollary 8] and the complete proof can be found in Ref. [1, Section 2.4.2 and Appendix B]. Although the complete analytical proof of this result uses resource monotones, it is also possible to check the incomparability of any pair of assemblages $\mathbf{\Sigma}^{\theta_1}_{\mathbb{A}|\mathbb{X}}$ and $\mathbf{\Sigma}^{\theta_2}_{\mathbb{A}|\mathbb{X}}$ with $\theta_1 \neq \theta_2$ using SDP 1.

3.1.3 Multipartite EPR scenario

In Ref. [1], we also introduce a resource theory of multipartite EPR scenarios with the free operations being LOSR. Since the techniques employed in the multipartite case resemble those used in the bipartite scenario, we omit the specific details here. In Ref. [1], we conceptualize multipartite EPR scenarios as common-cause resources. We derive the most general LOSR operation for the multipartite EPR assemblage in Ref. [1, Eq. (24)], and show that resource conversion in this scenario can be evaluated with a single instance of a semi-definite program, which we specify in Ref. [1, SDP 10]. In Ref. [1, Section 3.4] we derive resource monotones that certify that the pre-order of multipartite assemblages contains an infinite number of incomparable resources.

3.1.4 Comparison to prior work

Comparison of the results of Ref. [1] to previous literature allows us to get a deeper understanding of the nonclassicality of EPR scenarios. In Ref. [1, Section 4.1], we discuss the resource theory of 'steering' under stochastic local operations and one-way classical communication (S-1W-LOCC) introduced in Ref. [33]. We show that the two resource theories (under LOSR and under S-1W-LOCC) have different pre-orders. Moreover, in Ref. [1, Sections 4.2 and 4.3], we discuss the importance of consistency between the set of free operations and the set of free resources. We show that the principled approach given by a resource-theoretic framework allows us to give a well-motivated definition of the set of classically-explainable assemblages in the multipartite EPR scenario. Inconsistencies in defining these sets can result in apparent phenomena like 'steering exposure' [66].

3.2 The resource theory of nonclassicality of channel assemblages

Bipartite generalizations of the EPR scenario are common-cause processes that can exhibit post-quantumness. In Ref. [2], we develop a resource theory of nonclassicality of channel assemblages under local operations and shared randomness. The enveloping theory consists of all non-signaling channel assemblages (including the post-quantum ones), and the free subtheory is the set of assemblages generated by classical common causes. Special cases of this resource theory include the Bob-with-input EPR scenario and the measurement-device-independent EPR scenario, which we also study in detail. In this summary, we revisit each of the three scenarios, recalling the form of a free resource, specifying the most general LOSR operations, and discussing the pre-order of relevant assemblages.

3.2.1 Channel EPR scenario

A channel assemblage generated solely by LOSR operations is illustrated in Fig. 3.2(a). Alice's wing is the same as in the standard EPR scenario. Bob has access to a CPTP map $\mathcal{I}_\lambda^{B_{in}\to B_{out}}$ that depends on the value of the classical common cause λ. All LOSR-free channel assemblages admit a decomposition of the form of Eq. (2.8).

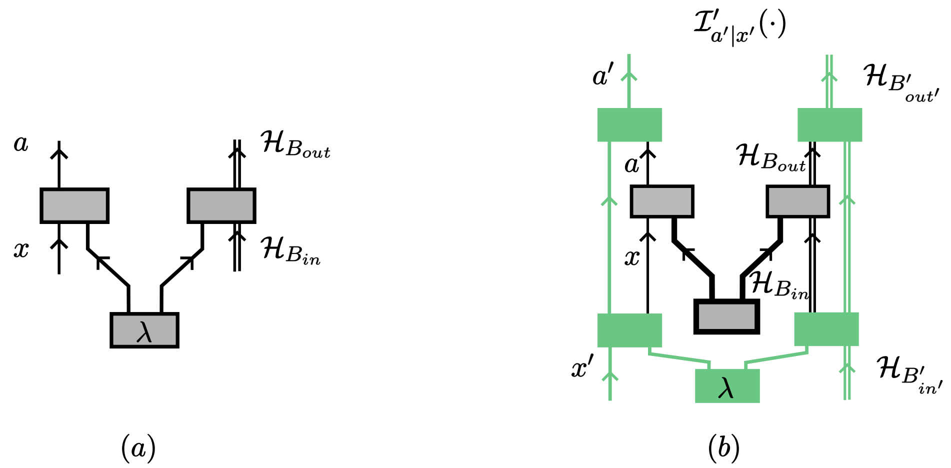

Figure 3.2: (a) LOSR-free channel assemblage. (b) The most general LOSR operation on a channel assemblage.

Similarly to the standard EPR scenario, the most general LOSR transformation of a channel assemblage consists of local combs correlated by shared randomness. However, the most general local processing on Bob's side is more complex than in the standard EPR scenario, as he now has access to quantum input and output systems. Bob's comb consists of pre- and post-processing stages that transform the input and output of the quantum channel, as illustrated in Fig. 3.2(b). For a fixed value of λ, let Bob's local comb be encoded in a single completely positive and trace non-increasing (CPTNI) map $\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$, with a Choi matrix $J_{\xi\,\lambda}$. The collection of maps $\{\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}\}_\lambda$ must then form a CPTP map given by $\sum_\lambda \xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$. As this map takes the systems $\mathcal{H}_{B'_{in'}} \otimes \mathcal{H}_{B_{out}}$ as inputs and outputs systems defined on $\mathcal{H}_{B'_{out'}} \otimes \mathcal{H}_{B_{in}}$, it must satisfy no-signaling conditions that reflect the causal structure of the comb. In particular, the input of Bob's post-processing $\mathcal{H}_{B_{out}}$ cannot have a direct causal influence on the output of Bob's pre-processing $\mathcal{H}_{B_{in}}$, a condition formalized as

$$\forall\,\lambda \quad \exists\; F_\lambda^{B'_{in'}\to B_{in}} \qquad \text{s.t.} \qquad \mathrm{tr}_{B'_{out'}}\left\{\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}\right\} = F_\lambda^{B'_{in'}\to B_{in}} \otimes \mathbb{I}_{B_{out}}\,. \qquad (3.10)$$

Then, the most general LOSR transformation on a channel assemblage transforms the elements of one assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ into a new assemblage $\mathbf{I}'_{\mathbb{A}'|\mathbb{X}'}$ with elements:

$$J'_{a'|x'} = \sum_{\lambda} \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,, \tag{3.11}$$

where the Choi states of the maps $\mathcal{I}'_{a'|x'}$ and $\mathcal{I}_{a|x}$, denoted by $J'_{a'|x'}$ and $J_{a|x}$, are composed using the link product $*$ defined in Ref. [47] as

$$J_{a|x} * J_{\xi\,\lambda} = d_{B_{in}}\, d_{B_{out}}\, \mathrm{tr}_{B_{in}B_{out}} \left\{ (\mathbb{I}_{B'_{in'}B'_{out'}} \otimes J_{a|x}^{T_{B_{out}}})\, J_{\xi\,\lambda}^{T_{B_{in}}} \right\} . \tag{3.12}$$

In Ref. [2, Section 2.2], we use this form of the LOSR transformation to derive a feasibility SDP that tests if a conversion between two channel assemblages is possible. We recall the program below.

SDP 5. *The channel assemblage $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}}$ can be converted to the channel assemblage $\boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations, denoted by $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$, if and only if the following SDP is feasible:*

$$\begin{aligned}
\textit{given} \quad & \{J_{a|x}\}_{a,x}\,,\ \{J'_{a'|x'}\}_{a',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
\textit{find} \quad & \{(J_{\xi\,\lambda})_{B_{in}B'_{in'}B_{out}B'_{out'}}\}_{\lambda}\,,\ \{(J_{F\,\lambda})_{B_{in}B'_{in'}}\}_{\lambda} \\
\textit{s.t.} \quad & \begin{cases}
J_{\xi\,\lambda} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} \propto \mathbb{I}_{B_{out}B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} = \frac{1}{d_{B_{out}}d_{B'_{in'}}}\,\mathbb{I}_{B_{out}B'_{in'}}\,, \\
J_{F\,\lambda} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} \propto \mathbb{I}_{B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} = \frac{1}{d_{B'_{in'}}}\,\mathbb{I}_{B'_{in'}}\,, \\
\mathrm{tr}_{B'_{out'}}\{J_{\xi\,\lambda}\} = J_{F\,\lambda} \otimes \frac{1}{d_{B_{out}}}\,\mathbb{I}_{B_{out}} \quad \forall\lambda\,, \\
J'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,.
\end{cases}
\end{aligned} \tag{3.13}$$

When the conversion is not possible, we denote it by $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$.

In the channel EPR scenario, the total number of deterministic strategies that λ encodes is given by $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|}$. It is worth mentioning that in Ref. [2, Appendix B], we converted SDP 5 into an optimization problem (where the optimization is performed relative to the objective function).

The code of SDP 5 in Matlab [58] is available in an online repository [63]. In Ref. [2], we use SDP 5 to study conversions between channel assemblages. For the subset of quantum channel assemblages, one interesting result is that we find an infinite number of equivalence classes. Moreover, we define two post-quantum assemblages that are incomparable under LOSR operations. These results are described in Ref. [2, Section 2.3.1] and Ref. [2, Section 2.3.2], respectively.

3.2.2 Bob-with-input EPR scenario

A Bob-with-input assemblage generated only with free operations, which can be decomposed as in Eq. (2.17), is illustrated in Fig. 3.3(a). Here, Bob's quantum state $\rho_{\lambda,y}$ is generated depending on the value of Bob's input y and shared randomness λ.

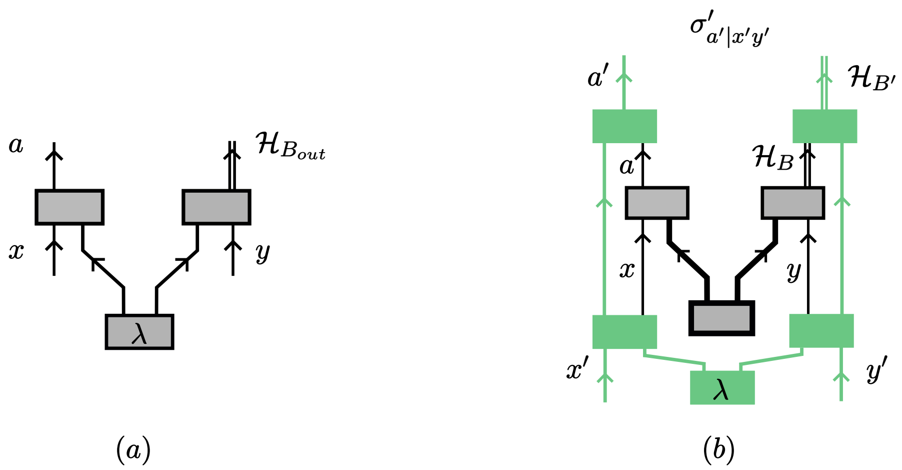

Figure 3.3: (a) LOSR-free Bob-with-input assemblage. (b) The most general LOSR operation on a Bob-with-input assemblage.

Let us now specify the most general LOSR processing of a Bob-with-input assemblage, which is illustrated in Fig. 3.3(b). In the Bob-with-input scenario, Bob again has a pre- and post-processing of his input and output systems. As Bob's input is a classical variable, the form of his local comb can be significantly simplified from the one in the channel EPR scenario. Let $J_{\xi\,\lambda\,y'}$ be a Choi matrix that corresponds to Bob's post-processing stage. In Ref. [2, Section 3.1], we show that the most general LOSR operation on a Bob-with-input assemblage takes the elements of one assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ into a new assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ with elements:

$$\sigma'_{a'|x'y'} = \sum_{\lambda} \sum_{a,x,y} D(x|x',\lambda) D(a'|a,x',\lambda) D(y|y',\lambda)\, d_B \,\mathrm{tr}_B \left\{ J_{\xi\,\lambda\,y'} \left(\mathbb{I}_{B'} \otimes \sigma^T_{a|xy}\right)\right\} . \tag{3.14}$$

Writing the LOSR transformation in this form allows us then to construct a semidefinite program that tests whether a conversion between two Bob-with-input assemblages is possible.

SDP 6. $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$.
The assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be converted into the assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ under LOSR operations if and only if the following SDP is feasible:

$$\begin{aligned}
\textit{given}\quad & \{\sigma_{a|xy}\}_{a,x,y}\,,\ \{\sigma'_{a'|x'y'}\}_{a',x',y'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(y|y',\lambda)\}_{\lambda,y,y'} \\
\textit{find}\quad & \{(J_{\xi\,\lambda\,y'})_{BB'}\}_{\lambda,y'} \\
\textit{s.t.}\quad & \begin{cases}
J_{\xi\,\lambda\,y'} \geq 0 \quad \forall\,\lambda, y'\,, \\
\mathrm{tr}_{B'}\left\{J_{\xi\,\lambda\,y'}\right\} \propto \mathbb{I}_B \quad \forall\,\lambda, y'\,, \\
\sum_\lambda \mathrm{tr}_{B'}\left\{J_{\xi\,\lambda\,y'}\right\} = \frac{1}{d}\,\mathbb{I}_B \quad \forall\, y'\,, \\
\mathrm{tr}_{B'}\left\{J_{\xi\,\lambda\,y'_1}\right\} = \mathrm{tr}_{B'}\left\{J_{\xi\,\lambda\,y'_2}\right\} \quad \forall\,\lambda\,, y'_1, y'_2\,, \\
\sigma'_{a'|x'y'} = \sum_\lambda \sum_{a,x,y} D(x|x',\lambda)\, D(a'|a,x',\lambda)\, D(y|y',\lambda)\, d_B \,\mathrm{tr}_B\left\{J_{\xi\,\lambda\,y'}\left(\mathbb{I}_{B'} \otimes \sigma^T_{a|xy}\right)\right\} .
\end{cases}
\end{aligned} \tag{3.15}$$

When the conversion is not possible, we denote it by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} \overset{LOSR}{\not\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$.

In SDP 6, the total number of deterministic strategies labeled by λ is $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|} \times |\mathbb{Y}|^{|\mathbb{Y}'|}$.

In Ref. [2, Section 3.2], we study the pre-order of Bob-with-input assemblages. The main result regards the incompatibility of two post-quantum resources, which we introduce below.

Consider the post-quantum Bob-with-input assemblage introduced in Ref. [45, Eq. (6)] that can be written as

$$\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{PTP}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}}\,, \tag{3.16}$$

$$\text{with} \quad \begin{cases}
\sigma^{PTP}_{a|xy} = \xi_y\left\{\mathrm{tr}_{\mathrm{A}}\left\{\left(M_{a|x} \otimes \mathbb{I}_B\right)\,|\phi\rangle\,\langle\phi|\right\}\right\}, \\
M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2}\,, \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a\sigma_y}{2}\,, \quad M_{a|3} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2}\,.
\end{cases}$$

Here, $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and σ_x, σ_y, and σ_z are the Pauli operators. Bob's local map ξ_y depends on the value of y. For $y = 0$, Bob applies identity, and for $y = 1$, Bob transposes his local system[1]. A different post-quantum Bob-with-input assemblage introduced in Ref. [45, Eq. (5)] is the following:

$$\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{PR}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}}\,, \tag{3.17}$$
$$\text{with}\quad \sigma^{PR}_{a|xy} = \begin{cases} |a\oplus xy\rangle\langle a\oplus xy| & \text{if } x\in\{0,1\} \\ \frac{\mathbb{I}}{2}a & \text{if } x = 2. \end{cases}$$

In Ref. [2], we analytically prove that $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ are incomparable resources (this result can be checked with SDP 6).

Corollary 7. *The two post-quantum assemblages $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ are unordered resources in the LOSR resource theory of common-cause assemblages.*

The proof of Corollary 7, given in Ref. [2, Section 3.2 and Appendix E], relies on two observations. Firstly, $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can generate post-quantum correlations when Bob decides to measure his quantum system. This is not the case for $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$. Secondly, the 'steering' functional constructed in Ref. [45, Eq. (D3)] is minimized by $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and cannot be minimized by $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$.

This result is particularly interesting for the following reason: In Ref. [67], the relation between $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ was studied relative to local operations and shared entanglement (LOSE). LOSE is a much more powerful set than LOSR, where two parties are allowed to share quantum common causes for free. In Ref. [67], it was shown that $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be transformed to $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ with LOSE operations[2]. Therefore, a conversion between post-quantum resources, that is impossible under LOSR, is possible when the parties share entanglement. This result shows that quantum entanglement is a powerful resource, even in post-quantum theories.

[1]The assemblage $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ is post-quantum; hence, it does not admit a quantum realization. Here we use the transpose operation (which is not CPTP) simply for convenience, to describe the elements of the assemblage mathematically.

[2]Technically, the authors of Ref. [67] have shown that a PR-box can be transformed to $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$. It is straightforward to show that the assemblage $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be transformed to a PR-box with local operations; hence $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be transformed to $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ with LOSE.

3.2.3 Measurement-device-independent EPR scenario

A free measurement-device-independent assemblage is depicted in Fig. 3.4(a). In this EPR scenario, Bob has access to a measurement channel $\mathcal{N}_{b,\lambda}$ which depends on the value of λ and is indexed by the classical outcome b. Such assemblages admit a realization of the form of Eq. (2.12).

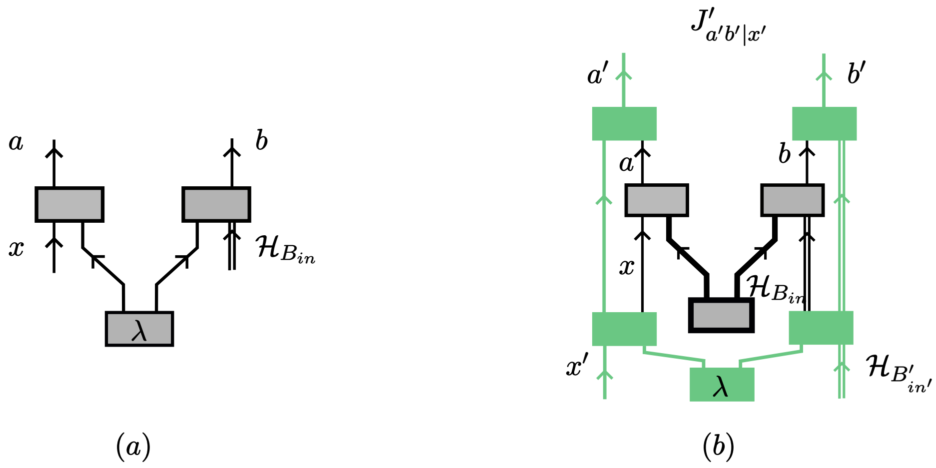

Figure 3.4: (a) LOSR-free measurement-device-independent assemblage. (b) The most general LOSR operation on a measurement-device-independent assemblage.

The most general LOSR processing of a measurement-device-independent assemblage consists of two local combs correlated by shared randomness, as illustrated in Fig. 3.4(b). Similarly to other generalized EPR scenarios, Bob's processing can be expressed as a single CPTNI map $\zeta_{b\,b'\,\lambda}^{B'_{in'}\to B_{in}}$, which is non-signaling from Bob's classical output b to his input state B_{in}, i.e., $\sum_{\lambda,b'}\zeta_{b\,b'\,\lambda}^{B'_{in'}\to B_{in}}$ is a CPTP map. Let $J'_{a'b'|x'}$, $J_{ab|x}$ and $J_{\zeta\,b\,b'\,\lambda}$ correspond to the Choi states of maps $\mathcal{N}'_{a'b'|x'}$, $\mathcal{N}_{ab|x}$ and $\zeta_{b\,b'\,\lambda}$, respectively. In Ref. [2, Section 4.1], we show that transforming an assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ with the most general LOSR operations results in a new assemblage $\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$ with elements:

$$J'_{a'b'|x'} = \sum_{\lambda}\sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{ab|x} * J_{\zeta\,b\,b'\,\lambda}\,, \tag{3.18}$$

with the link product given by

$$J_{ab|x} * J_{\zeta\,b\,b'\,\lambda} = d_{B_{in}}\ \mathrm{tr}_{B_{in}}\left\{(\mathbb{I}_{B'_{in'}} \otimes J_{ab|x})\, J_{\zeta\,b\,b'\,\lambda}^{T_{B_{in}}}\right\}. \tag{3.19}$$

This form of the transformation is then suitable to be a constraint in a semidefinite program, which decides whether one assemblage can be converted to another with LOSR transformations.

SDP 8. *The MDI assemblage $\boldsymbol{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ can be converted to the MDI assemblage $\boldsymbol{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$ under LOSR operations, denoted by $\boldsymbol{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}} \xrightarrow{LOSR} \boldsymbol{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$, if and only if the following SDP is feasible:*

$$
\begin{aligned}
\textit{given} \quad & \{J_{ab|x}\}_{a,b,x}\,,\ \{J'_{a'b'|x'}\}_{a',b',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
\textit{find} \quad & \{J_{\zeta\, b\, b'\, \lambda}\}_{b,b',\lambda} \\
\textit{s.t.} \quad & \begin{cases}
J_{\zeta\, b\, b'\, \lambda} \geq 0 \quad \forall\, b, b', \lambda\,, \\
\sum_{b'} \mathrm{tr}_{B_{in}} \{J_{\zeta\, b\, b'\, \lambda}\} \propto \mathbb{I}_{B'_{in'}} \quad \forall\, b, \lambda\,, \\
\sum_{\lambda, b'} \mathrm{tr}_{B_{in}} \{J_{\zeta\, b\, b'\, \lambda}\} = \frac{1}{d_{B'_{in'}}} \mathbb{I}_{B'_{in'}} \quad \forall\, b\,, \\
\sum_{b'} J_{\zeta\, b\, b'\, \lambda} \geq 0 \quad \forall\, b, \lambda\,, \\
J'_{a'b'|x'} = \sum_{\lambda} \sum_{a,b,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{ab|x} * J_{\zeta\, b\, b'\, \lambda}\,.
\end{cases}
\end{aligned}
\tag{3.20}
$$

When the conversion is not possible, we denote it by $\boldsymbol{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \boldsymbol{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$.

In the MDI scenario, the total number of deterministic strategies that λ encodes is given by $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|}$.

To study the pre-order of post-quantum resources, one needs to first construct a method for certifying if a measurement-device-independent assemblage is post-quantum or admits a quantum realization. In Ref. [2, Appendix D] we outline an SDP method that tests a membership of a MDI assemblage to the relaxation of the quantum set. Then, we run it to certify post-quantumness of a particular MDI assemblage and use SDP 8 to study the pre-order of post-quantum resources in the MDI scenario in Ref. [2, Section 4.2].

3.3 Activation of post-quantumness in bipartite generalized EPR scenarios

In Ref. [3], we develop protocols for certification of post-quantumness in bipartite Bob-with-input, measurement-device-independent, and channel EPR scenarios. One method to verify whether an assemblage is post-quantum is by transforming it into a Bell-type scenario (with classical inputs and outputs) and testing if the correlations generated in this set-up admit a quantum realization.

The main idea of the protocol is to embed the bipartite assemblage in a larger network and test the generated correlations with tailored Bell inequalities.

There exist post-quantum assemblages that can only generate quantum correlations when transformed to bipartite Bell-like scenarios [3, 52]. The protocols introduced in Ref. [3] can be used to *activate* their post-quantumness. Therefore, the results of Ref. [3] provide a mapping between post-quantum assemblages and post-quantum Bell-like correlations. The protocol is inspired by an activation protocol for the multipartite EPR scenario introduced in Ref. [18], which is based on the results of Refs. [68, 69].

In this section, for the sake of simplicity, we focus on the case of qubit assemblages (where Bob's output state is a qubit system). We recall the protocols for the Bob-with-input and measurement-device-independent scenarios. The protocol for the channel EPR scenario is a combination of the two.

3.3.1 Bob-with-input EPR scenario

There exist post-quantum assemblages in the Bob-with-input scenario which can only lead to quantum correlations if Bob decides to measure his local quantum state. Formally, when Bob performs a measurement $\{N_b^B\}$ on a post-quantum assemblage with elements $\{\sigma_{a|xy}\}$, it is possible that the observed correlations $p(ab|xy) = \text{tr}\left\{N_b \sigma_{a|xy}\right\}$ belong to the quantum set. We denote this set of assemblages $\mathbf{\Sigma}^{QC}$. In the activation protocol, a bipartite post-quantum assemblage is distributed in a multipartite network such that it is possible to generate post-quantum correlations in this new set-up. If the Bob-with-input assemblage embedded in the protocol belongs to the set $\mathbf{\Sigma}^{QC}$, we refer to this phenomenon as activation of post-quantumness, as it activates the post-quantum nature of the underlying assemblage.

The set-up of the protocol is illustrated in Fig. 3.5. It consists of three parties: Alice, Bob and Charlie. Alice and Bob share a (possibly post-quantum) bipartite Bob-with-input assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$. Bob and Charlie share a standard quantum assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$, with $\mathbb{C} := \{0, 1\}$ and $\mathbb{W} := \{1, 2, 3\}$ defined on $\mathcal{H}_{B'}$. Moreover, Bob has access to a measurement device $\{\mathrm{M}_{b|z}^{BB'}\}$ with input and output sets $b \in \mathbb{B} := \{0, 1\}$ and $z \in \mathbb{Z} := \{1, 2, 3, 4, \star\}$. The protocol involves two steps:

Step 1: *Self-testing of* $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$.

In this step of the protocol, we use the results of Refs. [69, 70] to certify that the state

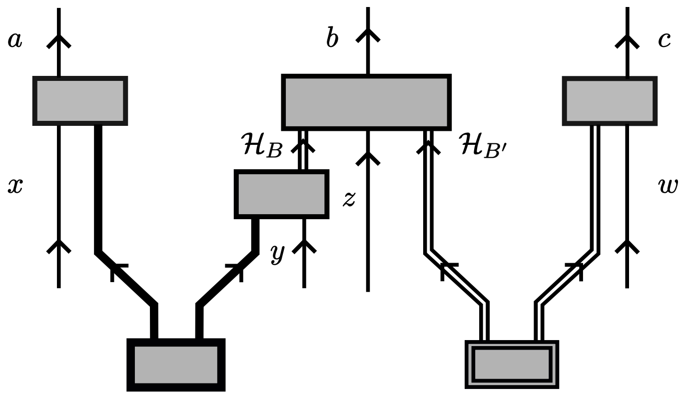

Figure 3.5: Set-up of the activation protocol for the Bob-with-input scenario.

of Bob's quantum system $\mathcal{H}_{B'}$ is prepared in the assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$ with elements $\sigma^{(r)}_{c|w} = r\widetilde{\sigma}_{c|w} + (1-r)(\widetilde{\sigma}_{c|w})^T$. Here, $0 \leq r \leq 1$ is an unknown parameter, and $\widetilde{\sigma}_{c|1} = (\mathbb{I} + (-1)^c Z)/2$, $\widetilde{\sigma}_{c|2} = (\mathbb{I} + (-1)^c X)/2$, $\widetilde{\sigma}_{c|3} = (\mathbb{I} + (-1)^c Y)/2$, with X, Y, Z being the Pauli operators[3]. This step corresponds to $z \in \{1, 2, 3, 4\}$.

Step 2: *Certification of post-quantumness.*

The second step of the protocol is to measure the system on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ and evaluate an appropriate Bell functional to certify post-quantumness of the correlations. This step relies on the following Theorem:

Theorem 9. *There always exists a Bell functional of the form*

$$I^*_{BwI}[\boldsymbol{p}] = \frac{1}{2} \sum_{a,x,y,c,w} \xi^{axy}_{cw} p(a, b = 0, c|x, y, z = \star, w), \tag{3.21}$$

for which $I^*_{BwI}[\boldsymbol{p}] \geq 0$ when evaluated on any correlations $\boldsymbol{p} = \{p(a, b, c|x, y, z, w)\}$ arising from quantum Bob-with-input assemblage with elements $\{\sigma_{a|xy}\}_{a,x,y}$ as $p(a, b = 0, c|x, y, z = \star, w) = \text{tr}\left\{M^{BB'}_{b=0|z=\star}(\sigma_{a|xy} \otimes \widetilde{\sigma}_{c|w})\right\}$.

In Ref. [3, Section 2.2], we provide a step-by-step derivation of the functional specified in Eq. (3.21) and we prove that its quantum bound is equal to zero under the assumption that the self-testing stage of the protocol was successful. Hence, observing $I^*_{BwI}[\mathbf{p}] < 0$ certifies that the underlying assemblage is post-quantum. Here, we outline the proof without recalling all the technical details.

[3]In the previous sections, Pauli operators are denoted by $\sigma_x, \sigma_y, \sigma_z$. In this section, we change the notation to be consistent with Ref. [3].

Consider a fixed post-quantum Bob-with-input assemblage with elements $\{\sigma^*_{a|xy}\}_{a,x,y}$. There always exists an EPR functional defined by Hermitian operators $\{F_{axy}\}_{a,x,y}$ with a quantum bound equal to zero. Therefore, $\mathrm{tr}\left\{\sum_{a,x,y} F_{axy}\sigma_{a|xy}\right\} \geq 0$ for any quantum assemblage $\{\sigma_{a|xy}\}_{a,x,y}$, and the functional evaluated on the particular post-quantum assemblage gives

$$\mathrm{tr}\left\{\sum_{a,x,y} F_{axy}\sigma^*_{a|xy}\right\} < 0. \tag{3.22}$$

The operators $\{F_{axy}\}_{a,x,y}$ can be decomposed as $F_{axy} = \sum_{c,w} \xi^{axy}_{cw}\pi_{c|w}$ for some real numbers $\{\xi^{axy}_{cw}\}_{a,x,y,c,w}$ and $\{\pi_{c|w}\}_{c,w}$ being a basis of the set of Hermitian operators in a two-dimensional Hilbert space. Here, we take the following basis: for $w \in \{1, 2, 3\}$, $\pi_{c|w}$ is the projector onto the eigenspace of the eigenstates of Pauli Z, X and Y operators with eigenvalue $(-1)^c$. Define the following Bell functional on the correlations generated in the set-up of the activation protocol:

$$I^*_{BwI}[\mathbf{p}] \equiv \sum_{a,x,y,c,w} \xi^{axy}_{cw}\, p(a, b = 0, c|x, y, z = \star, w). \tag{3.23}$$

Under the assumptions that Bob's measurement is given by $M^{BB'}_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$, with $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and Charlie's state is self-tested to be prepared in the assemblage $\{\tilde{\sigma}_{c|w}\}_{c,w}$, one can show that

$$\begin{aligned} I^*_{BwI}[\mathbf{p}] &= \sum_{a,x,y,c,w} \xi^{axy}_{cw}\mathrm{tr}\left\{M^{BB'}_{b=0|z=\star}(\sigma_{a|xy}\otimes\tilde{\sigma}_{c|w})\right\}, \\ &= \sum_{a,x,y,c,w} \xi^{axy}_{cw}\frac{1}{2}\mathrm{tr}\left\{(\tilde{\sigma}_{c|w})^T\sigma_{a|xy}\right\}, \\ &= \frac{1}{4}\sum_{a,x,y}\mathrm{tr}\left\{\left(\sum_{c,w}\xi^{axy}_{cw}\pi_{c|w}\right)\sigma_{a|xy}\right\}, \\ &= \frac{1}{4}\mathrm{tr}\left\{\sum_{a,x,y} F_{axy}\sigma_{a|xy}\right\}. \end{aligned} \tag{3.24}$$

By noticing the relation between the Bell functional in Eq. (3.24) and the EPR functional in Eq. (3.22), one can see that the quantum bound of the Bell functional (under the assumptions given above) is equal to zero. Furthermore, the Bell functional evaluated on the fixed post-quantum assemblage $\{\sigma^*_{a|xy}\}_{a,x,y}$ evaluates to $I^*_{BwI}[\mathbf{p}^*] < 0$. To complete the proof of Theorem 9, it is necessary to show that the quantum bound does not change when $M^{BB'}_{b=0|z=\star} \neq |\phi^+\rangle\langle\phi^+|$, which we prove in Ref. [3, Appendix C]. Moreover, we assumed that Charlie's state is self-tested to be prepared in the assemblage $\{\tilde{\sigma}_{c|w}\}$. As the first step of the protocol can only self-test the assemblage $\{\sigma^{(r)}_{c|w}\}_{c,w}$, not $\{\tilde{\sigma}_{c|w}\}_{c,w}$, in Ref. [3, Appendix B] we show that this assumption cannot create a false-positive detection in the protocol.

3.3.2 Measurement-device-independent EPR scenario

In the measurement-device-independent EPR scenario, there exist post-quantum assemblages that can only lead to quantum correlations if Bob's input consists of a set of fixed quantum states $\{\rho_y\}_y$. Formally, there exist post-quantum assemblages with elements $\mathcal{N}_{ab|x}(\cdot)$ such that the observed correlations $p(ab|xy) = \mathcal{N}_{ab|x}(\rho_y)$ belong to the quantum set.

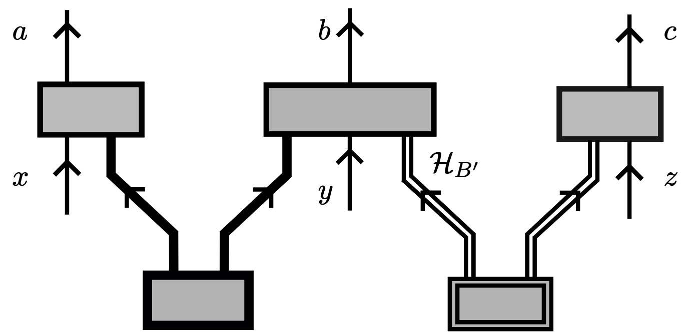

Figure 3.6: Set-up of the activation protocol for the measurement-device-independent scenario.

The activation protocol for the MDI scenario is presented in Fig. 3.6. It consists of three parties – Alice, Bob and Charlie. Alice and Bob share a bipartite measurement-device-independent assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$. Bob's input state is prepared by Charlie through a standard quantum assemblage $\boldsymbol{\Sigma}_{\mathbb{C}|\mathbb{Z}}$ on the Hilbert space $\mathcal{H}_{B'}$, with $\mathbb{C} := \{0, 1\}$ and $\mathbb{W} := \{1, 2, 3\}$. Finally, Bob has a measurement device $\{\mathrm{M}_{b|y}^{BB'}\}$ with $b \in \mathbb{B}$ and $y \in \mathbb{Y} := \{1, 2, 3, 4, \star\}$. Similarly to the Bob-with-input scenario, the protocol consists of two steps:

Step 1: *Self-testing of* $\boldsymbol{\Sigma}_{\mathbb{C}|\mathbb{Z}}$.

For $y \in \{1, 2, 3, 4\}$, Bob measures the quantum system $\mathcal{H}_{B'}$ to certify that Bob and Charlie share an assemblage $\boldsymbol{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{Z}}$ with elements $\sigma^{(r)}_{c|z} = r\tilde{\sigma}_{c|z} + (1 - r)(\tilde{\sigma}_{c|z})^T$.

Step 2: *Certification of post-quantumness.*

The second step of the protocol is to evaluate the MDI assemblage with the reduced state of $\boldsymbol{\Sigma}_{\mathbb{C}|\mathbb{Z}}$ as Bob's quantum input. Then, one can calculate an appropriate Bell functional to certify post-quantumness of the correlations based on the following Theorem:

Theorem 10. *There always exists a Bell functional of the form*

$$I^*_{MDI}[\boldsymbol{p}] = \sum_{a,b,x,c,z} \xi^{abx}_{cz} p(a, b, c|x, y = \star, z), \tag{3.25}$$

for which $I^*_{MDI}[\boldsymbol{p}] \geq 0$ *when evaluated on any correlations* $\boldsymbol{p} = \{p(a,b,c|x,y,z)\}$ *arising from quantum measurement-device-independent assemblage with elements* $\{\mathcal{N}_{ab|x}(\cdot)\}_{a,b,x}$ *as* $p(a,b,c|x,y=\star,z) = \mathcal{N}_{ab|x}(\widetilde{\sigma}_{c|z})$.

The derivation of the functional given in Eq. (3.25), as well as a proof of its quantum bound (when the self-testing stage of the protocol is successful), is given in Ref. [3, Section 2.2]. Theorem 10 implies that observing $I^*_{MDI}[\mathbf{p}] < 0$ certifies that the underlying assemblage is post-quantum. The proof of Theorem 10 is based on the same idea as the proof for the Bob-with-input scenario, as we outline below.

Let $\{\mathcal{N}^*_{ab|x}(\cdot)\}_{a,b,x}$ be a fixed post-quantum MDI assemblage. One can always find an EPR functional defined by Hermitian operators $\{F_{abx}\}_{a,b,x}$ with a quantum bound equal to zero, which certifies post-quantumness of the assemblage $\{\mathcal{N}^*_{ab|x}(\cdot)\}_{a,b,x}$ as

$$\mathrm{tr}\left\{\sum_{a,b,x} F_{abx}\, J(\mathcal{N}^*_{ab|x})\right\} < 0. \tag{3.26}$$

Similarly to the Bob-with-input scenario, we can decompose the operators $\{F_{abx}\}_{a,b,x}$ as $F_{abx} = \sum_{c,z} \xi^{abx}_{cz} \pi_{c|z}$. Then, consider the following Bell functional on the correlations $\mathbf{p} = \{p(a,b,c|x,y = \star, z)\}$:

$$I^*_{MDI}[\mathbf{p}] \equiv \sum_{a,b,x,c,z} \xi^{abx}_{cz} p(a,b,c|x,y=\star,z). \tag{3.27}$$

Under the assumption that the assemblage that Bob and Charlie share is self-tested to be $\{\widetilde{\sigma}_{c|w}\}_{c,w}$, Eq. (3.27) can be rewritten as

$$\begin{aligned} I^*_{MDI}[\mathbf{p}] &= \sum_{a,b,x,c,z} \xi^{abx}_{cz} \mathcal{N}_{ab|x}(\widetilde{\sigma}_{c|z}), \\ &= \sum_{a,b,x,c,z} \xi^{abx}_{cz} \mathrm{tr}\left\{\widetilde{M}^C_{c|z}\, J(\mathcal{N}_{ab|x})\right\}, \\ &= \sum_{a,b,x} \mathrm{tr}\left\{\left(\sum_{c,z} \xi^{abx}_{cz} \pi_{c|z}\right) J(\mathcal{N}_{ab|x})\right\}, \\ &= \mathrm{tr}\left\{\sum_{a,b,x} F_{abx}\, J(\mathcal{N}_{ab|x})\right\}. \end{aligned} \tag{3.28}$$

By comparing the Bell functional in Eq. (3.28) and the EPR functional in Eq. (3.26), it is easy to see that the quantum bound (when the self-testing stage is successful) of $I^*_{MDI}[\mathbf{p}]$ is equal to zero.

Moreover, for the fixed post-quantum assemblage $\{\mathcal{N}^*_{ab|x}(\cdot)\}_{a,b,x}$, the Bell functional evaluates to $I^*_{MDI}[\mathbf{p^*}] < 0$. In Ref. [3, Appendix B], we show that the assumption that assemblage self-tested in the first step of the protocol is given by $\{\widetilde{\sigma}_{c|w}\}_{c,w}$, and not $\{\sigma^{(r)}_{c|w}\}_{c,w}$, does not limit the application of the protocol.

3.3.3 Other results

In this section, we presented the activation protocols for the Bob-with-input and measurement-device-independent EPR scenarios. The protocol for the channel EPR scenario consists of applying the BwI protocol to the output of Bob's channel and the MDI protocol to the input of Bob's channel. As the methods used in this protocol are very similar to the ones described above, we do not present it here. The protocol, its technical description, and all related results are given in Ref. [3, Section 4].

Finally, one last simplification that we employed in this section was focusing only on qubit assemblages. All the protocols described above can be extended to the beyond-qubit case by adjusting the self-testing stage of the protocol. If Bob's local quantum system is not a qubit, the basis of the EPR functional is not defined simply by Pauli operators, but by their tensor product. Let us illustrate it for the Bob-with-input scenario. We start by embedding Bob's system in a higher-dimensional Hilbert space, where each qudit lives on a Hilbert space that results from a parallel composition of n qubits. Then, the self-testing stage of the protocol must guarantee that the self-tested assemblage is of the form

$$\sigma^{(r)}_{\mathbf{c}|\mathbf{w}} = r \bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i} + (1-r) \bigotimes_{i=1}^{n} (\widetilde{\sigma}_{c_i|w_i})^T. \tag{3.29}$$

Here, $\mathbf{c} = (c_1, ..., c_n)$ with $c_i \in \{0, 1\}$ and $\mathbf{w} = (w_1, ..., w_n)$ with $w_i \in \{1, 2, 3\}$. Self-testing of this assemblage is possible due to results of Ref. [69]. When the first step of the protocol certifies an assemblage of the form given in Eq. (3.29), the second step can be adjusted to construct a Bell functional for the beyond-qubit assemblages. All the technical details can be found in Refs. [3, 18, 69].

4 Discussion

EPR correlations are one of the fundamental resources of nonclassicality in quantum information theory. In this dissertation, we explored how viewing EPR assemblages as common-cause resources shapes our understanding of their nonclassicality. This approach is the key motivation behind the resource theory of standard EPR assemblages under LOSR, which we developed in Ref. [1], and the resource theory of channel assemblages under LOSR, developed in Ref. [2]. A resource-theoretic framework provides the tools to clearly define the nonclassicality of assemblages and study their pre-order. For all types of EPR scenarios of interest, we derived semidefinite programs for assessing resource conversions. The form and complexity of the programs significantly differ depending on the type of Bob's local input and output systems. These semidefinite programs are the first tools that allow one to explore the conversions between assemblages systematically. Moreover, we derived multiple analytical results regarding the pre-order in different EPR scenarios, both for quantum and post-quantum assemblages.

Going forward, there are many relevant questions one can ask in the resource-theoretic framework. For example, deriving resource monotones that quantify how useful particular assemblages are in certain quantum information processing protocols is a valuable direction to study. Moreover, as all the resource theories introduced in this dissertation form one coherent resource theory with LOSR as the set of free operations, it would be interesting to explore the properties of the pre-order of assemblages when type-changing LOSR operations are allowed. Finally, some concepts from quantum information theory can be translated into the language of resource theories, one example being self-testing [10]. It would be interesting to explore this idea for self-testing of quantum assemblages [70, 71, 72] and see if the SDP programs we developed in Refs. [1, 2] could be used to improve existing results.

Generalized EPR scenarios provide a playground for studying post-quantumness in bipartite non-signaling resources. To study whether there is a mapping between post-quantumness in

EPR scenarios and post-quantum Bell-like correlations, we examined their interplay in Ref. [3]. We demonstrated that post-quantumness of bipartite assemblages can be certified by embedding them in larger networks that produce Bell-like correlations. We derived tailored Bell inequalities with a quantum bound violated by the correlations generated in the protocol. This protocol activates post-quantumness for assemblages that can only generate quantum correlations in bipartite Bell scenarios. Therefore, as of now, there are no known examples of post-quantum assemblages that do not generate post-quantum correlations in Bell-like set-ups.

Although the protocols we derived do not rely on shared randomness and solely utilize local operations, their structure is well-suited to the study of common-cause resources. It would be interesting to explore whether other common-cause resources exist that do not demonstrate post-quantumness in bipartite Bell scenarios and investigate activation protocols for them. Furthermore, a natural extension of our work would be to investigate how the protocols change for multipartite Bob-with-input, measurement-device-independent, and channel EPR scenarios.

Bibliography

[1] Beata Zjawin, David Schmid, Matty J Hoban, and Ana Belén Sainz. "Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages". *Quantum* 7 (2023), 926. DOI: https://doi.org/10.22331/q-2023-02-16-926.

[2] Beata Zjawin, David Schmid, Matty J. Hoban, and Ana Belén Sainz. "The resource theory of nonclassicality of channel assemblages". *Quantum* 7 (2023), 1134. ISSN: 2521-327X. DOI: 10.22331/q-2023-10-10-1134.

[3] Beata Zjawin, Matty J Hoban, Ana Belén Sainz, and Paul Skrzypczyk. "Activation of post-quantumness in bipartite generalised EPR scenarios". *arXiv preprint arXiv:2406.10697* (2024). DOI: https://arxiv.org/pdf/2406.10697.

[4] John S Bell. "On the Einstein Podolsky Rosen paradox". *Physics Physique Fizika* 1.3 (1964), 195. DOI: https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195.

[5] Bob Coecke, Tobias Fritz, and Robert W Spekkens. "A mathematical theory of resources". *Information and Computation* 250 (2016), 59–86. DOI: https://doi.org/10.1016/j.ic.2016.02.008.

[6] Eric Chitambar and Gilad Gour. "Quantum resource theories". *Reviews of Modern Physics* 91.2 (2019), 025001. DOI: https://doi.org/10.1103/RevModPhys.91.025001.

[7] David Schmid, Denis Rosset, and Francesco Buscemi. "The type-independent resource theory of local operations and shared randomness". *Quantum* 4 (2020), 262. DOI: https://doi.org/10.22331/q-2020-04-30-262.

[8] Denis Rosset, David Schmid, and Francesco Buscemi. "Type-Independent Characterization of Spacelike Separated Resources". *Physical Review Letters* 125.21 (2020), 210402. DOI: https://doi.org/10.1103/PhysRevLett.125.210402.

[9] Elie Wolfe, David Schmid, Ana Belén Sainz, Ravi Kunjwal, and Robert W Spekkens. "Quantifying Bell: The resource theory of nonclassicality of common-cause boxes". *Quantum* 4 (2020), 280. DOI: https://doi.org/10.22331/q-2020-06-08-280.

[10] David Schmid, Thomas C Fraser, Ravi Kunjwal, Ana Belen Sainz, Elie Wolfe, and Robert W Spekkens. "Understanding the interplay of entanglement and nonlocality: motivating and developing a new branch of entanglement theory". *Quantum* 7 (2023), 1194. DOI: https://doi.org/10.22331/q-2023-12-04-1194.

[11] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. "Quantum entanglement". *Rev. Mod. Phys.* 81 (2 2009), 865–942. DOI: https://link.aps.org/doi/10.1103/RevModPhys.81.865.

[12] Albert Einstein, Boris Podolsky, and Nathan Rosen. "Can quantum-mechanical description of physical reality be considered complete?" *Physical review* 47.10 (1935), 777. DOI: https://doi.org/10.1103/PhysRev.47.777.

[13] Erwin Schrödinger. "Discussion of probability relations between separated systems". *Mathematical Proceedings of the Cambridge Philosophical Society* 31.4 (1935), 555–563. DOI: https://doi.org/10.1017/S0305004100013554.

[14] Eric Gama Cavalcanti, Steve J Jones, Howard M Wiseman, and Margaret D Reid. "Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox". *Physical Review A* 80.3 (2009), 032112. DOI: https://doi.org/10.1103/PhysRevA.80.032112.

[15] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. "Proposed experiment to test local hidden-variable theories". *Physical review letters* 23.15 (1969), 880. DOI: https://doi.org/10.1103/PhysRevLett.23.880.

[16] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. "Bell nonlocality". *Reviews of Modern Physics* 86.2 (2014), 419. DOI: https://doi.org/10.1103/RevModPhys.86.419.

[17] Christopher J Wood and Robert W Spekkens. "The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning". *New Journal of Physics* 17.3 (2015), 033002. DOI: https://doi.org/10.1088/1367-2630/17/3/033002.

[18] Matty J Hoban, Paul Skrzypczyk, and Ana Belén Sainz. "Activation of post-quantum steering". *arXiv preprint arXiv:2406.10570* (2024). DOI: https://arxiv.org/pdf/2406.10570.

[19] Carlos Palazuelos. "Superactivation of quantum nonlocality". *Physical review letters* 109.19 (2012), 190401. DOI: https://doi.org/10.1103/PhysRevLett.109.190401.

[20] Iman Marvian and Robert W Spekkens. "How to quantify coherence: Distinguishing speakable and unspeakable notions". *Physical Review A* 94.5 (2016), 052324. DOI: https://doi.org/10.1103/PhysRevA.94.052324.

[21] Iman Marvian, Robert W Spekkens, and Paolo Zanardi. "Quantum speed limits, coherence, and asymmetry". *Physical Review A* 93.5 (2016), 052331. DOI: https://doi.org/10.1103/PhysRevA.93.052331.

[22] Andreas Winter and Dong Yang. "Operational resource theory of coherence". *Physical review letters* 116.12 (2016), 120404. DOI: https://doi.org/10.1103/PhysRevLett.116.120404.

[23] Stephen D Bartlett, Terry Rudolph, and Robert W Spekkens. "Reference frames, superselection rules, and quantum information". *Reviews of Modern Physics* 79.2 (2007), 555. DOI: https://doi.org/10.1103/RevModPhys.79.555.

[24] Fernando GSL Brandao, Michał Horodecki, Jonathan Oppenheim, Joseph M Renes, and Robert W Spekkens. "Resource theory of quantum states out of thermal equilibrium". *Physical review letters* 111.25 (2013), 250404. DOI: https://doi.org/10.1103/PhysRevLett.111.250404.

[25] Paul Skrzypczyk, Anthony J Short, and Sandu Popescu. "Work extraction and thermodynamics for individual quantum systems". *Nature communications* 5.1 (2014), 1–8. DOI: https://doi.org/10.1038/ncomms5185.

[26] Dominik Janzing, Pawel Wocjan, Robert Zeier, Rubino Geiss, and Th Beth. "Thermodynamic cost of reliability and low temperatures: tightening Landauer's principle and the second law". *International Journal of Theoretical Physics* 39.12 (2000), 2717–2753. DOI: https://doi.org/10.1023/A:1026422630734.

[27] Michał Horodecki and Jonathan Oppenheim. "Fundamental limitations for quantum and nanoscale thermodynamics". *Nature communications* 4.1 (2013), 1–6. DOI: https://doi.org/10.1038/ncomms3059.

[28] Gilad Gour, Markus P Müller, Varun Narasimhachar, Robert W Spekkens, and Nicole Yunger Halpern. "The resource theory of informational nonequilibrium in thermodynamics". *Physics Reports* 583 (2015), 1–58. DOI: https://doi.org/10.1016/j.physrep.2015.04.003.

[29] Zoë Holmes, Erick Hinds Mingo, Calvin Y-R Chen, and Florian Mintert. "Quantifying athermality and quantum induced deviations from classical fluctuation relations". *Entropy* 22.1 (2020), 111. DOI: https://doi.org/10.3390/e22010111.

[30] Michael A Nielsen. "Conditions for a class of entanglement transformations". *Physical Review Letters* 83.2 (1999), 436. DOI: https://doi.org/10.1103/PhysRevLett.83.436.

[31] Charles H Bennett, Herbert J Bernstein, Sandu Popescu, and Benjamin Schumacher. "Concentrating partial entanglement by local operations". *Physical Review A* 53.4 (1996), 2046. DOI: https://doi.org/10.1103/PhysRevA.53.2046.

[32] Yuval Rishu Sanders and Gilad Gour. "Necessary conditions for entanglement catalysts". *Physical Review A* 79.5 (2009), 054302. DOI: https://doi.org/10.1103/PhysRevA.79.054302.

[33] Rodrigo Gallego and Leandro Aolita. "Resource theory of steering". *Physical Review X* 5.4 (2015), 041008. DOI: https://doi.org/10.1103/PhysRevX.5.041008.

[34] Masaki Owari and Masahito Hayashi. "Two-way classical communication remarkably improves local distinguishability". *New Journal of Physics* 10.1 (2008), 013006. DOI: https://doi.org/10.1088/1367-2630/10/1/013006.

[35] Michael Nathanson. "Three maximally entangled states can require two-way local operations and classical communication for local discrimination". *Physical Review A* 88.6 (2013), 062316. DOI: https://doi.org/10.1103/PhysRevA.88.062316.

[36] Eric Chitambar and Min-Hsiu Hsieh. "Round complexity in the local transformations of quantum and classical states". *Nature communications* 8.1 (2017), 2086. DOI: https://doi.org/10.1038/s41467-017-01887-5.

[37] Eric M Rains. "Entanglement purification via separable superoperators". *arXiv preprint quant-ph/9707002* (1997). DOI: https://doi.org/10.48550/arXiv.quant-ph/9707002.

[38] Francesco Buscemi. "All entangled quantum states are nonlocal". *Physical review letters* 108.20 (2012), 200401. DOI: https://doi.org/10.1103/PhysRevLett.108.200401.

[39] Bob Coecke and Aleks Kissinger. "Picturing quantum processes: A first course on quantum theory and diagrammatic reasoning". *Diagrammatic Representation and Inference: 10th International Conference, Diagrams 2018, Edinburgh, UK, June 18-22, 2018, Proceedings 10*. Springer. 2018, 28–31. DOI: https://doi.org/10.1017/9781316219317.

[40] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011. ISBN: 9781107002173. DOI: https://doi.org/10.1017/CBO9780511976667.

[41] David Schmid, Katja Ried, and Robert W. Spekkens. "Why initial system-environment correlations do not imply the failure of complete positivity: A causal perspective". *Phys. Rev. A* 100 (2 2019), 022112. DOI: https://link.aps.org/doi/10.1103/PhysRevA.100.022112.

[42] Nicolas Brunner and Paul Skrzypczyk. "Nonlocality distillation and postquantum theories with trivial communication complexity". *Physical review letters* 102.16 (2009), 160403. DOI: https://doi.org/10.1103/PhysRevLett.102.160403.

[43] Marco Piani. "Channel steering". *JOSA B* 32.4 (2015), A1–A7. DOI: https://doi.org/10.1364/JOSAB.32.0000A1.

[44] Eric G Cavalcanti, Michael JW Hall, and Howard M Wiseman. "Entanglement verification and steering when Alice and Bob cannot be trusted". *Physical Review A* 87.3 (2013), 032306. DOI: https://doi.org/10.1103/PhysRevA.87.032306.

[45] Ana Belén Sainz, Matty J Hoban, Paul Skrzypczyk, and Leandro Aolita. "Bipartite postquantum steering in generalized scenarios". *Physical Review Letters* 125.5 (2020), 050404. DOI: https://doi.org/10.1103/PhysRevLett.125.050404.

[46] Qiongyi Y He and Margaret D Reid. "Genuine multipartite Einstein-Podolsky-Rosen steering". *Physical Review Letters* 111.25 (2013), 250403. DOI: https://doi.org/10.1103/PhysRevLett.111.250403.

[47] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. "Theoretical framework for quantum networks". *Physical Review A* 80.2 (2009), 022339. DOI: https://doi.org/10.1103/PhysRevA.80.022339.

[48] Roope Uola, Ana CS Costa, H Chau Nguyen, and Otfried Gühne. "Quantum steering". *Reviews of Modern Physics* 92.1 (2020), 015001. DOI: https://doi.org/10.1103/RevModPhys.92.015001.

[49] Matthew F Pusey. "Negativity and steering: A stronger Peres conjecture". *Physical Review A* 88.3 (2013), 032313. DOI: https://doi.org/10.1103/PhysRevA.88.032313.

[50] Nicolas Gisin. "Stochastic quantum dynamics and relativity". *Helvetica Physica Acta* 62.4 (1989), 363–371. DOI: http://doi.org/10.5169/seals-116034.

[51] Lane P Hughston, Richard Jozsa, and William K Wootters. "A complete classification of quantum ensembles having a given density matrix". *Physics Letters A* 183.1 (1993), 14–18. DOI: https://doi.org/10.1016/0375-9601(93)90880-9.

[52] Ana Belén Sainz, Nicolas Brunner, Daniel Cavalcanti, Paul Skrzypczyk, and Tamás Vértesi. "Postquantum steering". *Physical review letters* 115.19 (2015), 190403. DOI: https://doi.org/10.1103/PhysRevLett.115.190403.

[53] Daniel Cavalcanti and Paul Skrzypczyk. "Quantum steering: a review with focus on semidefinite programming". *Reports on Progress in Physics* 80.2 (2016), 024001. DOI: https://doi.org/10.1088/1361-6633/80/2/024001.

[54] Lieven Vandenberghe and Stephen Boyd. "Semidefinite programming". *SIAM review* 38.1 (1996), 49–95. DOI: https://doi.org/10.1137/1038003.

[55] Piotr Mironowicz. "Semi-definite programming and quantum information". *Journal of Physics A: Mathematical and Theoretical* (2023). DOI: https://doi.org/10.1088/1751-8121/ad2b85.

[56] Man-Duen Choi. "Completely positive linear maps on complex matrices". *Linear algebra and its applications* 10.3 (1975), 285–290. DOI: https://doi.org/10.1016/0024-3795(75)90075-0.

[57] Andrzej Jamiołkowski. “Linear transformations which preserve trace and positive semidefiniteness of operators”. *Reports on Mathematical Physics* 3.4 (1972), 275–278. DOI: https://doi.org/10.1016/0034-4877(72)90011-0.

[58] *MATLAB*. 2010. URL: https://www.mathworks.com/.

[59] Michael Grant and Stephen Boyd. *CVX: MATLAB software for disciplined convex programming*. 2013. URL: http://cvxr.com/cvx.

[60] Michael Grant and Stephen Boyd. “Graph implementations for nonsmooth convex programs”. *Recent Advances in Learning and Control*. Ed. by V. Blondel, S. Boyd, and H. Kimura. Lecture Notes in Control and Information Sciences. Springer-Verlag Limited, 2008, 95–110.

[61] K. C. Toh, M. J. Todd, and R. H. Tütüncü. “SDPT3 — A Matlab software package for semidefinite programming, Version 1.3”. *Optimization Methods and Software* 11.1-4 (1999), 545–581. DOI: https://doi.org/10.1080/10556789908805762.

[62] Nathaniel Johnston. *QETLAB: a MATLAB toolbox for quantum entanglement*. 2015. URL: http://qetlab.com.

[63] Beata Zjawin, David Schmid, Matty J. Hoban, and Ana Belén Sainz. DOI: https://doi.org/10.5281/zenodo.10657023.

[64] Cédric Bamps and Stefano Pironio. “Sum-of-squares decompositions for a family of Clauser-Horne-Shimony-Holt-like inequalities and their application to self-testing”. *Physical Review A* 91.5 (2015), 052111. DOI: https://doi.org/10.1103/PhysRevA.91.052111.

[65] Tomáš Gonda and Robert W Spekkens. “Monotones in general resource theories”. *Compositionality: the open-access journal for the mathematics of composition* 5 (2023). DOI: https://doi.org/10.32408/compositionality-5-7.

[66] Márcio M Taddei, Thais de Lima Silva, Ranieri V. Nery, Gabriel Aguilar, Stephen P Walborn, and Leandro Aolita. “Exposure of subtle multipartite quantum nonlocality”. *npj Quantum Information* 7.1 (2021), 1–8. DOI: https://doi.org/10.1038/s41534-021-00402-5.

[67] David Schmid, Haoxing Du, Maryam Mudassar, Ghi Coulter-de Wit, Denis Rosset, and Matty J Hoban. "Postquantum common-cause channels: the resource theory of local operations and shared entanglement". *Quantum* 5 (2021), 419. DOI: https://doi.org/10.22331/q-2021-03-23-419.

[68] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti, and Antonio Acín. "Device-independent entanglement certification of all entangled states". *Physical review letters* 121.18 (2018), 180503. DOI: https://doi.org/10.1103/PhysRevLett.121.180503.

[69] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti, and Antonio Acín. "Self-testing of Pauli observables for device-independent entanglement certification". *Physical Review A* 98.4 (2018), 042336. DOI: https://doi.org/10.1103/PhysRevA.98.042336.

[70] Shin-Liang Chen, Huan-Yu Ku, Wenbin Zhou, Jordi Tura, and Yueh-Nan Chen. "Robust self-testing of steerable quantum assemblages and its applications on device-independent quantum certification". *Quantum* 5 (2021), 552. DOI: https://doi.org/10.22331/q-2021-09-28-552.

[71] Ivan Šupić and Matty J Hoban. "Self-testing through EPR-steering". *New Journal of Physics* 18.7 (2016), 075006. DOI: https://doi.org/10.1088/1367-2630/18/7/075006.

[72] Alexandru Gheorghiu, Petros Wallden, and Elham Kashefi. "Rigidity of quantum steering and one-sided device-independent verifiable quantum computation". *New Journal of Physics* 19.2 (2017), 023043. DOI: https://doi.org/10.1088/1367-2630/aa5cff.

Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages

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Einstein-Podolsky-Rosen (EPR) steering is often (implicitly or explicitly) taken to be evidence for spooky action-at-a-distance. An alternative perspective on steering is that Alice has no causal influence on the physical state of Bob's system; rather, Alice merely updates her knowledge of the state of Bob's system by performing a measurement on a system correlated with his. In this work, we elaborate on this perspective (from which the very term 'steering' is seen to be inappropriate), and we are led to a resource-theoretic treatment of correlations in EPR scenarios. For both bipartite and multipartite scenarios, we develop the resulting resource theory, wherein the free operations are local operations and shared randomness (LOSR). We show that resource conversion under free operations in this paradigm can be evaluated with a single instance of a semidefinite program, making the problem numerically tractable. Moreover, we find that the structure of the pre-order of resources features interesting properties, such as infinite families of incomparable resources. In showing this, we derive new EPR resource monotones. We also discuss advantages of our approach over a pre-existing proposal for a resource theory of 'steering', and discuss how our approach sheds light on basic questions, such as which multipartite assemblages are classically explainable.

1 Introduction

The notion of Einstein-Podolsky-Rosen (EPR) 'steering' [1–3] refers to a form of nonclassical correlations that arise when one considers measurements performed on half of a bipartite system prepared on an entangled state. Such correlations have multiple applications in quantum information [4, 5]; for example, they allow for the certification of entanglement under relaxed assumptions [6, 7], and constitute an information-theoretic resource for various cryptographic tasks [8, 9]. For these reasons, many previous works [3, 10–13] have begun the process of characterizing the resourcefulness of such correlations.

In this paper, we develop a novel resource-theoretic approach to quantifying the nonclassicality of a given correlation in an EPR scenario. Our approach is conceptually motivated by a particular perspective on EPR correlations arising from the language of causal modelling [14, 15]. In this approach, it is assumed that the relevant causal structure contains no directed causal

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influences between the parties that share the quantum system, which implies that 'steering' is not a form of action-at-a-distance, but rather is a form of inference through a nonclassical common cause. This view leads to a 'resource theory of steering' which is distinct from that proposed in previous works [13]. Our approach follows the recent developments of resource theories for studying nonclassicality of common-cause processes wherein the free operations are local operations and shared randomness (LOSR) [16–23].

We now motivate our approach and compare it to prior approaches. Readers already familiar with the conceptual framework of Refs. [20, 21] may wish to skip to Section 1.4, where we summarize our main technical results.

1.1 Reimagining (and renaming) EPR 'steering'

Consider a scenario wherein two parties (Alice and Bob) share a bipartite quantum system prepared in an entangled state, and where Alice then chooses a measurement, performs it on her share of the system, and obtains an outcome. Conditioned on the outcome of the measurement, Alice updates her description of the quantum state that corresponds to Bob's system. Consequently, each of Alice's possible measurements leads to an ensemble of potential updated states of Bob's system together with their associated probabilities of arising. The collection of these ensembles – an 'ensemble of ensembles' – is termed an *assemblage* [10]. Depending on *which* measurement she chooses to carry out, then, Alice chooses from which of these ensembles the quantum state of Bob's system will be drawn from.

Schrödinger considered this dependence of the wavefunction describing Bob's system on Alice's measurement choice to be 'magic' [2], and termed this phenomena *quantum steering*. This terminology is motivated by the idea that Alice's choice of measurement exerts a *causal influence* on Bob's system, even when the two are at an arbitrary distance. If one instead considers the quantum state to be merely a representation of one's *information* about a system, then Alice's ability to update this information conditioned on her measurement outcome is not in itself surprising. When Alice *learns* something about the true state of Bob's system, by virtue of her measurement on a system correlated with it, it is only natural that she would adjust her description of it (as a wavefunction), accordingly.

In this work we do not view 'quantum steering' as evidence for nonlocal causal influences. Since the term 'steering' has an intrinsic bias towards action-at-a-distance, in this manuscript we forgo its use in favor of more neutral terminology.[1] For example, we will refer to a 'steering scenario' as an *EPR scenario*. This attitude towards terminology follows the lines of Ref. [20], which avoided the use of the term 'nonlocality' since it similarly suggests the existence of superluminal causal influences. We will only use the term 'steering' when referring to prior work, and even then we will write it between quotations marks.

With this in mind, without loss of generality, we recast the EPR scenario as follows. Firstly, the fundamental causal structure underpinning an EPR scenario is that depicted in Fig. 1(a), wherein Alice and Bob are connected only by a *common cause*, with no causal influence between them. This is exactly the causal structure suggested by relativity theory. Second, we assume Alice has access to a classical input system and a classical output system, while Bob has access to a quantum output system [2]. A process with these two features defines an assemblage, and can be depicted as in Fig. 1(a).

[1] The importance of thinking clearly about the distinction between causation and inference has been argued (and demonstrated) in many previous works [14, 24–29].

[2] Considering Bob to have access to a quantum system can be thought of as Bob having a characterized (or 'trusted') quantum device.

Figure 1: Depiction of a bipartite EPR scenario. Quantum systems are represented by double lines, while classical systems are depicted as single lines. (a) A generic quantum assemblage requires Alice and Bob to share a quantum common cause. (b) Free assemblages are those which can be generated when Alice and Bob share a classical common cause.

A fundamental question about assemblages in an EPR scenario is which assemblages admit a classical description, that is, one in terms of an underlying classical system that generates the observed correlations and that always takes a definite value. In the next section, we discuss that there exist assemblages which cannot be understood as being prepared with a shared classical random variable while setting up the relevant resource theory of nonclassicality in EPR scenarios.

1.2 The resource theory of nonclassicality in EPR scenarios

We follow the standard approach to quantifying the resourcefulness of given processes, namely, the framework of resource theories [30, 31]. In this approach, one determines two defining features of the resource theory: the *enveloping theory*, which is the set of all possible resources one can produce in the setup; and the *free subtheory* – the set of operations on the resources that are considered to be freely available, as dictated by the physical constraints in the scenario under study. From this, one can directly determine the relative value of any pair of resources: if resource R can be converted to resource S, then R is at least as valuable as S, since (in the context of these free operations) R can clearly be used for any purpose that S can be used for. There are by now many useful resource theories, including those for entanglement (relative to local operations and classical communication [32–34] or relative to local operations and shared randomness [21, 35]), for nonclassicality in Bell scenarios [20] and other common-cause processes [22, 23], for post-quantumness [36–38], for coherence [39–41], for athermality [42–47], and so on.

Historically, the EPR scenario has been viewed as a means of indirectly studying the properties of entangled states [4]. More recently, researchers have begun to study assemblages as processes in and of themselves, since these contain all the relevant information for characterizing an EPR scenario. Here, we will follow this latter approach[3], and so the resources in our

[3]To quantify resources in EPR scenarios, there are three particularly natural options. (i) One may quantify the value of bipartite quantum states, as these are necessary resources for achieving nonclassical assemblages; this has been studied in Ref. [21]. (ii) One may quantify the value of incompatible measurements, as these are also necessary resources for achieving nonclassical assemblages; this resource theory has been studied in Ref. [48]. (iii) One may quantify the value of assemblages as resources in and of themselves. This is in some sense the most direct way of studying resources for nonclassicality, as motivated in the previous paragraph, and this is the sort of resource theory we develop in this paper.

resource theory are taken to be assemblages. We will only consider assemblages which can be realized within quantum theory (as opposed to those which can be realized with post-quantum resources [36, 49, 50]), and so we take our enveloping theory to be given by assemblages that can be generated by quantum common causes as in Fig. 1(a).

The critical step in defining any resource theory is to determine the relevant set of free operations. It has previously been argued that the relevant free operations for studying nonclassicality in common-cause scenarios are given by the set of local operations and shared randomness [20–23]. In this work, we also adopt this approach. In other words, free resources and free transformations on resources are those which can be generated by classical common causes. In doing so, our approach unifies the study of 'steering' resources with resources of 'nonlocality' and (LOSR-)entanglement, showing that these are all simply different manifestations of nonclassicality of common-cause processes.

We now briefly reiterate the basic motivations for taking LOSR as free operations, while specializing these arguments to the case of EPR scenarios. Our choice of free operations is guided by our assumptions about the causal structure, depicted in Fig. 1(a). Firstly, we do not allow Alice or Bob to freely exert causal influences on one another, thus ruling out both classical and quantum communication. We place no limitations on the local quantum operations that each can carry out on the systems in their lab. Finally, as we wish to quantify the *non*classical properties of assemblages, we restrict the set of common causes which are taken to be free to be the *classical* common causes, so that a resource may be valuable only by virtue of having a nonclassical common cause. In other words, we take the set of free operations to include local operations and classical common causes. Since a more common term synonymous to 'classical common cause' is 'shared randomness', we refer to the set of free operations as *local operations and shared randomness*.

From the perspective endorsed in this manuscript, then, the notion of resourcefulness embodied in a useful assemblage is *nonclassicality of its common-cause*. The free (classically explainable) assemblages are all and only those which can be generated by classical common causes, i.e., those of the form shown in Fig. 1(b). As we will see, this class of assemblages coincides exactly with the set of "unsteerable" assemblages—those that admit of so-called "hidden-state models", i.e., classical common-cause explanations. For such assemblages, Bob's system can always be viewed as locally prepared in a state that depends on a classical random variable which always takes a definite classical value, and Alice's choice of measurement can then be viewed as simply providing her information about this classical value, which in turn allows her to refine her description of the state of Bob's quantum system.

In contrast, the common cause generating any nonfree assemblages is necessarily nonclassical (i.e., cannot be viewed as a shared classical random variable), and so Alice's refinements of her knowledge about Bob's system are *not* compatible in this sense. However, there is currently no agreed-upon framework for formalizing Alice's refinements of her knowledge in this sense. Indeed, to fully understand how inferences are made through such a nonclassical common cause would presumably require a nonclassical generalization of the classical theory of inference. While some first attempts at such a program have been made [24, 27, 51], arguably with some success when applied specifically to EPR scenarios, this ambitious problem remains very much unsolved. A convenient feature of our resource theoretic approach is that it does not require us to resolve these difficult issues. This is because the structure of the resource theory is entirely determined by the free set of *classical* common-cause processes, which are well-understood even without such a novel theory of inference.

1.3 Comparison to prior work

A resource theory of 'steering' in bipartite scenarios, distinct from ours, was defined a few years ago [13]. Therein, the free operations were taken to be (stochastic) local operations together with classical communication from Bob to Alice. This set of free operations was motivated by one particular application for which assemblages are known to serve as resources – namely, this set is the most general one that does not compromise the security of one-sided device-independent quantum key distribution protocols. In contrast, our resource theory is developed based on the natural physical limitations arising in any common-cause scenario. As such, our approach follows a recent body of work [20–23] which demonstrates the importance of studying nonclassicality in common-cause scenarios (like the Bell scenario) within a resource-theoretic perspective underpinned by LOSR operations.

We compare these two approaches further in Section 4.1.2. In particular, we note that our approach has the technical advantage that its set of free transformations is simpler to characterize and study. More importantly, our resource theory has the conceptual advantage that it allows for the unification of every type of nonclassical correlation in Bell-like scenarios, since all of these (including entanglement, 'steering', and 'nonlocality') can be viewed as resources of nonclassicality of common-cause processes. We expect this unification to be useful for better understanding the relationships between entanglement, EPR nonclassicality, and Bell nonclassicality, as well as for understanding the possibilities for interconversion between these (and many other [22, 23]) forms of nonclassicality in common-cause scenarios. Furthermore, the principled approach we follow here to distill the essence of nonclassicality allows us to directly and uniquely generalize our framework to the case of multipartite EPR scenarios (which we provide a resource theory for in this work) and to Bob-with-input EPR scenarios [50] and channel EPR scenarios [52][4] (which we provide a resource theory for in a follow-up work).

1.4 Summary of main results

In this paper, we construct a resource theory of bipartite and multipartite 'steering' under LOSR operations. This is the first time a resource-theoretic approach has been applied to the latter scenario. Even within the traditional bipartite scenario, our approach differs from previous approaches to resource theories of 'steering' [13], and clarifies some issues which arose within them.

This paper is divided into three main sections, where we discuss the definition and implications of the LOSR resource-theoretic approach to bipartite EPR scenarios (Section 2), multipartite EPR scenarios (Section 3), and the relation of this resource theory to previous work (Section 4). In Sections 2 and 3, we first recall the definition of the scenario of interest, and explicitly specify its most general LOSR processing of the corresponding EPR assemblage. Next, we show that resource conversion under free operations in the corresponding scenario can be evaluated with a single instance of a semidefinite program. We conclude each of these sections by highlighting properties of the pre-order of resources. In Section 4, we discuss the advantages of our LOSR approach in relation to the problem of choosing the set of free operations in a resource theory (Section 4.1.2) and consistently defining the set of free resources (Sections 4.2 and 4.3).

[4] Motivated by our common-cause description of an EPR scenario, we refer to so-called channel 'steering' scenarios [52] as channel EPR scenarios. Bob-with-input EPR scenarios [50] and channel EPR scenarios [52] are generalizations of the traditional EPR scenario in which Bob is also allowed to have a (classical or quantum) input that further influences the state preparation of his quantum system.

2 The bipartite EPR scenario

In this section, we define the resource theory of nonclassicality of bipartite assemblages under LOSR operations. We begin by formalizing the set of free (LOSR) operations. Then, we explicitly show that a resource conversion in this paradigm can be decided with a single instance of a semidefinite program. This is the first tool that allows one to systematically determine the relative nonclassicality of assemblages, that is, the convertibility relations that hold among them. We use this semidefinite program to study the possible conversions among a family of infinite assemblages. Moreover, we define new EPR measures and study the properties of the pre-order analytically.

2.1 Definition of the scenario and free assemblages

The bipartite EPR scenario (see Fig. 2(a)) consists of two distant parties, Alice and Bob, that share a physical system and perform local actions on it. Alice performs (possibly incompatible) measurements on her subsystem: upon choosing a measurement setting denoted by x, she obtains a classical outcome a with probability $p(a|x)$. We denote by $\mathbb{X}$ the set of classical labels that denote Alice's choice of measurement, and by $\mathbb{A}$ the set of labels for her measurement outcomes. Without loss of generality, we assume all the measurements to have the same outcome cardinality. By measuring her system, Alice refines her knowledge of Bob's system, which is now described by a conditional marginal state $\rho_{a|x}$. The relevant object of study is the ensemble of ensembles of quantum states, dubbed an *assemblage* [10], that contains all the information characterizing an EPR scenario. It is defined as $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\{\sigma_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$, where each unnormalised state $\sigma_{a|x}$ is given by $\sigma_{a|x} := p(a|x)\rho_{a|x}$. That is, the probability that an updated state arises is given by the normalization factor for the corresponding state in the assemblage.

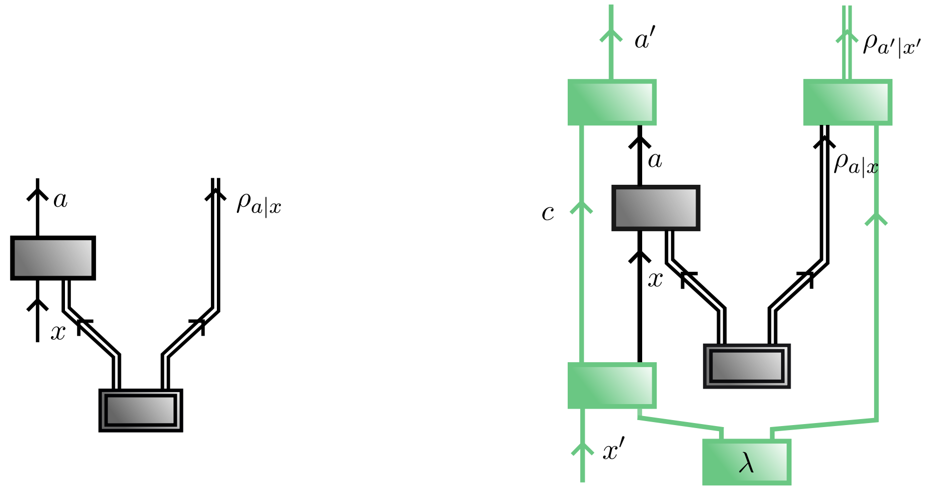
One can depict an assemblage as in Fig. 2(a), where the classical and quantum systems are depicted with single and double lines, respectively. It is worth noticing that, even though we have chosen our enveloping theory to consist only of quantumly-realizable assemblages, the results derived in this section also apply to broader enveloping theories including those generated by arbitrary (e.g. post-quantum) common-causes. In fact, in the bipartite case, there is no difference between the assemblages[5] that can be realized by quantum common causes and those that can be realized by arbitrary common causes. This follows from the GHJW theorem, proven by Gisin [53] and Hughston, Jozsa, and Wootters [54], which (in our causal language) shows that *all* bipartite common-cause assemblages are quantumly-realizable.

In quantum theory, the state of the shared system can be described by a density matrix ρ and Alice implements generalised measurements, i.e., positive operator-valued measures (POVMs), which we denote by $\{\{M_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$. In this case, the unnormalised states admit a quantum realisation of the form $\sigma_{a|x} = \mathrm{tr}_A\{(M_{a|x}\otimes\mathbb{I})\rho\}$. This is formalised as follows:

Definition 1. ***Quantumly-realizable assemblage.***
An assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\{\sigma_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$ has a quantum realisation iff there exists a Hilbert space $\mathcal{H}_A$, a state ρ in $\mathcal{H}_A\otimes\mathcal{H}_B$, and POVMs $\{\{M_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$ on $\mathcal{H}_A$ such that

$$\sigma_{a|x} = \mathrm{tr}_A\{(M_{a|x}\otimes\mathbb{I})\rho\} \tag{1}$$

[5] It is worth noticing that in this definition of an EPR scenario – which is just a generalisation of EPR's original thought experiment – the system in Bob's laboratory remains quantum, or in other words, effectively admits a quantum description. Hence, even if post-quantum common causes are allowed, Bob's system does not require a post-quantum treatment. This is formally the same requirement as when Alice and Bob share a classical common cause: we still describe Bob's system with the language of quantum theory. Of course, one can define other types of experiments, where now Bob's system is allowed to be post-quantum.



(a) (b)

Figure 2: Depiction of a bipartite EPR scenario. Quantum systems are represented by double lines, while classical systems are depicted as single lines. (a) Quantum assemblage: Alice and Bob share a quantum common cause. (b) The most general LOSR operation on an assemblage in a bipartite EPR scenario.

for all $x \in \mathbb{X}$ *and* $a \in \mathbb{A}$*.*

Finally, we consider the question of which assemblages admit a classical explanation: that is, which assemblages can be understood as a classical common-cause process. As argued in the introduction, these are all and only those that can be constructed freely from LOSR operations. Assemblages of this form are depicted in Fig. 1(b). Now, one can associate:

- a probability distribution $p(\lambda)$ with the box representing the state of the common cause (where we imagine that λ is sampled according to this distribution, copied, and shared with both parties),
- a conditional probability distribution $p(a|x, \lambda)$ with the box representing the process by which Alice's outcome is generated (depending on her setting and the value of λ),
- a normalized quantum state ρ_λ with the process by which Bob's quantum state is locally generated (depending on λ).

Hence, the elements of an LOSR-free assemblage can be written as $\sigma_{a|x} = \sum_\lambda p(\lambda)p(a|x, \lambda)\rho_\lambda$, with $p(a|x, \lambda)$ being valid conditional probability distributions for all values of λ, which is the classical variable corresponding to the shared classical common cause. Traditionally, these LOSR-free assemblages have been referred to as 'unsteerable' assemblages and are mathematically expressed in terms of the so-called 'local hidden states' $\sigma_\lambda := p(\lambda)\,\rho_\lambda$[6].

[6]The 'local hidden states' terminology is motivated by the 'local hidden variable' terminology in Bell scenarios. In our view, this terminology is not ideal. Firstly, the states σ_λ are local in the sense that Bob produces them locally, but the word local itself has a connotation in the context of 'quantum nonlocality' and so could be misleading. Secondly, it is irrelevant for the freeness of an assemblage whether or not the state describing Bob's system is 'hidden'. As such, it is better to refer to free resources as 'classical common-cause resources', as this highlights precisely which feature of them is critical to their nonfreeness.

2.2 LOSR transformations between bipartite assemblages

The most general LOSR transformation of an assemblage is illustrated in Fig. 2(b). It consists of a comb (which locally pre- and post-processes the classical variables in Alice's wing) [55] and a completely positive [28, 56] trace preserving (CPTP) map (which post-processes Bob's quantum system). Both operations are correlated by a classical variable λ. Formally, a generic LOSR transformation illustrated in Fig. 2(b) transforms one assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ into a new assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ as follows:

$$\sigma'_{a'|x'} = \sum_{\lambda,c} \sum_{a,x} p(\lambda)\, p(c,x|x',\lambda) p(a|x) p(a'|a,c) \mathcal{E}_\lambda(\rho_{a|x})\,, \tag{2}$$

where

- $p(c,x|x',\lambda)$ encodes the classical pre-processing of Alice's input x as a function of x' and the shared classical randomness λ. Here c denotes the variable to be transmitted through Alice's classical side channel toward the post-processing stage.
- $p(a'|a,c)$ encodes the classical post-processing of Alice's output a, as a function of the classical information c kept from the pre-processing stage. The output of the process is Alice's new outcome a'.
- $\mathcal{E}_\lambda[\cdot]$ is the CPTP map corresponding to Bob's local post-processing of his quantum system, as a function of the shared classical randomness λ.

Notice that if $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ is free, then $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ is free as well – that is, the set of free assemblages is closed under LOSR operations. This highlights the basic property of a resource theory: applying a free transformation on a free resource cannot create a resourceful object. Any assemblage that cannot be realised by local operations and shared randomness is nonfree and constitutes a resource of LOSR nonclassicality.

The local pre- and post-pocessings of Alice's classical variables are represented by the product $p(c,x|x',\lambda)\, p(a'|a,c)$ in Eq. (2). Notice, however, that the classical variable c is a function only of the classical values x' and λ, and that we have no constraints on the dimension of the classical variable c. Therefore, Eq. (2) can be written as

$$\sigma'_{a'|x'} = \sum_{\lambda} \sum_{a,x} p(\lambda)\, p(a'|a,x',\lambda)\, p(x|x',\lambda)\, p(a|x)\, \mathcal{E}_\lambda(\rho_{a|x})\,. \tag{3}$$

Notice that this expression satisfies the condition of no-retrocausation – the variable a cannot influence the value of the variable x.

A final remark pertains to a particular way to express a generic LOSR transformation, based on Fine's argument [57] and discussed in Ref. [20]. In the central point of this expression is the fact that the set of LOSR operations is convex and its extremal elements are deterministic [20]. This implies that Alice's pre- and post-processing can be decomposed as a convex combination of deterministic operations:

$$p(a'|a,x',\lambda)\, p(x|x',\lambda) = \sum_{\tilde{\lambda}} p(\tilde{\lambda}|\lambda) D(a'|a,x',\tilde{\lambda})\, D(x|x',\tilde{\lambda}). \tag{4}$$

Here, $D(a'|a,x',\tilde{\lambda})$ assigns a fixed outcome a' for each possible choice of a, x', and $\tilde{\lambda}$, i.e., $D(a'|a,x',\tilde{\lambda}) = \delta_{a',f_{\tilde{\lambda}}(a,x')}$. Similarly, $D(x|x',\tilde{\lambda})$ assigns a fixed outcome x for each measurement x' and value of $\tilde{\lambda}$, i.e., $D(x|x',\tilde{\lambda}) = \delta_{x,g_{\tilde{\lambda}}(x')}$. Let us define a new completely positive and trace non-increasing (CPTNI) map

$$\tilde{\mathcal{E}}_{\tilde{\lambda}}(\sigma_{a|x}) = \sum_{\lambda} p(\lambda) p(\tilde{\lambda}|\lambda) \mathcal{E}_\lambda(p(a|x)\rho_{a|x}). \tag{5}$$

Notice that $\sum_{\tilde{\lambda}} \tilde{\mathcal{E}}_{\tilde{\lambda}}(\sigma_{a|x})$ forms a CPTP map. We are now in the position to rewrite Eq. (3) as follows

$$\sigma'_{a'|x'} = \sum_{\tilde{\lambda}} \sum_{a,x} D(a'|a, x', \tilde{\lambda})\, D(x|x', \tilde{\lambda}) \tilde{\mathcal{E}}_{\tilde{\lambda}}(\sigma_{a|x}), \tag{6}$$

which yields the simplified characterisation of a generic LOSR transformation that we will use throughout.

2.3 A semidefinite test for deciding resource conversions

An assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ can be converted into a different assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations if and only if there exist a collection of CPTNI maps $\tilde{\mathcal{E}}_{\lambda}$ such that $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ can be decomposed as in Eq. (6). Therefore, deciding whether $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations is equivalent to checking whether this decomposition is possible. We will now show that this can be decided with a single instance of a semidefinite program (SDP).

Recall that due to Choi-Jamiołkowski isomorphism [58, 59] every CPTP map $\mathcal{E} : \mathcal{H}_B \to \mathcal{H}_{B'}$ can be associated with an operator W on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ such that $\mathcal{E}(\rho_B) = d_B \operatorname{tr}_B \left\{ W \left(\mathbb{I}_{B'} \otimes \rho_B^T \right) \right\}$, where d_B is the dimension of the system ρ_B. Conversely, the operator W can be written as $W = (\mathcal{E} \otimes \mathbb{I}_{B'}) \left|\Omega\right\rangle \left\langle\Omega\right|$, with $\left|\Omega\right\rangle = \frac{1}{\sqrt{d_B}} \sum_{i=1}^{d_B} \left|ii\right\rangle$. This isomorphism is crucial for reformulating our problem as an SDP. For Eq. (6) to hold, each $\sigma'_{a'|x'}$ must admit the following decomposition

$$\sigma'_{a'|x'} = \sum_{\lambda} \sum_{a,x} D(a'|a, x', \lambda)\, D(x|x', \lambda)\, d_B \operatorname{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma_{a|x}^T \right) \right\} . \tag{7}$$

Here, the map $\tilde{\mathcal{E}}_\lambda(\sigma_{a|x})$ is written in the operator form with W_λ being the Choi state, i.e., $\tilde{\mathcal{E}}_\lambda(\sigma_{a|x}) = d_B \operatorname{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma_{a|x}^T \right) \right\}$, with W_λ being a $(d_B \times d_{B'})$ by $(d_B \times d_{B'})$ matrix. Notice also that Eq. (7) involves only finite sums, since the variable λ enumerates the finitely-many deterministic distributions $D(a'|a, x', \lambda)$ and $D(x|x', \lambda)$. The total number of the deterministic strategies encoded in λ is equal to $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|}$. This correspondence between elements $\sigma'_{a'|x'}$ and $\sigma_{a|x}$ enables us to construct an SDP that checks whether $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations:

SDP 2. $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$.
The assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ *can be converted into the assemblage* $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ *under LOSR operations, denoted by* $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$*, if and only if the following SDP is feasible:*

$$\begin{aligned} &\textit{given} \quad \{\{\sigma_{a|x}\}_a\}_x\,,\ \{\{\sigma'_{a'|x'}\}_{a'}\}_{x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\ &\textit{find} \quad \{(W_\lambda)_{BB'}\}_\lambda \\ &\textit{s.t.} \quad \begin{cases} W_\lambda \geq 0\,, \\ \operatorname{tr}_{B'}\{W_\lambda\} \propto \frac{1}{d}\,\mathbb{I}_B \quad \forall \lambda\,, \\ \sum_\lambda \operatorname{tr}_{B'}\{W_\lambda\} = \frac{1}{d}\,\mathbb{I}_B\,, \\ \sigma'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, d_B \operatorname{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma_{a|x}^T \right) \right\} . \end{cases} \end{aligned} \tag{8}$$

When the conversion is not possible, we denote it by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$.

SDP 2 is a feasibility problem, i.e., it checks if the feasible set is equal to the empty set. If this is the case, the primal optimal value is equal to $-\infty$, which means that there exists no set $\{W_\lambda\}_\lambda$ that satisfies the constraints specified by Eq. (6). If the feasible set is not equal to the empty set, the optimal value is equal to zero, and the problem is feasible. Therefore, checking whether $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations requires a single instance of an SDP.

2.4 Properties of the pre-order

When one assemblage can be freely converted into another, then the former is said to be *at least as nonclassical* as the latter. Therefore, the pre-order of assemblages gives information about their relative nonclassicality. In this section, we study the nonclassicality among members of an infinite family of assemblages defined below.

Consider an EPR scenario where $\mathbb{A} = \mathbb{X} = \{0, 1\}$, and Bob's dimension is 2. Imagine Alice and Bob share an entangled state of the form $|\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, where $\theta \in (0, \pi/4]$. The measurements that Alice performs on her share of the entangled state are given by $\widetilde{M}_{a|0} = \frac{1}{2}\{\mathbb{I} + (-1)^a\sigma_z\}$ and $\widetilde{M}_{a|1} = \frac{1}{2}\{\mathbb{I} + (-1)^a\sigma_x\}$, with σ_z and σ_x being Pauli matrices. Then, the assemblage elements can be written as

$$\sigma^\theta_{a|x} = \mathrm{tr}_{\mathrm{A}}\left\{\widetilde{M}_{a|x} \otimes \mathbb{I}\,|\theta\rangle\,\langle\theta|\right\}. \tag{9}$$

This infinite family of assemblages is indexed by one parameter – the angle $\{\theta\}$. We will now introduce one more parameter to this family. The new parameter $\{p\}$ is responsible for mixing the elements $\sigma^\theta_{a|x}$ with noise. Let us define a family of assemblages S as:

$$\mathsf{S} = \left\{\mathbf{\Sigma}^{\theta,p}_{\mathbb{A}|\mathbb{X}}\,\middle|\,\theta \in (0, \pi/4]\,, p \in [0,1]\right\}\,, \tag{10}$$
$$\text{where} \quad \mathbf{\Sigma}^{\theta,p}_{\mathbb{A}|\mathbb{X}} = \left\{\left\{p\,\sigma^\theta_{a|x} + (1-p)\,\frac{\mathbb{I}}{4}\right\}_{a\in\mathbb{A}}\right\}_{x\in\mathbb{X}}.$$

This family of assemblages has an infinite number of elements. Each element is indexed by two parameters – the angle and the probability, $\{\theta, p\}$ respectively. We will sometimes focus on a subset of S with a fixed value of p. In such cases, for $p = \varrho$, we define $\mathsf{S}_\varrho = \left\{\mathbf{\Sigma}^{\theta,\varrho}_{\mathbb{A}|\mathbb{X}}\,\middle|\,\theta \in (0, \pi/4]\,, p = \varrho\right\}$. For example, for $p = 1$, we have the following family of assemblages:

$$\mathsf{S}_1 = \left\{\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}\,\middle|\,\theta \in (0, \pi/4]\,, p = 1\right\}\,, \tag{11}$$
$$\text{where} \quad \mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}} = \left\{\left\{\sigma^\theta_{a|x}\right\}_{a\in\mathbb{A}}\right\}_{x\in\mathbb{X}}.$$

For simplicity, we denote $\mathbf{\Sigma}^{\theta,p}_{\mathbb{A}|\mathbb{X}} = \mathbf{\Sigma}^{\theta,p}$ in this section.

We now study the properties of the pre-order among the resources in S. First, we run the SDP 2 and test which conversions between resources parametrized by different values of $\{\theta, p\}$ are possible. Second, in Section 2.4.2, we focus on S_1 and confirm the results obtained from the SDP analytically.

To test whether two assemblages in S can be converted into each other, we check if they satisfy the constraints specified by Eq. (2). The solutions to the SDP (computed in Matlab [60], using the software CVX [61, 62], the solver SDPT3 [63] and the toolbox QETLAB [64]; see the code at [65]) are illustrated in Fig. 3. In this figure, each dot represents one assemblage $\mathbf{\Sigma}^{\theta,p}$. For example, the point indexed by $\{\theta = \pi/6, p = 0.9\}$ corresponds to

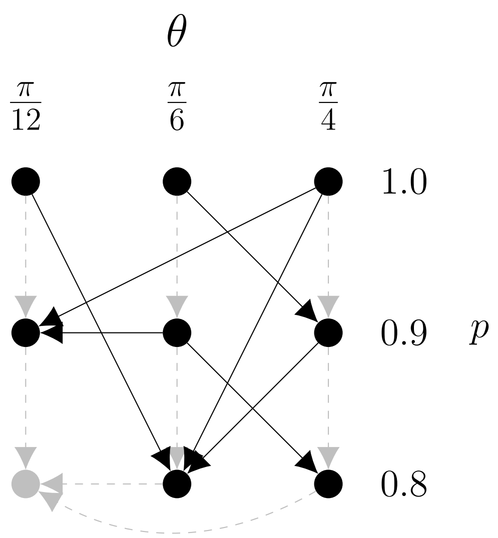


Figure 3: Possible conversions between elements of S. The black dots represent the assemblages $\mathbf{\Sigma}^{\theta,p}$. The arrows represent possible conversions.

$\mathbf{\Sigma}^{\pi/6,0.9} = \left\{ \left\{ 0.9\, \sigma_{a|x}^{\pi/6} + 0.1\, \frac{\mathbb{I}}{4} \right\}_{a\in\mathbb{A}} \right\}_{x\in\mathbb{X}}$. The black dots correspond to nonfree assemblages, and the grey dot represents a free assemblage. Note that the assemblage $\mathbf{\Sigma}^{\pi/4,1}$ is one obtained when Alice and Bob share a quantum system prepared in a 'maximally entangled'[7] state, and Alice performs two suitable Pauli measurements in her share of the system. The arrows represent possible conversions, e.g., the arrow pointing from $\mathbf{\Sigma}^{\pi/4,1}$ to $\mathbf{\Sigma}^{\pi/6,0.8}$ means that $\mathbf{\Sigma}^{\pi/4,1}$ can be converted into $\mathbf{\Sigma}^{\pi/6,0.8}$ under LOSR operations. The grey, dashed lines represent trivial conversions. For the sake of simplicity, Fig. 3 represents only nine assemblages, which is already sufficient to illustrate some interesting features of the pre-order among assemblages in S.

Fig. 3 displays some elements of S_1 that are unordered in our LOSR resource theory of assemblages; e.g., notice that no conversions are possible among assemblages where $p = 1$. In contrast, some conversions *are* possible among assemblages for which $p = 0.9$, and among assemblages for which $p = 0.8$. Moreover, this figure is not symmetric: conversions of assemblages with higher value of θ to those with lower value of θ are more common. For example, $\mathbf{\Sigma}^{\pi/4,1}$ can be converted into $\mathbf{\Sigma}^{\pi/12,0.9}$, but $\mathbf{\Sigma}^{\pi/12,1}$ cannot be converted into $\mathbf{\Sigma}^{\pi/4,0.9}$. Finally, all arrows point in one direction only, suggesting that there are no equivalent assemblages in this family.

Notice that we used the SDP 2 to study the pre-order of assemblages in S, where $\mathbb{A} = \mathbb{X} = \{0, 1\}$, and Bob's dimension is 2. However, the SDP is not constrained to such simple examples. In principle, resource conversion can be evaluated via the SDP 2 for families of assemblages with an arbitrary cardinality of $\mathbb{A}$ and $\mathbb{X}$, and arbitrary Bob's dimension.

2.4.1 EPR monotones

A common approach to unraveling the pre-order of resources in a resource theory is by employing so-called *resource monotones*. Formally, a resource monotone is a function which is monotonic under free operations. Resource monotones enable comparison of resources and identification of

[7] Here, by a 'maximally entangled' quantum state we mean one where entanglement is quantified as per a resource theory based on local operations and classical communication as the free operations [21, 66], which is the standard approach. In the case of two qubits this may be the singlet state.

conversion relations and equivalence classes. That is, if the real numbers that resource monotone M assigns to the pair of resources R_1 and R_2 satisfy $M(R_1) < M(R_2)$, then one can conclude that the monotone M witnesses that $R_1 \not\longrightarrow R_2$. Strictly speaking, hence, a resource monotone usually gives only partial information about the pre-order of resources, and one generically may need a collection of resource monotones (called a *complete set* of them) to have the complete information to fully specify the pre-order of resources based on them. Nevertheless, resource monotones are friendly ways to explore the pre-order of resources, and the real numbers in their co-domain are interpreted as a (possibly incomplete) *quantification* of the nonfreeness of the resources in its domain.

In the particular case of our resource theory, an EPR monotone is a function from the space of assemblages into real numbers, whose value does not increase under LOSR operations. Among the existing measures of bipartite assemblages, the *'steerable' weight* [11], *robustness of 'steering'* [12], and *relative entropy* [13] were shown to be EPR monotones in a resource theory of 'steering' with stochastic local operations assisted by one-way classical communication (S-1W-LOCC) being the set of free operations [13]. The set of S-1W-LOCC operations allows, in particular, for one party to generate randomness and share it with the second party by classical communication; hence, the set of S-1W-LOCC operations strictly includes the set of LOSR operations. It follows that all EPR monotones defined for S-1W-LOCC are therefore also monotones for LOSR. It has moreover been shown that the first two of these monotones – 'steerable' weight and robustness of 'steering' – can be reformulated in a type-independent form, i.e., they can be used to compare the LOSR-nonclassicality of resources of arbitrary types (where an EPR assemblage is one specific type of a resource) [23]. Let us now recall their formal definitions for the case of study in this section.

Definition 3 ('Steerable' weight [11])**.** *The 'steerable' weight of an assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ is given by the minimum μ such that $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ can be decomposed as*

$$\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \mu\, \mathbf{\Sigma}^{S}_{\mathbb{A}|\mathbb{X}} + (1-\mu)\, \mathbf{\Sigma}^{\text{free}}_{\mathbb{A}|\mathbb{X}}, \tag{12}$$

where $\mathbf{\Sigma}^{S}_{\mathbb{A}|\mathbb{X}}$ is an arbitrary nonfree assemblage and $\mathbf{\Sigma}^{\text{free}}_{\mathbb{A}|\mathbb{X}}$ is an arbitrary free assemblage.

We provide the following intuition for this definition. Imagine Alice wants to prepare a given assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ using the minimal amount of a nonfree resource. To achieve her goal, she prepares a free assemblage $\mathbf{\Sigma}^{\text{free}}_{\mathbb{A}|\mathbb{X}}$ most of the time (that is, a $1-\mu$ fraction of the rounds), and sometimes (a μ fraction of the rounds), she prepares a nonfree assemblage $\mathbf{\Sigma}^{S}_{\mathbb{A}|\mathbb{X}}$. On average, hence she prepares $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$. The 'steerable' weight quantifies the minimal amount of $\mathbf{\Sigma}^{S}_{\mathbb{A}|\mathbb{X}}$ needed to generate the desired assemblage in this way.

Definition 4 (Robustness of 'steering' [12])**.** *The 'steering' robustness of an assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ is given by the minimum ν such that the assemblage*

$$\mathbf{\Sigma}^{\text{free}}_{\mathbb{A}|\mathbb{X}} = \frac{1}{1+\nu}\, \mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} + \frac{\nu}{1+\nu}\, \mathbf{\Sigma}'_{\mathbb{A}|\mathbb{X}} \tag{13}$$

is free, with $\mathbf{\Sigma}'_{\mathbb{A}|\mathbb{X}}$ being an arbitrary assemblage.

Robustness of 'steering' can be understood as follows. Imagine Alice holds an assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ and she wants to make it free by mixing it with some other assemblage $\mathbf{\Sigma}'_{\mathbb{A}|\mathbb{X}}$. Robustness of 'steering' quantifies the minimal amount of mixing needed to make $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ a free assemblage. If one optimises $\mathbf{\Sigma}'_{\mathbb{A}|\mathbb{X}}$ over all assemblages, this leads to mixing $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ with worst-case noise. If one restricts the type of noise that can be added to the original assemblage, this measure is referred to as generalized robustness or random robustness, depending on the type of noise added.

We conjecture that relative entropy could be defined in a type-independent way as well, as similar measures exist for states and Bell scenarios [18].

2.4.2 New EPR monotones

We now develop a method for obtaining EPR monotones from Bell inequalities and use it to construct a family of monotones that certify the incomparability of elements of S_1. We only focus on S_1 in this section, hence we drop the index $p = 1$ and denote $\mathbf{\Sigma}^{\theta,1} = \mathbf{\Sigma}^{\theta}$.

To prove that the elements of S_1 are unordered as per the LOSR resource theory of assemblages, it suffices to find a set of EPR LOSR monotones $\mathsf{M} = \{\mathrm{M}_j\}$ such that, for every pair (θ_1, θ_2) there exists a pair $(\mathrm{M}_{\theta_1}, \mathrm{M}_{\theta_2})$ with the following properties:

$$\begin{cases} \mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_1}) > \mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_2}) & \text{– which implies } \mathbf{\Sigma}^{\theta_2} \overset{\text{LOSR}}{\not\longrightarrow} \mathbf{\Sigma}^{\theta_1} \,, \\ \mathrm{M}_{\theta_2}(\mathbf{\Sigma}^{\theta_1}) < \mathrm{M}_{\theta_2}(\mathbf{\Sigma}^{\theta_2}) & \text{– which implies } \mathbf{\Sigma}^{\theta_1} \overset{\text{LOSR}}{\not\longrightarrow} \mathbf{\Sigma}^{\theta_2} \,. \end{cases} \tag{14}$$

To construct such a family of monotones, we will make use of the EPR inequalities [3] constructed from the Bell inequalities presented in Ref. [67, Eq. (1)]. These Bell inequalities, which we recall in the Appendix A, are uniquely maximized by the states $|\theta\rangle$ (via a self-testing result [67]). In the Appendix B, we use these inequalities to construct an EPR functional, which we denote $S_\eta[\mathbf{\Sigma}]$. This functional defines a family of EPR inequalities that are uniquely maximized by assemblages living in S_1. For a formalization of this statement and definition of $S_\eta[\mathbf{\Sigma}]$, see Appendix B and Definition 15 therein.

We construct the monotone M_η from the EPR functional $S_\eta[\mathbf{\Sigma}]$ following a yield-based construction:

Definition 5. *The resource monotone* M_η*, for* $\eta \in (0, \pi/4]$*, is defined as*

$$\mathrm{M}_\eta[\mathbf{\Sigma}] := \max_{\widetilde{\mathbf{\Sigma}}} \{ S_\eta[\widetilde{\mathbf{\Sigma}}] : \mathbf{\Sigma} \xrightarrow{\text{LOSR}} \widetilde{\mathbf{\Sigma}} \}. \tag{15}$$

We will now show that each of the monotones $\mathrm{M}_\eta[\mathbf{\Sigma}]$, when evaluated on assemblages in S_1, is uniquely maximized by $\mathbf{\Sigma}^\eta$. First, note that

$$\mathrm{M}_\eta[\mathbf{\Sigma}] \leq S_\eta[\mathbf{\Sigma}^\eta] \quad \forall\, \eta \in (0, \pi/4] \,, \tag{16}$$

since all assemblages in this traditional bipartite scenario admit a quantum realisation. Hence, $S_\eta[\widetilde{\mathbf{\Sigma}}]$ is upperbounded by its maximum quantum violation, which is given by $S_\eta[\mathbf{\Sigma}^\eta]$. Moreover,

$$\mathrm{M}_\eta[\mathbf{\Sigma}^\eta] = S_\eta[\mathbf{\Sigma}^\eta] \,. \tag{17}$$

We now show that that equality in Eq. (17) only holds when M_η and S_η are evaluated on the same assemblage.

Theorem 6. *Let* M_η *be an EPR monotone from Definition 5 and* S_η *be an EPR functional given in Definition 15. Then, if* $\theta \neq \eta$, $\mathrm{M}_\eta[\mathbf{\Sigma}^\theta] < S_\eta[\mathbf{\Sigma}^\eta]$.

Proof. Let us prove this by contradiction. Our starting assumption is that there exists a pair $(\theta\,, \eta)$ with $\theta \neq \eta$, such that $\mathrm{M}_\eta[\mathbf{\Sigma}^\theta] = S_\eta[\mathbf{\Sigma}^\eta]$. Then, one of the two should happen:

First case: $\mathrm{M}_\eta[\mathbf{\Sigma}^\theta] = S_\eta[\mathbf{\Sigma}^\theta]$.
In this case, the solution to the maximisation problem in the computation of M_η is achieved by $\mathbf{\Sigma}^\theta$ itself.
Our starting assumption then tells us that $S_\eta[\mathbf{\Sigma}^\theta] = S_\eta[\mathbf{\Sigma}^\eta] = S_\eta^{\max}$.
From Remark 17 (see Appendix B) it follows that necessarily $\theta = \eta$, which contradicts our initial condition.

Second case: $\mathrm{M}_\eta[\mathbf{\Sigma}^\theta] = S_\eta[\widetilde{\mathbf{\Sigma}}]$, with $\mathbf{\Sigma}^\theta \xrightarrow{\text{LOSR}} \widetilde{\mathbf{\Sigma}}$.
In this case, the solution to the maximisation problem in the computation of M_η is achieved by an LOSR processing of $\mathbf{\Sigma}^\theta$.
Our starting assumption then tells us that $S_\eta[\widetilde{\mathbf{\Sigma}}] = S_\eta[\mathbf{\Sigma}^\eta] = S_\eta^{\max}$.
Let ρ and $\{\{M_{a|x}\}_{a\in A}\}_{x\in X}$ be any quantum state and measurements that realise the quantum assemblage $\widetilde{\mathbf{\Sigma}}$. From Remark 18 (see Appendix B), we know that ρ is equivalent to $|\eta\rangle$ up to local isometries. But since local isometries are free LOSR operations, this means that $|\theta\rangle$ is more LOSR-entangled[8] than $|\eta\rangle$, which contradicts the result of Ref. [21] that all two-qubit pure entangled states are LOSR-inequivalent.

□

We can now prove that S_1 is composed of pairwise unordered resources by identifying a pair of monotones that satisfies Eqs. (14). As we see next, this is achieved by choosing $\mathrm{M}_j = \mathrm{M}_{\theta_j}$.

Theorem 7. *For every pair $\theta_1 \neq \theta_2$, the monotones M_{θ_1} and M_{θ_2} given by Definition 5 satisfy*

$$\mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_1}) > \mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_2})\,, \tag{18}$$
$$\mathrm{M}_{\theta_2}(\mathbf{\Sigma}^{\theta_1}) < \mathrm{M}_{\theta_2}(\mathbf{\Sigma}^{\theta_2})\,. \tag{19}$$

Proof. Let us first prove Eq. (18). On the one hand, $\mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_1}) = S_{\theta_1}[\mathbf{\Sigma}^{\theta_1}]$. On the other hand, since $\theta_1 \neq \theta_2$, Theorem 6 implies that $\mathrm{M}_{\theta_1}(\mathbf{\Sigma}^{\theta_2}) < S_{\theta_1}[\mathbf{\Sigma}^{\theta_1}]$. Therefore, Eq. (18) follows.
The proof of Eq. (19) follows similarly. □

Corollary 8. *The infinite family of EPR monotones $\mathsf{M} = \{\mathrm{M}_\eta \,|\, \eta \in (0, \pi/4]\}$ certifies that the infinite family of assemblages S_1 is composed of pairwise unordered resources.*

We defined new measures of EPR assemblages and showed that they certify that the elements of S_1 are unordered in the resource theory of assemblages under LOSR operations. As we showed using the SDP 2, this property of the pre-order appears to be unique to assemblages $\mathbf{\Sigma}^{\theta,p}$ with $p = 1$.

3 The multipartite EPR scenario

We now develop a resource theory for multipartite EPR scenarios under LOSR operations. This is the first time multipartite EPR scenarios have been studied in a resource-theoretic framework. Similarly to the bipartite case, we show that resource conversion can be decided with a single instance of an SDP in this paradigm. Moreover, we define new measures of multipartite nonclassicality and use them to analytically study the properties of the pre-order. Our results show that multipartite scenarios are easier and more natural to understand from an LOSR perspective rather than an LOCC one; we elaborate on this in Section 4.1.2.

In this paper, we focus on one particular type of multipartite EPR scenario with multiple Alices and one Bob. However, the number of quantum parties (Bobs) can also be increased. Indeed, one possible multipartite setup consists of a single Alice and multiple Bobs. In such a scenario, results from bipartite scenarios may sometimes directly generalise. However, this generalisation might not be straight-forward and a detailed analysis of this multipartite scenario is beyond the scope of this work. For a description of a scenario with multiple Bobs, we refer the reader to Refs. [6, 68]. Focusing on scenarios with more-than-one Alice is however necessary

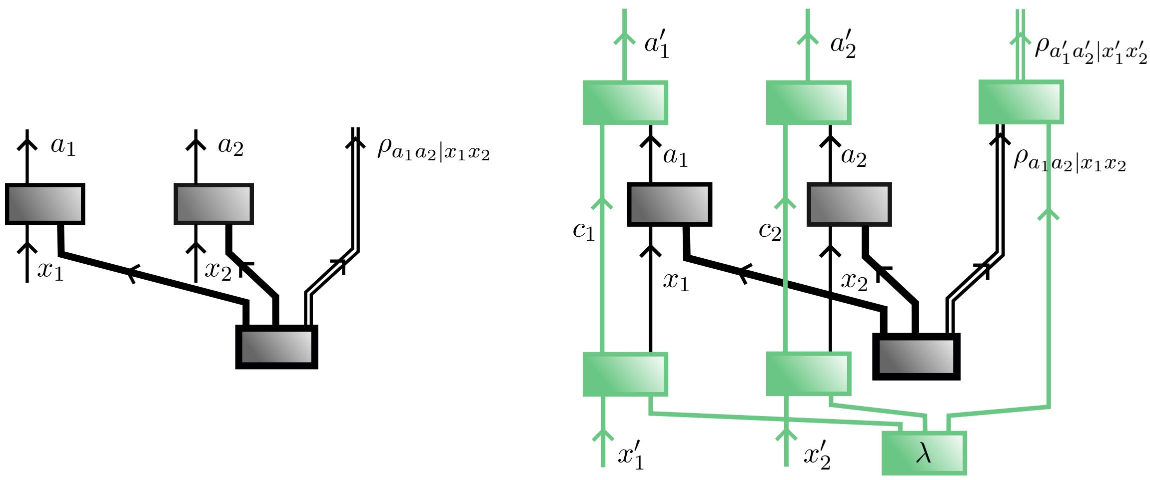
[8]We say that a state $|\theta\rangle$ is more LOSR-entangled than a different state $|\eta\rangle$ if $|\theta\rangle$ can be converted freely to $|\eta\rangle$ with LOSR operations.

when one wants to allow for post-quantum assemblages, which is crucial when exploring ways to single out quantum phenomena from basic principles – a.k.a. studying quantum ‘from the outside’.

3.1 Definition of the scenario and free assemblages

In the multipartite EPR scenario of interest to us, $k+1$ separated parties share a physical system. In analogy to the bipartite scenario, we consider k parties called Alices that hold measurement devices. Each Alice, labeled by $A_{i\in\{1...k\}}$, decides on a classical input x_i from the set $x_i \in \{1, ..., m_A\} =: \mathbb{X}$ and generates a classical outcome a_i from the set $a_i \in \{0, ..., o_A - 1\} =: \mathbb{A}$ with probability $p^i(a_i|x_i)$. Without loss of generality, we will take the sets $\mathbb{X}$ and $\mathbb{A}$ to be the same for all the Alices. When Alices perform the measurements on their subsystems, they update their knowledge about Bob's subsystem which is now described by a conditional marginal state $\rho_{a_1...a_k|x_1...x_k}$, which depends on all Alices inputs and outputs. In this scenario, the elements of the assemblage $\mathbf{\Sigma}_{\mathbb{A}_1...\mathbb{A}_k|\mathbb{X}_1...\mathbb{X}_k} = \{\{\sigma_{a_1...a_k|x_1...x_k}\}_{a_1...a_k}\}_{x_1...x_k}$ are given by $\sigma_{a_1...a_k|x_1...x_k} := p(a_1...a_k|x_1...x_k)\rho_{a_1...a_k|x_1...x_k}$. The multipartite EPR scenario is illustrated in Fig. 4(a) for $k = 2$ (two Alices, one Bob).

In this paper, we are interested in studying the common-cause processes between multiple parties, hence we focus on the global object given by the assemblage, and not on the individual agents. Nevertheless, we would like to note that it is always possible to recover the individual parties by tracing out the irrelevant subsystems. It would be interesting to study what inferences each Alice can make about the other parties. However, this problem is related to the quantum marginal problem [69] and is beyond the scope of our framework.



(a) (b)

Figure 4: Depiction of a multipartite EPR scenario for $k = 2$ (two Alices, one Bob). Quantum systems are depicted by double lines, classical systems by single lines. A system that may be classical or quantum is depicted by a thick line. (a) Quantum assemblage: Bob receives a quantum system, and the Alices receive systems that may be quantum or classical. Each Alice performs measurements labeled by classical variables x_1 and x_2, and obtains classical measurement outputs a_1 and a_2. (b) The most general LOSR operation on a quantum assemblage in a multipartite EPR scenario.

In a quantum common-cause multipartite EPR scenario, then, Bob and the Alices share a quantum system, and for each $i \in \{1, \ldots, k\}$, the i-th Alice is allowed to perform on her share of the system generalised measurements represented by the set $\{\{M_{a_i|x_i}\}_{a_i}\}_{x_i}$. Then, Bob's unnormalised states can be realised as $\sigma_{a_1 \ldots a_k|x_1 \ldots x_k} = \mathrm{tr}_{A_1 \ldots A_k}\{(M_{a_1|x_1} \otimes \ldots \otimes M_{a_k|x_k} \otimes \mathbb{I})\rho\}$, where ρ is the quantum state of the shared system. These quantumly-realised assemblages are formalised as follows:

Definition 9. ***Multipartite quantum assemblage.***
An assemblage $\mathbf{\Sigma}_{\mathbb{A}_1 \ldots \mathbb{A}_k|\mathbb{X}_1 \ldots \mathbb{X}_k}$ has a quantum realisation iff there exist Hilbert spaces $\mathcal{H}_{A_i}$ with $i \in \{1, \ldots, k\}$, a state ρ in $\mathcal{H}_{A_1} \otimes \ldots \otimes \mathcal{H}_{A_k} \otimes \mathcal{H}_B$, and a POVM $\{M_{a_i|x_i}\}_{a_i \in \mathbb{A}}$ on $\mathcal{H}_{A_i}$ for each $x_i \in \mathbb{X}$ and $i \in \{1, \ldots, k\}$, such that

$$\sigma_{a_1 \ldots a_k|x_1 \ldots x_k} = tr_{A_1 \ldots A_k}\{(M_{a_1|x_1} \otimes \ldots \otimes M_{a_k|x_k} \otimes \mathbb{I})\rho\} \tag{20}$$

for all $a_1, \ldots, a_k \in \mathbb{A}$ and $x_1, \ldots, x_k \in \mathbb{X}$.

In analogy to the bipartite case, a multipartite assemblage is LOSR-free if the parties (Alices and Bob) can generate it using local operations (classical and quantum) and classical shared randomness. The elements of an LOSR-free assemblage can be decomposed as $\sigma_{a_1 \ldots a_k|x_1 \ldots x_k} = \sum_\lambda p(\lambda) p^1(a_1|x_1, \lambda) \ldots p^k(a_k|x_k, \lambda)\, \sigma_\lambda$, where $p^j(a_j|x_j, \lambda)$ is a conditional probability distribution for the j-th Alice for every λ, and σ_λ are unnormalised quantum states which, similarly to the bipartite case, satisfy $\mathrm{tr}\left\{\sum_\lambda \sigma_\lambda\right\} = 1$. Notice that this definition of a free multipartite assemblage coincides with the definition of an "unsteerable multipartite assemblage" in Refs. [6, 68, 70], in the sense of "fully unsteerable"[9]. Therefore our LOSR underpinning of the resource-theoretic understanding of assemblages coincides with what people in the literature understand as a "totally-useless assemblage". Notice, however, that unlike in Ref. [71], we do not render unsteerable assemblages where the correlations observed among the Alices' measurement outputs are non-classical. In Section 4.2, we review different definitions of a multipartite free assemblage and compare them to our approach.

3.2 LOSR transformations between multipartite assemblages

The most general LOSR transformation of a multipartite assemblage consists of a comb for each Alice (which locally pre- and post-processes the variables on each Alice's wing), and a CPTP map (which post-processes Bob's quantum system), with all these actions being coordinated by a classical random variable λ. Formally, a generic LOSR operation is illustrated in Fig. 4(b), and transforms an assemblage $\mathbf{\Sigma}_{\mathbb{A}_1 \ldots \mathbb{A}_k|\mathbb{X}_1 \ldots \mathbb{X}_k}$ into a new one as follows:

$$\begin{aligned}\sigma'_{a'_1 \ldots a'_k|x'_1 \ldots x'_k} = \sum_{\substack{\lambda \\ c_1 \ldots c_k}} \sum_{\substack{a_1 \ldots a_k \\ x_1 \ldots x_k}} \quad & p(c_1, x_1|x'_1, \lambda) \ldots p(c_k, x_k|x'_k, \lambda) \quad p(a'_1|a_1, c_1) \ldots p(a'_k|a_k, c_k) \\ & p(\lambda) \quad p(a_1, \ldots, a_k|x_1, \ldots, x_k) \quad \mathcal{E}_\lambda(\rho_{a_1 \ldots a_k|x_1 \ldots x_k}),\end{aligned} \tag{21}$$

where, similarly to the bipartite case,

- $p(c_i, x_i|x'_i, \lambda)$ encodes the classical pre-processing of the i-th Alice's input x_i as a function of x'_i and the shared classical randomness λ. Here, c_i denotes the variable to be transmitted through the i-th Alice's classical side channel towards her post-processing stage.
- $p(a'_i|a_i, c_i)$ encodes the classical post-processing of the i-th Alice's output a_i, as a function of the classical information c_i kept from the pre-processing stage. The output of the process is the i-th Alice's new outcome a'_i.

[9]This is in analogy to a fully-separable quantum state [66].

- $\mathcal{E}_\lambda[\cdot]$ is the CPTP map corresponding to Bob's local post-processing of his quantum system, as a function of the shared classical randomness λ.

Now recall that, similarly to the bipartite case, an LOSR-free assemblage is one that can be created from local operations and shared randomness. Hence, classical assemblages are all and only the ones that can be generated through the free operations of choice, consistent with the unifying assessment of 'free of cost' that this resource-theoretic underpinning brings. Moreover, if $\mathbf{\Sigma}_{\mathbb{A}_1\ldots\mathbb{A}_k|\mathbb{X}_1\ldots\mathbb{X}_k}$ is free, $\mathbf{\Sigma}'_{\mathbb{A}'_1\ldots\mathbb{A}'_k|\mathbb{X}'_1\ldots\mathbb{X}'_k}$ is free as well, hence the set of free multipartite assemblages (our free resources) is closed under LOSR operations, as it should be.

A final remark pertains to a particular way to express a generic LOSR transformation, similarly to the discussion in the bipartite scenario. Let us represent each Alice's local comb with a single probability distribution $p(a'_i, x_i|a_i, x'_i, \lambda)$. By Fine's argument [57] and discussion in Ref. [20], such local combs can be decomposed as a convex combination of deterministic combs as follows

$$p(a'_i, x_i|a_i, x'_i, \lambda) = \sum_{\tilde{\lambda}} p(\tilde{\lambda}|\lambda) D(x_i|x'_i, \tilde{\lambda})\, D(a'_i|a_i, x'_i, \tilde{\lambda}). \tag{22}$$

Here, each deterministic probability distribution assigns a fixed outcome a'_i (resp. x_i) for each possible choice of a_i, x'_i, and $\tilde{\lambda}$ (resp. x'_i and $\tilde{\lambda}$). Let us now use this observation to rewrite Eq. (21); for clarity in the presentation let us focus on the tripartite case (two Alices, one Bob). Define the CPTNI map $\tilde{\mathcal{E}}_{\tilde{\lambda}}$ as:

$$\tilde{\mathcal{E}}_{\tilde{\lambda}}(\sigma_{a_1a_2|x_1x_2}) = \sum_{\lambda} p(\lambda) p(\tilde{\lambda}|\lambda) \mathcal{E}_\lambda(p(a_1a_2|x_1x_2)\rho_{a_1a_2|x_1x_2}). \tag{23}$$

A generic LOSR operation transforms an assemblage $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2}$ into a new one as follows:

$$\sigma'_{a'_1a'_2|x'_1x'_2} = \sum_{\tilde{\lambda}} \sum_{\substack{a_1a_2\\x_1x_2}} D(x_1|x'_1, \tilde{\lambda})\, D(a'_1|a_1, x'_1, \tilde{\lambda})\, D(x_2|x'_2, \tilde{\lambda})\, D(a'_2|a_2, x'_2, \tilde{\lambda})\, \tilde{\mathcal{E}}_{\tilde{\lambda}}(\sigma_{a_1a_2|x_1x_2}). \tag{24}$$

Eq. (24) provides the simplified characterisation of a generic multipartite LOSR transformation that we will use throughout.

3.3 Resource conversion as a semidefinite test

For clarity in the presentation, we will still focus on the specific multipartite scenario, illustrated in Fig. 4(a). Our method, however, extends to scenarios with an arbitrary number of Alices. Given two assemblages generated in this setup, $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2}$ and $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$, testing whether $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$ under LOSR operations amounts to checking if $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$ admits a decomposition as per Eq. (24) for the case of two Alices. Similarly to the bipartite scenario, the possibility of the conversion can be decided with a single instance of an SDP.

First, notice that the map $\tilde{\mathcal{E}}_\lambda(\sigma_{a_1a_2|x_1x_2})$ can be represented in the operator form in terms of its Choi state W_λ as follows:

$$\tilde{\mathcal{E}}_\lambda(\sigma_{a_1a_2|x_1x_2}) = d_B \operatorname{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a_1a_2|x_1x_2} \right) \right\}, \tag{25}$$

where d_B is the dimension of Bob's Hilbert space. Therefore, for $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$ to decompose as in Eq. (24), each $\sigma'_{a'_1 a'_2|x'_1 x'_2}$ must admit the following decomposition:

$$\sigma'_{a'_1 a'_2|x'_1 x'_2} = \sum_{\substack{\lambda \\ c_1,c_2}} \sum_{\substack{a_1 a_2 \\ x_1 x_2}} D(x_1|x'_1,\lambda)\, D(a'_1|a_1,x'_1,\lambda)\, D(x_2|x'_2,\lambda)\, D(a'_2|a_2,x'_2,\lambda)$$
$$d_B\, \mathrm{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a_1 a_2|x_1 x_2} \right) \right\} . \tag{26}$$

This relation between elements $\sigma'_{a'_1 a'_2|x'_1 x'_2}$ and $\sigma_{a_1 a_2|x_1 x_2}$ can be tested via an SDP. The SDP that tests whether $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$ under LOSR operations reads as follows.

SDP 10. $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$.

The assemblage $\mathbf{\Sigma}_{\mathbb{A}_1\mathbb{A}_2|\mathbb{X}_1\mathbb{X}_2}$ *can be converted into the assemblage* $\mathbf{\Sigma}'_{\mathbb{A}'_1\mathbb{A}'_2|\mathbb{X}'_1\mathbb{X}'_2}$ *under LOSR operations if and only if the following SDP is feasible*

$$\begin{aligned}
\textit{given} \quad & \{\{\sigma_{a_1 a_2|x_1 x_2}\}_{a_1,a_2}\}_{x_1,x_2} \, , \; \{\{\sigma'_{a'_1 a'_2|x'_1 x'_2}\}_{a'_1,a'_2}\}_{x'_1,x'_2} \, , \\
& \{D(a'_1|a_1,x'_1,\lambda)\}_{\lambda,a'_1,a_1,x'_1} \, , \; D(x_1|x'_1,\lambda)\}_{\lambda,x_1,x'_1} \, , \\
& \{D(a'_2|a_2,x'_2,\lambda)\}_{\lambda,a'_2,a_2,x'_2} \, , \; D(x_2|x'_2,\lambda)\}_{\lambda,x_2,x'_2} \\
\textit{find} \quad & \{(W_\lambda)_{BB'}\}_\lambda \\
\textit{s.t.} \quad & \begin{cases}
W_\lambda \geq 0 \, , \\
\mathrm{tr}_{B'}\{W_\lambda\} \propto \frac{1}{d}\mathbb{I}_B \quad \forall \lambda \, , \\
\sum_\lambda \mathrm{tr}_{B'}\{W_\lambda\} = \frac{1}{d}\mathbb{I}_B \, , \\
\sigma'_{a'_1 a'_2|x'_1 x'_2} = \sum_\lambda \sum_{\substack{a_1 a_2 \\ x_1 x_2}} \quad D(a'_1|a_1,x'_1,\lambda)\, D(x_1|x'_1,\lambda)\, D(a'_2|a_2,x'_2,\lambda) \\
\qquad\qquad D(x_2|x'_2,\lambda)\, d_B\, \mathrm{tr}_B \left\{ W_\lambda \left(\mathbb{I}_{B'} \otimes \sigma^T_{a_1 a_2|x_1 x_2} \right) \right\} .
\end{cases}
\end{aligned} \tag{27}$$

Similarly to SDP 2, SDP 10 is a feasibility problem.

3.4 Properties of the pre-order

The properties of the pre-order of multipartite assemblages can be studied numerically, with the SDP 10, or analytically, with EPR LOSR monotones. The SDP 10 can be used to test conversions between multipartite assemblages, and a plot similar to that in Fig. 3 can be made to illustrate the pre-order of any multipartite family. Hence, we will not repeat this analysis for the multipartite case. In this section, we study the pre-order analytically, but the results can be easily verified with SDP 10. Although our methods apply to general assemblages, here we focus on quantumly-realisable assemblages, and study the properties of the pre-order for a particular family of resources therein.

Consider an EPR scenario with $N = k+1$ parties, where $\mathbb{A}_i = \mathbb{X}_i = \{0,1\}$ for $i \in \{1, ..., N-1\}$. Assume all parties share the state ρ_N^θ defined as

$$\begin{aligned}
\rho_N^\theta &= |GHZ_N^\theta\rangle \langle GHZ_N^\theta| \, , \\
\text{with} \quad |GHZ_N^\theta\rangle &= \cos\theta\, |0\rangle^{\otimes N} + \sin\theta\, |1\rangle^{\otimes N} \, ,
\end{aligned} \tag{28}$$

and the measurements that Alices perform upon an input $x_i \in \mathbb{X}$ are given by

$$\widetilde{M}_{a_i|0} = \frac{\mathbb{I} + (-1)^{a_i}\sigma_z}{2} \, , \quad \widetilde{M}_{a_i|1} = \frac{\mathbb{I} + (-1)^{a_i}\sigma_x}{2} \, . \tag{29}$$

Let us define a family of assemblages $\mathsf{S}^{(N)}$ as:

$$\mathsf{S}^{(N)} = \left\{ \mathbf{\Sigma}^{\theta}_{\mathbb{A}_1 \dots \mathbb{A}_{N-1} | \mathbb{X}_1 \dots \mathbb{X}_{N-1}} \,\middle|\, \theta \in (0, \pi/4] \right\} , \tag{30}$$

$$\text{where} \quad \mathbf{\Sigma}^{\theta}_{\mathbb{A}_1 \dots \mathbb{A}_{N-1} | \mathbb{X}_1 \dots \mathbb{X}_{N-1}} = \left\{ \left\{ \sigma^{\theta}_{a_1 \dots a_{N-1} | x_1 \dots x_{N-1}} \right\}_{a_i \in \mathbb{A}_i} \right\}_{x_i \in \mathbb{X}_i} ,$$

$$\text{with} \quad \sigma^{\theta}_{a_1 \dots a_{N-1} | x_1 \dots x_{N-1}} = \mathrm{tr}_{A_1 \dots A_{N-1}} \left\{ \widetilde{M}_{a_1|x_1} \otimes \dots \otimes \widetilde{M}_{a_{N-1}|x_{N-1}} \otimes \mathbb{I}\, \rho^{\theta}_N \right\} .$$

For simplicity in the notation, we here denote $\mathbf{\Sigma}^{\theta}_N := \mathbf{\Sigma}^{\theta}_{\mathbb{A}_1 \dots \mathbb{A}_{N-1} | \mathbb{X}_1 \dots \mathbb{X}_{N-1}}$ and $\mathbf{\Sigma}_N := \mathbf{\Sigma}_{\mathbb{A}_1 \dots \mathbb{A}_{N-1} | \mathbb{X}_1 \dots \mathbb{X}_{N-1}}$.

This family $\mathsf{S}^{(N)}$, for fixed N, has an infinite number of elements. We claim that the elements of $\mathsf{S}^{(N)}$ are unordered as per the LOSR resource theory of assemblages. To show this, we define a set of EPR monotones $\{S^{(N)}_{\eta}\}_{\eta}$ using the Bell inequalities derived in Ref. [72, Eq. (13)]. This procedure is presented in the Appendix D, and the EPR functional is given in Definition 20. For each value of η, we construct the monotone $\mathrm{M}^{(N)}_{\eta}$ from the EPR functional $S^{(N)}_{\eta}$ following a yield-based construction:

Definition 11. *The EPR monotone* $\mathrm{M}^{(N)}_{\eta}$*, for* $\eta \in (0, \pi/4]$*, is defined as*

$$\mathrm{M}^{(N)}_{\eta}[\mathbf{\Sigma}_N] := \max_{\widetilde{\mathbf{\Sigma}}_N} \{ S^{(N)}_{\eta}[\widetilde{\mathbf{\Sigma}}_N] : \mathbf{\Sigma}_N \xrightarrow{\text{LOSR}} \widetilde{\mathbf{\Sigma}}_N \}. \tag{31}$$

In Appendix D, we show that the monotones $\mathrm{M}^{(N)}_{\eta}[\mathbf{\Sigma}_N]$ satisfy properties analogous to these given in Eqs. (16) and (17), and Theorems 6 and 7, for the monotones $\mathrm{M}_{\eta}[\mathbf{\Sigma}]$. It follows that:

Corollary 12. *The infinite family of EPR monotones* $\mathsf{M}^{(N)} = \{\mathrm{M}^{(N)}_{\eta} \,|\, \eta \in (0, \pi/4]\}$ *certifies that the infinite family of assemblages* $\mathsf{S}^{(N)}$ *is composed of pairwise unordered resources.*

This result shows how, when LOSR are considered to be free operations in a resource theory, methods used in the bipartite scenarios can be leveraged to the multipartite case.

4 Related Work

Although the causal approach we take in this paper is not the standard description of the EPR scenario, the objects that we study, i.e., assemblages, have been widely investigated for their information-theoretic and foundational relevance. In particular, their role in one-side device-independent quantum key distribution (1S-DI-QKD) protocols motivated a formulation of the resource theory of 'steering', where the set of free operations is deemed to be stochastic local operations assisted by one-way classical communication (S-1W-LOCC) [13]. This set of operations – just like LOSR – is a valid set of free operations in a resource theory of assemblages, as it maps free resources to free resources. However, the conceptual underpinnings of our resource theory of nonclassicality of assemblages and of the resource theory of 'steering' under S-1W-LOCC operations are very different. In this section, we discuss some conceptual and technical differences between these two resource theories. Moreover, we show that the choice of free operations in a resource theory has significant consequences for the definition of a free multipartite assemblage.

4.1 S-1W-LOCC as the set of free operations

4.1.1 Formalisation of S-1W-LOCC operations

To start with, let us recall the formal definition of a S-1W-LOCC operation [13] which is illustrated in Fig. 5. Consider a bipartite EPR scenario. First, Bob performs an instrument on his subsystem that produces both a classical system represented by the variable ω, and a quantum system. This action corresponds to a completely-positive, trace-non-increasing map $\mathcal{E}_\omega$ on Bob's quantum system. Next, he communicates a classical message that depends on ω to Alice; without loss of generality, we can take the communicated message to be ω itself. In the second laboratory, Alice at first generates the classical input variable x'. Once she has received ω, she generates the new input variable x which can depend on both x' and ω. Then, she performs the measurement labelled by x on her share of the system; she will then obtain a measurement outcome a. Finally, Alice classically processes the variables x', ω and a, to produce the classical output a' of her measurement process. In the S-1W-LOCC operations, the instrument does not necessary need to be complete, i.e., if the probability for each outcome ω is P_ω, then the only guarantee on the sum of these probabilities is $0 < \sum_\omega P_\omega \leq 1$. It means that the total S-1W-LOCC operation does not need to happen with certainty, but with some non-zero probability. It was shown in Ref. [13] that these bipartite S-1W-LOCC operations map free assemblages to free assemblages in bipartite EPR scenarios; hence, they are formally a valid set of free operations in a resource theory of assemblages therein.

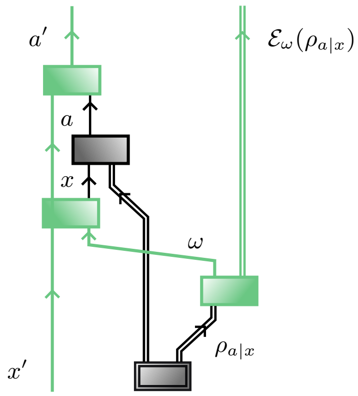


Figure 5: The most general 1W-LOCC operation (in green) applied to an EPR assemblage (in black).

4.1.2 Comparing S-1W-LOCC and LOSR

The resource theory of 'steering' under S-1W-LOCC and our resource theory of nonclassicality of assemblages are both formally valid resource theories that quantify nonclassicality in the bipartite EPR scenario. In this subsection, we will argue that the causal view on assemblages has several advantages over the S-1W-LOCC approach.

The set of free operations in any resource theory should be motivated by natural physical constraints in the scenario of interest. The set of S-1W-LOCC operations is hence a meaningful choice in scenarios wherein one-way classical communication is possible. Indeed, the choice of S-1W-LOCC as the set of free operations was motivated by the fact that it is the most general set that does not compromise the security of 1S-DI-QKD protocols [13]. In contrast, the motivation behind our LOSR approach is fundamentally different. Rather than using a methodology which singles out one particular task for which assemblages are known to be useful, we take a principled approach that applies to any scenario with a common-cause causal structure. In particular, we follow a general construction that can be applied to study nonclassicality in arbitrary causal networks, as first introduced in Ref. [20] (see in particular Appendix A3 therein). Indeed, our resource theory of EPR assemblages is simply a type-specific instance of the type-independent resource theory introduced in Ref. [22]. This is the most significant conceptual advantage of our LOSR approach – it unifies the study of nonclassicality of assemblages with the study of nonclassicality of arbitrary processes in common-cause (e.g., Bell) scenarios. This unified view leads to a deeper understanding of nonclassical correlations, as well as to new technical tools for studying them [21–23].

It is also worth noting that, at a technical level, LOSR conversions of assemblages are much easier to characterize and study than the S-1W-LOCC conversions.

The fact that we have taken a principled approach to constructing our resource theory also implies a final advantage: our framework extends naturally and uniquely to more general scenarios, including multipartite EPR scenarios, as well as EPR scenarios where the processes about which Alice is learning are dynamic (e.g., general channels or Bob-with-input processes[10]). It is not clear how prior approaches generalize to these cases; indeed, in past work there is no consensus even on the simplest question of how to define free assemblages in multipartite EPR scenarios (we discuss this point further in the following sections).

Finally, for completeness, we note that the LOSR and S-1W-LOCC resource theories do in fact lead to different pre-orders, and hence give different answers to questions about the relative value of assemblages.

Proposition 13. *The pre-order of assemblages under LOSR operations is different than the pre-order of assemblages under S-1W-LOCC operations.*

The proof of this proposition is given in Appendix F. The main idea is to find a conversion that is possible with S-1W-LOCC operations and is impossible with LOSR operations. In the proof we focus on the family $\mathbf{\Sigma}^{\theta,p}_{\mathbb{A}|\mathbb{X}}$ defined in Eq. (10). We show that for S-1W-LOCC operations, the assemblage $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ is above all others in the family; whereas in LOSR, all assemblages $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$ are incomparable for $\theta \in (0, \pi/4]$. In Appendix F we provide an S-1W-LOCC map that maps $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ to $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$ for $\theta \in (0, \pi/4)$. This result is analogous to the difference in the pre-order in the resource theories of LOCC-entanglement and LOSR-entanglement [21][11].

In proving Proposition 13, we note a deficiency in Theorem 5 in Ref. [13]. The theorem claims that there is no universal assemblage in a particular setting from which we obtain any other assemblage in the same setting with S-1W-LOCC maps. The proof applies if one assumes that the universal assemblage is not $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$, thus limiting the applicability of the theorem. Therefore, the proof given in Ref. [13] is incomplete.

4.2 Multipartite free assemblages

Historically, a bipartite assemblage would be considered nonfree or nonclassical[12] if Alice and Bob need to share a quantum system prepared in an entangled state in order to produce it. If the assemblage could be generated by performing measurements on a system prepared on a separable state, then the assemblage is free.

Both for bipartite *and* multipartite scenarios, this is exactly the distinction between free and nonfree in our approach: an assemblage is free if and only if it can be generated by measurements on a fully-separable state. Although the definition of a free multipartite assemblage coincides

[10] Bob-with-input processes are those in which Bob can locally influence the state preparation of his system [50].

[11] There are two different resource theories of entanglement established in the literature. If one allows local operations and classical communication (LOCC) as the set of free operations, it is well known that a maximally entangled state can be converted into any partially entangled pure state of the same Schmidt rank [66]. However, if one takes LOSR to be the set of free operations, such states are incomparable [21]. Here, we use the terms LOCC-entanglement and LOSR-entanglement to distinguish these two resource theories [21, 66].

[12] In the literature, nonfree assemblages are referred to as 'steerable' assemblages.

with our approach in some past literature [6, 68, 70], alternative approaches to studying multipartite EPR scenarios also exist . We now discuss some of the different sets of assemblages that have been proposed as free sets in the multipartite case, and contrast them with our proposal.

For simplicity, we will focus on the case of a tripartite EPR scenario with two Alices and one Bob, although the three definitions easily generalise to EPR scenarios with more than two Alices. In this setup, one proposal is given in Ref. [5], where an assemblage is called free if it can be decomposed as

$$\sigma_{a_1 a_2|x_1 x_2} = \sum_\lambda p(\lambda) p(a_1, a_2|x_1, x_2, \lambda) \rho_\lambda, \tag{32}$$

where $p(\lambda)$ is a normalised probability distribution on λ, ρ_λ is a normalised quantum state for each λ, and $p(a_1, a_2|x_1, x_2, \lambda)$ is only required to be a valid conditional probability distribution. Indeed, here $p(a_1, a_2|x_1, x_2, \lambda)$ is allowed to even be a signalling correlation between the Alices, for each λ, as long as this signalling is averaged out in Eq. (32) to give rise to a valid quantum assemblage $\{\{\sigma_{a_1 a_2|x_1 x_2}\}_{a_1,a_2\in\mathbb{A}}\}_{x_1,x_2\in\mathbb{X}}$. This condition for $\{p(a_1, a_2|x_1, x_2, \lambda)\}$ is a form of fine-tuning [15], and is conceptually problematic. Furthermore, it is clear that any assemblage that is nontrivially fine-tuned in this manner is not consistent with our hypothesis about the causal structure of an EPR scenario. Note also that, by this definition, there exist assemblages that are free and yet for which the correlations shared by the Alices (when one marginalizes over Bob's system) are highly nonclassical– e.g., a Popescu-Rohrlich box correlations [73]. As such, this approach is not suited to studying nonclassicality in general, but could only be suitable for studying nonclassicality *across the Alices-vs-Bob* partition. A different proposal for the definition of a free assemblage is given in Ref. [71], where an assemblage is deemed free if it can be expressed as

$$\begin{aligned} \sigma_{a_1 a_2|x_1 x_2} &= \sum_\lambda p(\lambda) p^{A_1\to A_2}(a_1, a_2|x_1, x_2, \lambda) \rho_\lambda \\ &= \sum_\lambda p'(\lambda) p^{A_2\to A_1}(a_1, a_2|x_1, x_2, \lambda) \rho'_\lambda, \end{aligned} \tag{33}$$

where for each value of λ, the probability distribution $p^{A_1\to A_2}(a_1, a_2|x_1, x_2, \lambda)$ is non-signalling from Alice$_2$ to Alice$_1$, and the probability distribution $p^{A_2\to A_1}(a_1, a_2|x_1, x_2, \lambda)$ is non-signalling from Alice$_1$ to Alice$_2$. Such a decomposition is called the time-ordered local hidden state model (TO-LHS) of an assemblage. This set has been motivated by the fact that it is the largest set of free assemblages for which an anomalous phenomenon termed 'exposure of steering' [71] does not occur (under particular assumptions about the set of free operations, distinct from those we have advocated for), and we elaborate further on this point in Section 4.3. In particular, we note that in any standard resource theory (i.e., in the sense of Ref. [30]), both the free resources and the free transformations on them should be derivable from a single set of free operations. What this implies here is that it should be the case that the assemblages which admit of TO-LHS models constitute *all and only* those assemblages which can be constructively generated using S-1W-LOCC operations. This is, however, not the case, since the set of TO-LHS assemblages includes some post-quantum tripartite assemblages (e.g., where the Alices share a Popescu-Rohrlich box; see the next subsection), and these cannot be generated by a S-1W-LOCC protocol.

These two definitions of free assemblage contrast with our LOSR resource theory of nonclassicality of assemblages, where an assemblage is LOSR-free if it can be written as

$$\sigma_{a_1 a_2|x_1 x_2} = \sum_\lambda p(\lambda) p(a_1|x_1, \lambda) p(a_2|x_2, \lambda) \sigma_\lambda. \tag{34}$$

Classicality in our resource-theoretic approach is captured by the structure of the network, and imposes that the assemblage is free, i.e., classical, if the parties – Alices and Bob – are related solely by a shared classical common cause. This definition of a free assemblage has indeed appeared in the literature before in the context of EPR scenarios, and not from a resource-theoretic perspective: in Refs. [6, 68, 70], a free assemblage is defined as one that arises when all parties share a quantum system prepared in a fully separable state, and the Alices perform local measurements on their subsystems – this leads to all and only assemblages of the form of Eq. (34).

Finally, it is worth noting that in this discussion we have focused on multipartite EPR scenarios with multiple Alices and one Bob. However, a possible multipartite EPR scenario comprises also multiple Bobs. Such a multipartite scenario is beyond the scope of this work, however, it would be interesting to see which other approaches to defining free assemblages it gives rise to[13].

4.3 Exposure of 'steering'

Exposure of 'steering' refers to a process in which a resourceful assemblage in an EPR scenario can be created from a free assemblage in a multipartie EPR scenario, by performing a global operation on the Alices. Notice that such an operation may change the type of scenario under study; e.g., it may transform a tripartite assemblage into a bipartite one.

A type-changing transformation of this nature is considered in Ref. [71], where Taddei *et al.* focus on a tripartite EPR scenario with two Alices and one Bob. As a case study, they take a free tripartite assemblage to be defined as per Eq. (32), when no extra constraints are imposed on $p(a_1, a_2|x_1, x_2, \lambda)$ other than the implicit fine tuning condition. The global operation on the two Alices is a wiring operation – the measurement outcome of one Alice is taken as the choice of measurement for the other Alice. Taddei *et al.* argue that this global operation should not increase the Alices' capability to remotely 'steer' Bob's system, since no global action is performed across the Bob-vs-Alices divide, i.e., this global operation should be a free operation (at least relative to the Bob-vs-Alices divide). Despite this intuition, they find that a wiring between the Alices in a tripartite free assemblage may give rise to a new bipartite assemblage that is nonfree.

This then raises important conceptual questions, since a well-defined *compositional* resource theory should not allow one to create nonfree resources by using free operations on free resources, *even if* the free transformation is type-changing. The set of free resources in any resource theory must be closed under free operations [30]. The spectre of 'steering' exposure emphasizes the importance of choosing one's set of free resources and free operations wisely and consistently.

As Taddei *et al.* observe, then, 'steering' exposure implies an inconsistency taking Eq. (32) to define the set of free resources (together with taking wirings-and-S-1W-LOCC as free operations). They then derive the largest set of assemblages that do not allow for 'steering' exposure under these free operations, and show this set to be those assemblages that admit a TO-LHS model. Hence, they argue that the set of free assemblages should be taken to be defined by Eq. (33).

One awkwardness of this proposal is that the set of TO-LHS assemblages includes assemblages that are not quantumly-realizable; i.e., that cannot be produced by Bob and the Alices sharing a multipartite quantum system and performing local measurements on it. An example

[13] In the case of our LOSR resource theory of nonclassical assemblages the situation is straightforward: all the parties share a classical common cause, and each Bob locally prepares a quantum system on a state that depends on the value of such shared classical randomness.

of this arises when $p^{A_1 \to A_2}(a_1, a_2|x_1, x_2, \lambda) = p^{A_2 \to A_1}(a_1, a_2|x_1, x_2, \lambda)$ is equal to 'a Popescu-Rohrlich box' [73] for all λ. This sort of definition, then, is clearly not suitable for studying generic nonclassicality in multipartite EPR scenarios, but rather could *only* be suitable as a way of studying *nonclassicality across the Alices-vs-Bob partition*.

Another problem with Taddei *et al.*'s approach is that it comes at the expense of introducing an arbitrary divide between free resources and free processes. That is, as argued in the previous section, the set of free resources defined by Eq. (33) does not constitute all and only those that can be constructively built out of S-1W-LOCC operations, as the standard approach to resource theories requires. Indeed, any TO-LHS assemblage for which the marginal correlations shared between the Alices is post-quantum will be a counterexample, since the classical communication from Bob to the Alices is not sufficient to set up such correlations.

Note that Taddei *et al.*'s approach is only *one* way to avoid exposure phenomena. A second way to avoid it is to consider a different set of free operations. Indeed, in our approach, neither the free operations *nor* the free resources are taken to be the sets that Taddei *et al.* considered. Furthermore, our approach does not allow for any exposure phenomena, nor does it face the issues we just outlined which arise in Taddei *et al.*'s TO-LHS approach. Note first that in our approach, wirings are not free LOSR operations, since our approach does not only aim to characterize nonclassicality between the Alices-vs-Bob partition, but rather *any* nonclassicality of common cause, shared between *any* of the parties. As such, wirings among black-box parties can of course increase the nonclassicality of a given common-cause process. In our approach, then, the question of exposure is whether nonfree transformations that act globally on the Alices—but do not act on Bob—are capable of transforming free resources to resources which require nonclassical common causes *shared across the Alices-vs-Bob divide*. But such exposure phenomena are not possible in our approach. Indeed, the set of LOSR-free assemblages is strictly contained within the set of TO-LHS assemblages, and hence (by Taddei *et al.*'s result) they cannot yield exposure.

For completeness, we further note that in our resource theory, it is indeed the case that the free resources are all and only those that can be constructed out of free (i.e., LOSR) operations, so that the consistency condition mentioned above is satisfied.

5 Conclusions and outlook

In this paper, we developed a resource theory of nonclassicality for assemblages in EPR scenarios, with its free operations defined as local operations and shared randomness. This choice of free processes is motivated by the causal structure of the EPR scenario, together with a viewpoint on EPR scenarios wherein Alice's actions allow her to make *inferences about* the physical state of Bob's system rather than to *remotely steer* (i.e., influence) it. This is the first resource theory for assemblages in multipartite EPR scenarios.

We proved that resource conversions in our resource theory can be tested using a single instance of a semidefinite program, in both bipartite and multipartite EPR scenarios. These semidefinite programs – which we give – are the first tools that allow one to explore systematically free conversions of assemblages. We also proved that the pre-order of (both bipartite and multipartite) assemblages contain an infinite number of incomparable assemblages. To prove this, we constructed new EPR monotones, which may be useful in their own right for other tasks.

The causal approach that we have endorsed here brings a new perspective to fundamental questions regarding EPR scenarios. Firstly, we have motivated a principled approach to defining the set of classically-explainable assemblages in multipartite EPR scenarios from a resource-theoretic perspective. In addition, our approach (in contrast to the approach of Ref. [71]) ensures that the free set of resources and the free set of operations are consistent, in the sense that the free resources are all and only those that can be constructed out of free operations. Finally, our approach ensures that there are no free type-changing operations that do not preserve the free set (i.e., that there is no 'steering exposure'), and sheds light on previous approaches to ensuring this property. As far as we know, this is the first resource theory defined for EPR assemblages that satisfies these (in our opinion) important properties, highlighting the value of taking our principled LOSR approach.

The resource theory of nonclassicality of assemblages is an instance of a general construction for studying the nonclassicality of common-cause processes. This principled approach was previously used to study LOSR-entanglement [21], nonclassical correlations in Bell scenarios [20], and indeed arbitrary types of common-cause processes [22, 23]. Our work continues the development of this program in the specific context of EPR scenarios, and in doing so provides a perspective on EPR 'steering' which unifies it with other pertinent notions of nonclassicality. A natural continuation of this work would be to study other EPR-like scenarios, wherein the process of Bob about which Alice is learning is not merely a quantum output system, but rather a dynamical process. Indeed, in our forthcoming paper we develop a resource theory for Bob-with-input scenarios and channel EPR scenarios.

Another relevant research avenue would be to extend the enveloping theory to include post-quantum common-cause processes [36, 49, 50] (e.g., described by processes in arbitrary generalized probabilistic theories [55, 74, 75]). Scenarios such as the multipartite EPR scenario and the Bob-with-Input scenario are of particular importance for these questions because – unlike the bipartite EPR scenario – they allow for post-quantum assemblages. In order to quantify the nonclassicality of such assemblages, one would again take the free set to be defined by LOSR operations, and apply all the tools and monotones developed herein. Alternatively, one could study not the nonclassicality, but rather the *post-quantumness* of these resources, by taking the free set to be defined by *local operations and shared entanglement* (LOSE operations), as argued in Ref. [36, 76]. In this latter resource theory, one would be studying not the possibilities allowed by quantum theory, but rather the limitations imposed by quantum theory.

Given that both entanglement and incompatibility are necessary resources for generating nonclassical assemblages, another interesting line of work would be to explore how our resource theory of EPR assemblages emerges from the resource theories of incompatibility [48] and entanglement [21].

Finally, an interesting question pertains to the notion of self-testing of quantum states, which is relevant for the certification of quantum devices in communication and information processing protocols. Ref. [21] shows that self-testing of states can indeed be defined in a resource-theoretic framework. It would be interesting to see how self-testing of quantum assemblages (cf. Refs. [77–79]) may be understood from the tools we developed in our work, such as the resource-conversion SDP tests.

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References

[1] Albert Einstein, Boris Podolsky, and Nathan Rosen. "Can quantum-mechanical description of physical reality be considered complete?". Physical review **47**, 777 (1935).

[2] Erwin Schrödinger. "Discussion of probability relations between separated systems". Mathematical Proceedings of the Cambridge Philosophical Society **31**, 555–563 (1935).

[3] Eric Gama Cavalcanti, Steve J Jones, Howard M Wiseman, and Margaret D Reid. "Experimental criteria for steering and the einstein-podolsky-rosen paradox". Physical Review A **80**, 032112 (2009).

[4] Howard M Wiseman, Steve James Jones, and Andrew C Doherty. "Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox". Physical review letters **98**, 140402 (2007).

[5] Roope Uola, Ana CS Costa, H Chau Nguyen, and Otfried Gühne. "Quantum steering". Reviews of Modern Physics **92**, 015001 (2020).

[6] Daniel Cavalcanti, Paul Skrzypczyk, Gabriel Aguilar, Ranieri V Nery, PH Souto Ribeiro, and Stephen Walborn. "Detection of entanglement in asymmetric quantum networks and multipartite quantum steering". Nature communications **6**, 1–6 (2015).

[7] Alejandro Máttar, Paul Skrzypczyk, Gabriel Aguilar, Ranieri V Nery, PH Souto Ribeiro, Stephen Walborn, and Daniel Cavalcanti. "Experimental multipartite entanglement and randomness certification of the w state in the quantum steering scenario". Quantum Science and Technology **2**, 015011 (2017).

[8] Cyril Branciard, Eric G Cavalcanti, Stephen P Walborn, Valerio Scarani, and Howard M Wiseman. "One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering". Physical Review A **85**, 010301 (2012).

[9] Yu Xiang, Ioannis Kogias, Gerardo Adesso, and Qiongyi He. "Multipartite gaussian steering: Monogamy constraints and quantum cryptography applications". Phys. Rev. A **95**, 010101 (2017).

[10] Matthew F Pusey. "Negativity and steering: A stronger Peres conjecture". Physical Review A **88**, 032313 (2013).

[11] Paul Skrzypczyk, Miguel Navascués, and Daniel Cavalcanti. "Quantifying Einstein-Podolsky-Rosen steering". Physical review letters **112**, 180404 (2014).

[12] Marco Piani and John Watrous. "Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering". Physical review letters **114**, 060404 (2015).

[13] Rodrigo Gallego and Leandro Aolita. "Resource theory of steering". Physical Review X **5**, 041008 (2015).

[14] Judea Pearl. "Causality". Cambridge University Press. (2009).

[15] Christopher J Wood and Robert W Spekkens. "The lesson of causal discovery algorithms for quantum correlations: causal explanations of bell-inequality violations require fine-tuning". New Journal of Physics **17**, 033002 (2015).

[16] Julio I De Vicente. "On nonlocality as a resource theory and nonlocality measures". Journal of Physics A: Mathematical and Theoretical **47**, 424017 (2014).

[17] Joshua Geller and Marco Piani. "Quantifying non-classical and beyond-quantum correlations in the unified operator formalism". Journal of Physics A: Mathematical and Theoretical **47**, 424030 (2014).

[18] Rodrigo Gallego and Leandro Aolita. "Nonlocality free wirings and the distinguishability between bell boxes". Physical Review A **95**, 032118 (2017).

[19] Antonio Acín, Serge Massar, and Stefano Pironio. "Randomness versus nonlocality and entanglement". Physical review letters **108**, 100402 (2012).

[20] Elie Wolfe, David Schmid, Ana Belén Sainz, Ravi Kunjwal, and Robert W Spekkens. "Quantifying Bell: The resource theory of nonclassicality of common-cause boxes". Quantum **4**, 280 (2020).

[21] David Schmid, Thomas C Fraser, Ravi Kunjwal, Ana Belén Sainz, Elie Wolfe, and Robert W Spekkens. "Understanding the interplay of entanglement and nonlocality: motivating and developing a new branch of entanglement theory" (2020).

[22] David Schmid, Denis Rosset, and Francesco Buscemi. "The type-independent resource theory of local operations and shared randomness". Quantum **4**, 262 (2020).

[23] Denis Rosset, David Schmid, and Francesco Buscemi. "Type-independent characterization of spacelike separated resources". Physical Review Letters **125**, 210402 (2020).

[24] David Schmid, John H Selby, and Robert W Spekkens. "Unscrambling the omelette of causation and inference: The framework of causal-inferential theories" (2020).

[25] E.T. Jaynes. "Probability in quantum theory". in Complexity, Entropy, and the Physics of Information edited by W. H. ZurekPage 381 (1990).

[26] Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens. "Reconstruction of Gaussian quantum mechanics from Liouville mechanics with an epistemic restriction". Phys. Rev. A **86**, 012103 (2012).

[27] Christopher A Fuchs. "Quantum mechanics as quantum information (and only a little more)" (2002).

[28] David Schmid, Katja Ried, and Robert W. Spekkens. "Why initial system-environment correlations do not imply the failure of complete positivity: A causal perspective". Phys. Rev. A **100**, 022112 (2019).

[29] David Schmid. "Guiding our interpretation of quantum theory by principles of causation and inference". PhD thesis, University of Waterloo (2021).

[30] Bob Coecke, Tobias Fritz, and Robert W Spekkens. "A mathematical theory of resources". Information and Computation **250**, 59–86 (2016).

[31] Eric Chitambar and Gilad Gour. "Quantum resource theories". Reviews of Modern Physics **91**, 025001 (2019).

[32] Michael A Nielsen. "Conditions for a class of entanglement transformations". Physical Review Letters **83**, 436 (1999).

[33] Charles H Bennett, Herbert J Bernstein, Sandu Popescu, and Benjamin Schumacher. "Concentrating partial entanglement by local operations". Physical Review A **53**, 2046 (1996).

[34] Yuval Rishu Sanders and Gilad Gour. "Necessary conditions for entanglement catalysts". Physical Review A **79**, 054302 (2009).

[35] Francesco Buscemi. "All entangled quantum states are nonlocal". Physical review letters **108**, 200401 (2012).

[36] David Schmid, Haoxing Du, Maryam Mudassar, Ghi Coulter-de Wit, Denis Rosset, and Matty J Hoban. "Postquantum common-cause channels: the resource theory of local operations and shared entanglement". Quantum **5**, 419 (2021).

[37] Jonathan Barrett, Noah Linden, Serge Massar, Stefano Pironio, Sandu Popescu, and David Roberts. "Nonlocal correlations as an information-theoretic resource". Physical review A **71**, 022101 (2005).

[38] Nicolas Brunner and Paul Skrzypczyk. "Nonlocality distillation and postquantum theories with trivial communication complexity". Physical review letters **102**, 160403 (2009).

[39] Iman Marvian and Robert W Spekkens. "How to quantify coherence: Distinguishing speakable and unspeakable notions". Physical Review A **94**, 052324 (2016).

[40] Iman Marvian, Robert W Spekkens, and Paolo Zanardi. "Quantum speed limits, coherence, and asymmetry". Physical Review A **93**, 052331 (2016).
[41] Andreas Winter and Dong Yang. "Operational resource theory of coherence". Physical review letters **116**, 120404 (2016).
[42] Fernando GSL Brandao, Michał Horodecki, Jonathan Oppenheim, Joseph M Renes, and Robert W Spekkens. "Resource theory of quantum states out of thermal equilibrium". Physical review letters **111**, 250404 (2013).
[43] Paul Skrzypczyk, Anthony J Short, and Sandu Popescu. "Work extraction and thermodynamics for individual quantum systems". Nature communications **5**, 1–8 (2014).
[44] Dominik Janzing, Pawel Wocjan, Robert Zeier, Rubino Geiss, and Th Beth. "Thermodynamic cost of reliability and low temperatures: tightening landauer's principle and the second law". International Journal of Theoretical Physics **39**, 2717–2753 (2000).
[45] Michał Horodecki and Jonathan Oppenheim. "Fundamental limitations for quantum and nanoscale thermodynamics". Nature communications **4**, 1–6 (2013).
[46] Gilad Gour, Markus P Müller, Varun Narasimhachar, Robert W Spekkens, and Nicole Yunger Halpern. "The resource theory of informational nonequilibrium in thermodynamics". Physics Reports **583**, 1–58 (2015).
[47] Zoë Holmes, Erick Hinds Mingo, Calvin Y-R Chen, and Florian Mintert. "Quantifying athermality and quantum induced deviations from classical fluctuation relations". Entropy **22**, 111 (2020).
[48] Francesco Buscemi, Eric Chitambar, and Wenbin Zhou. "Complete resource theory of quantum incompatibility as quantum programmability". Physical review letters **124**, 120401 (2020).
[49] Ana Belén Sainz, Nicolas Brunner, Daniel Cavalcanti, Paul Skrzypczyk, and Tamás Vértesi. "Postquantum steering". Physical review letters **115**, 190403 (2015).
[50] Ana Belén Sainz, Matty J Hoban, Paul Skrzypczyk, and Leandro Aolita. "Bipartite postquantum steering in generalized scenarios". Physical Review Letters **125**, 050404 (2020).
[51] Matthew S Leifer and Robert W Spekkens. "Towards a formulation of quantum theory as a causally neutral theory of bayesian inference". Physical Review A **88**, 052130 (2013).
[52] Marco Piani. "Channel steering". JOSA B **32**, A1–A7 (2015).
[53] Nicolas Gisin. "Stochastic quantum dynamics and relativity". Helvetica Physica Acta **62**, 363–371 (1989).
[54] Lane P Hughston, Richard Jozsa, and William K Wootters. "A complete classification of quantum ensembles having a given density matrix". Physics Letters A **183**, 14–18 (1993).
[55] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. "Theoretical framework for quantum networks". Physical Review A **80**, 022339 (2009).
[56] Michael A. Nielsen and Isaac L. Chuang. "Quantum computation and quantum information: 10th anniversary edition". Cambridge University Press. (2011).
[57] Arthur Fine. "Hidden Variables, Joint Probability, and the Bell Inequalities". Phys. Rev. Lett. **48**, 291–295 (1982).
[58] Man-Duen Choi. "Completely positive linear maps on complex matrices". Linear algebra and its applications **10**, 285–290 (1975).
[59] Andrzej Jamiołkowski. "Linear transformations which preserve trace and positive semidefiniteness of operators". Reports on Mathematical Physics **3**, 275–278 (1972).
[60] "Matlab". url: `https://www.mathworks.com/`.
[61] Michael Grant and Stephen Boyd. "Cvx: MATLAB software for disciplined convex programming". url: `http://cvxr.com/cvx`.

[62] Michael Grant and Stephen Boyd. “Graph implementations for nonsmooth convex programs”. In V. Blondel, S. Boyd, and H. Kimura, editors, Recent Advances in Learning and Control. Pages 95–110. Lecture Notes in Control and Information Sciences. Springer-Verlag Limited (2008).
[63] K. C. Toh, M. J. Todd, and R. H. Tütüncü. “SDPT3 — a matlab software package for semidefinite programming, version 1.3”. Optimization Methods and Software **11**, 545–581 (1999).
[64] Nathaniel Johnston. “QETLAB: a MATLAB toolbox for quantum entanglement”. url: `http://qetlab.com`.
[65] Beata Zjawin, David Schmid, Matty J. Hoban, and Ana Belén Sainz. code: beatazjawin/Quantifying-EPR.
[66] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. “Quantum entanglement”. Rev. Mod. Phys. **81**, 865–942 (2009).
[67] Cédric Bamps and Stefano Pironio. “Sum-of-squares decompositions for a family of Clauser-Horne-Shimony-Holt-like inequalities and their application to self-testing”. Physical Review A **91**, 052111 (2015).
[68] Eric G Cavalcanti, Qiongyi Y He, Margaret D Reid, and Howard M Wiseman. “Unified criteria for multipartite quantum nonlocality”. Physical Review A **84**, 032115 (2011).
[69] Alexander A Klyachko. “Quantum marginal problem and n-representability”. Journal of Physics: Conference Series **36**, 014 (2006).
[70] Daniel Cavalcanti and Paul Skrzypczyk. “Quantum steering: A review with focus on semidefinite programming”. Reports on Progress in Physics **80**, 024001 (2016).
[71] Márcio M Taddei, Thais de Lima Silva, Ranieri V. Nery, Gabriel Aguilar, Stephen P Walborn, and Leandro Aolita. “Exposure of subtle multipartite quantum nonlocality”. npj Quantum Information **7**, 1–8 (2021).
[72] Flavio Baccari, Remigiusz Augusiak, Ivan Šupić, Jordi Tura, and Antonio Acín. “Scalable Bell inequalities for qubit graph states and robust self-testing”. Physical review letters **124**, 020402 (2020).
[73] Sandu Popescu and Daniel Rohrlich. “Quantum nonlocality as an axiom”. Foundations of Physics **24**, 379–385 (1994).
[74] Lucien Hardy. “Quantum theory from five reasonable axioms” (2001).
[75] Jonathan Barrett. “Information processing in generalized probabilistic theories”. Physical Review A **75**, 032304 (2007).
[76] Matty J Hoban and Ana Belén Sainz. “A channel-based framework for steering, non-locality and beyond”. New Journal of Physics **20**, 053048 (2018).
[77] Ivan Šupić and Matty J Hoban. “Self-testing through EPR-steering”. New Journal of Physics **18**, 075006 (2016).
[78] Alexandru Gheorghiu, Petros Wallden, and Elham Kashefi. “Rigidity of quantum steering and one-sided device-independent verifiable quantum computation”. New Journal of Physics **19**, 023043 (2017).
[79] Shin-Liang Chen, Huan-Yu Ku, Wenbin Zhou, Jordi Tura, and Yueh-Nan Chen. “Robust self-testing of steerable quantum assemblages and its applications on device-independent quantum certification”. Quantum **5**, 552 (2021).

A The bipartite tilted Bell inequalities

Consider a bipartite Bell scenario with two dichotomic measurements per party. The Bell inequality studied in Refs. [19, 67] for this scenario, defined in [67, Eq. (1)], reads:

$$I_\alpha = \alpha\langle A_0\rangle + \langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle\,, \tag{35}$$

where $\alpha \in [0,2]$ is a real parameter. For $\alpha = 0$, I_0 correspond to the CHSH inequality.

The set $\mathsf{I} = \{I_\alpha \,|\, \alpha \in [0,2]\}$ defines a family of Bell inequalities for the bipartite Bell scenario, indexed by the parameter α. It was proved in Ref. [19] that the maximum quantum violation of I_α is given by $I_\alpha^{\max} = \sqrt{8+2\alpha^2}$, and the classical bound of I_α is given by $I_\alpha^{\mathrm{C}} = 2+\alpha$. The value $I_\alpha^{\max}$ is achieved by measuring the observables

$$\begin{aligned} A_0 &= \sigma_z\,, & B_0 &= \cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,,\\ A_1 &= \sigma_x\,, & B_1 &= \cos(\mu)\,\sigma_z - \sin(\mu)\,\sigma_x\,, \end{aligned} \tag{36}$$

on the quantum state

$$|\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle\ , \tag{37}$$

with the relation between the parameters α, θ, and μ being the following:

$$\alpha = \frac{2}{\sqrt{1+2\tan^2(2\theta)}}\,, \quad \tan(\mu) = \sin(2\theta)\,. \tag{38}$$

In Ref. [67, Section III and Appendix A], it was proved that the Bell inequalities specified by Eq. (35) provide a robust self-test for the reference states given in Eq. (37).

Remark 14. *Let $\widetilde{A}_x$, $\widetilde{B}_y$, and $|\widetilde{\psi}\rangle$ be physical observables and a physical quantum state that achieve the value $I_\alpha^{\max}$ for the Bell inequality I_α. Then, these are equal to (up to local isometries) the state and observables given in Eqs. (37) and (36).*

B Constructing bipartite EPR inequalities from Bell inequalities

We now show how to use the tilted Bell inequalities from the set I, defined in Appendix A, to construct a family of EPR inequalities [3]. Let us begin by transforming a Bell functional $I_\alpha \in \mathsf{I}$ into an EPR functional S. A generic EPR functional is defined as:

$$S[\mathbf{\Sigma}] = \mathrm{tr}\left\{\sum_{a\in\mathbb{A},\,x\in\mathbb{X}} F_{a,x}\,\sigma_{a|x}\right\}\,. \tag{39}$$

The Bell expression I_α can be expressed in terms of the elements of $\mathbf{\Sigma}$ as follows:

$$\begin{aligned} I_\alpha &= \alpha\langle A_0\rangle + \langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle\\ &= \alpha\mathrm{tr}\left\{(M_{0|0}-M_{1|0})\otimes \mathbb{I}_B\,\rho\right\} + \mathrm{tr}\left\{(M_{0|0}-M_{1|0})\otimes B_0\,\rho\right\} + \mathrm{tr}\left\{(M_{0|0}-M_{1|0})\otimes B_1\,\rho\right\}\\ &\quad + \mathrm{tr}\left\{(M_{0|1}-M_{1|1})\otimes B_0\,\rho\right\} - \mathrm{tr}\left\{(M_{0|1}-M_{1|1})\otimes B_1\,\rho\right\}\\ &= \mathrm{tr}\left\{\alpha\,\sigma_{0|0} - \alpha\,\sigma_{1|0} + B_0\,\sigma_{0|0} - B_0\,\sigma_{1|0} + B_1\,\sigma_{0|0} - B_1\,\sigma_{1|0} + B_0\,\sigma_{0|1} - B_0\,\sigma_{1|1}\right\}\\ &\quad + \mathrm{tr}\left\{-B_1\,\sigma_{0|1} + B_1\,\sigma_{1|1}\right\} \end{aligned}$$

From the above expansion of I_α, we then define an EPR functional $S[\mathbf{\Sigma}]$ by setting:

$$F_{0,0} = \alpha\mathbb{I} + B_0 + B_1 = -F_{1,0} = , \; F_{0,1} = B_0 - B_1 = -F_{1,1} \,. \tag{40}$$

In the expression for $S[\mathbf{\Sigma}]$ given by the operators of Eq. (40) there are plenty of things that still need to be specified: the value of the parameter α, as well as the operators B_0 and B_1. This is the freedom we will leverage to construct a family of EPR inequalities.

Definition 15. ***EPR functional*** $S_\eta[\mathbf{\Sigma}]$
We define the EPR functional $S_\eta[\mathbf{\Sigma}]$ via the operators of Eq. (40) by taking:

$$\begin{aligned} \alpha &= \frac{2}{\sqrt{1+2\tan^2(2\eta)}} \,, \\ B_0 &= \cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x \,, \\ B_1 &= \cos(\mu)\,\sigma_z - \sin(\mu)\,\sigma_x \,, \\ \tan(\mu) &= \sin(2\eta) \,. \end{aligned} \tag{41}$$

A few remarks regarding the properties of $S_\eta[\mathbf{\Sigma}]$ are in order.

Remark 16. *The maximum quantum value $S_\eta^{\max}$ of $S_\eta[\mathbf{\Sigma}]$ is given by*

$$S_\eta^{\max} = 2\sqrt{2}\sqrt{1+\frac{1}{1+2\tan^2(2\eta)}} \,. \tag{42}$$

Proof. First notice that 'optimising $S_\eta[\mathbf{\Sigma}]$ over quantum assemblages' is equivalent to 'optimising I_α over quantum correlations constrained on Bob's observables being those from Eq. (41), and on α and η being related as in Eq. (41)'.

Since the maximum quantum violation $I_\alpha^{\max}$ can indeed be achieved by the operators for Bob from Eq. (41), then $S_\eta^{\max} = I_\alpha^{\max}$. Using the relation between α and η we obtain Eq. (42). □

Remark 17. *The quantum assemblage $\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}$, defined in Eq. (11), satisfies $S_\eta[\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}] = S_\eta^{\max}$ if and only if $\eta = \theta$.*

Proof. The 'if' direction is straightforward to prove, by noticing that the state and measurements used to prepare $\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}$, together with Bob's observables from Eq. (41), produce correlations that achieve $I_\alpha^{\max}$ in the corresponding Bell experiment, where α and η are related as in Eq. (41).

The 'only if' direction follows from a proof by contradiction. Assume that there exists $\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}$ with $\theta \neq \eta$, with $S_\eta[\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}] = S_\eta^{\max}$. Then, the state and measurements used to prepare $\mathbf{\Sigma}^\theta_{\mathbb{A}|\mathbb{X}}$, together with Bob's observables from Eq. (41), become a quantum realisation of correlations that achieve $I_\alpha^{\max}$ for the Bell functional I_α, where $\alpha = \frac{2}{\sqrt{1+2\tan^2(2\eta)}}$.

By Remark 14 then one concludes that there exists a local isometry in Alice's lab that takes $|\theta\rangle$ to $|\eta\rangle$. This local isometry, however, can never exist, since $|\theta\rangle$ cannot be transformed into $|\eta\rangle$ by such local operations [21]. Hence, it must be the case that $\theta = \eta$. □

Remark 18. *Let $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}$ be a quantum assemblage, and let ρ and $\{\{M_{a|x}\}_{a\in\mathbb{A}}\}_{x\in\mathbb{X}}$ be a quantum state and measurements that realise it. Then, $S_\eta[\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}}] = S_\eta^{\max}$ if and only if:*
- The observables $A'_0 = M_{0|0} - M_{1|0}$ and $A'_1 = M_{0|1} - M_{1|1}$ are equivalent (up to local isometries) to A_0 and A_1, respectively, from Eq. (36),
- The quantum state ρ is equivalent, up to local isometries, to $|\eta\rangle$.

Proof. The 'if' direction follows trivially from Remark 17.
To prove the 'only if' direction, notice that ρ, $\{A'_j\}_{j=0,1}$, and Bob's observables $\{B_j\}_{j=0,1}$ from Eq. (41), provide a quantum realisation of correlations that achieve the maximum value of the Bell functional I_α, with α and η related as in Eq. (41). From Remark 14 it follows that ρ and $\{A'_j\}_{j=0,1}$ must be equal to (up to local isometries) the state and observables of Eqs. (37) and (36), which proves our claim. □

C The multipartite Bell inequalities

Consider N-partite Bell scenario with two dichotomic measurements per party. The Bell inequality studied in Ref. [72] for this scenario, defined in Ref. [72, Eq. (13)], reads:

$$\begin{aligned} I_\alpha^{(N)} &= (N-1)\langle (B_0+B_1)A_0^{(1)}...A_0^{(N-1)}\rangle + (N-1)\frac{\cos 2\alpha}{\sqrt{1-\cos^2 2\alpha}}(\langle B_0\rangle - \langle B_1\rangle) \\ &+ \frac{1}{\sqrt{1-\cos^2 2\alpha}}\sum_{i=1}^{N-1}\langle (B_0-B_1)A_1^{(i)}\rangle\,, \end{aligned} \tag{43}$$

where $\alpha \in (0,\pi/4]$.

The set $\mathsf{I}^{(N)} = \{I_\alpha^{(N)} \,|\, \alpha \in (0,\pi/4]\}$ defines, for fixed N, a family of Bell inequalities for the multipartite scenario, indexed by the parameter α. It was shown in Ref. [72] that the maximum quantum violation of $I_\alpha^{(N)}$ is given by $I_\alpha^{(N)\,\mathrm{max}} = 2\sqrt{2}(N-1)$, and the classical bound of $I_\alpha^{(N)}$ is given by $I_\alpha^{(N)\,\mathrm{C}} = (N-1)\frac{1-\cos 2\alpha}{\sqrt{1-\cos 2\alpha}}$. The value $I_\alpha^{(N)\,\mathrm{max}}$ is achieved by measuring the observables

$$A_0^{(i)} = \sigma_x\,, \qquad B_0 = \cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,, \tag{44}$$

$$A_1^{(i)} = \sigma_z\,, \qquad B_1 = -\cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,, \tag{45}$$

for $i \in \{1...\text{N-1}\}$, on the quantum state

$$|GHZ_N^\theta\rangle = \cos\theta\,|0\rangle^{\otimes N} + \sin\theta\,|1\rangle^{\otimes N}\,, \tag{46}$$

with the relation between the parameters α, θ, and μ being the following:

$$\alpha = \theta\,, \quad 2\sin^2(\mu) = \sin^2(2\theta)\,. \tag{47}$$

It was also proved in Ref. [72] that the inequality (43) can be used to self-test the the state in Eq. (46) for any $\theta \in (0,\pi/4]$.

Remark 19. *Let $\widetilde{A}_x^{(i)}$, $\widetilde{B}_y$, and $|\widetilde{\psi}\rangle$ be physical observables and a physical quantum state that achieve the value $I_\alpha^{(N)\,\mathrm{max}}$ for the Bell inequality $I_\alpha^{(N)}$. Then, these are equal to (up to local isometries) the state and observables of Eqs. (46), (44), and (45).*

D Constructing multipartite EPR inequalities from Bell inequalities

We now define a family of multipartite EPR functionals $S_\alpha^{(N)}$ from the multipartite Bell inequalities $I_\alpha^{(N)} \in \mathsf{I}^{(N)}$. A generic N-partite EPR functional is defined as:

$$S^{(N)}[\mathbf{\Sigma}_N] = \mathrm{tr}\left\{\sum_{\substack{a_i\in\mathbb{A},\\ x_i\in\mathbb{X}}} F_{a_1...a_{N-1},x_1...x_{N-1}}\,\sigma_{a_1...a_{N-1}|x_1...x_{N-1}}\right\}. \tag{48}$$

To transform the Bell functional to the EPR functional, notice the following relations. First, for any choice of $x_1 \dots x_{N-1}$, we can write:

$$\langle B \rangle = \mathrm{tr}\left\{B\,\rho_B\right\} = \sum_{a_1 \dots a_{N-1}} \mathrm{tr}\left\{B\,\sigma_{a_1 \dots a_{N-1}|x_1 \dots x_{N-1}}\right\} . \tag{49}$$

Moreover,

$$\begin{aligned} \langle (B_0 - B_1) A_1^{(i)} \rangle &= \mathrm{tr}\left\{\mathbb{I}^{(1)} \otimes \dots \otimes (\widetilde{M}_{0|1} - \widetilde{M}_{1|1})^{(i)} \otimes \dots \otimes \mathbb{I}^{(N-1)} \otimes (B_0 - B_1)\,\rho\right\} \\ &= \mathrm{tr}\left\{(B_0 - B_1)\,(\sigma_{0|1}^{(i)} - \sigma_{1|1}^{(i)})\right\} , \end{aligned} \tag{50}$$

where $\sigma_{a|x}^{(i)} = \sum_{a_j, j\neq i} \sigma_{a_1 \dots (a_i = a) \dots a_{N-1}|x_1 \dots (x_i = x) \dots x_{N-1}}$.

Lastly,

$$\begin{aligned} \langle (B_0 + B_1) A_0^{(1)} \dots A_0^{(N-1)} \rangle &= \mathrm{tr}\left\{(\widetilde{M}_{0|0} - \widetilde{M}_{1|0})^{(1)} \otimes \dots \otimes (\widetilde{M}_{0|0} - \widetilde{M}_{1|0})^{(N-1)} \otimes (B_0 + B_1)\,\rho\right\} \\ &= \sum_{a_1, \dots, a_{N-1}} (-1)^{a_1 + \dots + a_{N-1}} \mathrm{tr}\left\{\widetilde{M}_{a_1|0}^{(1)} \otimes \dots \otimes \widetilde{M}_{a_{N-1}|0}^{(N-1)} \otimes (B_0 + B_1)\,\rho\right\} \\ &= \sum_{a, \dots, a_{N-1}} (-1)^{a_1 + \dots + a_{N-1}} \mathrm{tr}\left\{(B_0 + B_1)\,\sigma_{a_1 \dots a_{N-1}|0 \dots 0}\right\} . \end{aligned} \tag{51}$$

Using the above relations, we define an EPR functional $S_\alpha^{(N)}[\boldsymbol{\Sigma}_N]$ by setting:

$$F_{a_1 \dots a_{N-1}, 0 \dots 0} = (N-1)(-1)^{a_1 + \dots + a_{N-1}}(B_0 + B_1) + (N-1)\frac{\cos 2\alpha}{\sqrt{1 - \cos^2 2\alpha}}(B_0 - B_1), \tag{52}$$

$$F_{a_1 \dots (a_i = a) \dots x_{N-1}, x_1 \dots (x_i = 1) \dots x_{N-1}} = (-1)^a \frac{1}{\sqrt{1 - \cos^2 2\alpha}}(B_0 - B_1).$$

From the operators of Eq. (52), we construct a family of EPR functionals as follows.

Definition 20. *EPR functional* $S_\eta^{(N)}[\boldsymbol{\Sigma}_N]$.
We define the EPR functional $S_\eta^{(N)}[\boldsymbol{\Sigma}_N]$ via the operators of Eq. (52) by taking:

$$\begin{aligned} \alpha &= \eta\,, \\ B_0 &= \cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,, \\ B_1 &= -\cos(\mu)\,\sigma_z + \sin(\mu)\,\sigma_x\,, \\ 2\sin^2(\mu) &= \sin^2(2\eta)\,. \end{aligned} \tag{53}$$

The properties of the multipartite EPR functional from Definition 20 can be studied in a similar manner to the properties of the bipartite EPR functional from Definition 15. We now make a few remarks about these properties.

Remark 21. *The maximum quantum value $S_\eta^{(N)\max}$ of $S_\eta^{(N)}[\boldsymbol{\Sigma}_N]$ is given by*

$$S_\eta^{(N)\max} = 2\sqrt{2}\,(N-1)\,. \tag{54}$$

Proof. The problem of optimising $S_\eta^{(N)}[\boldsymbol{\Sigma}_N]$ over quantum assemblages is equivalent to the problem optimising $I_\alpha^{(N)}$ over quantum correlations constrained on Bob's observables being those from Eq. (53) for $\alpha = \eta$. Since the maximum quantum violation $I_\alpha^{(N)\max}$ can indeed be achieved by the operators for Bob from Eq. (53), then $S_\eta^{(N)\max} = I_\alpha^{(N)\max}$. □

In order to explore quantum assemblages that achieve this maximum violation, first we need to prove a claim about the interconvertibility of the quantum states of the form given in Eq. (28) under LOSR operations. Let us recall Corollary 8 in [21].

Remark 22 ([21, Corollary 8])**.** *An n-partite pure state $|\psi\rangle$ can be converted to an n-partite pure state $|\phi\rangle$ by LOSR only if*

$$\exists\,|\zeta\rangle\,,\forall\beta : (\lambda_\psi^{(\beta)})^\downarrow = (\lambda_\phi^{(\beta)} \otimes \lambda_\zeta^{(\beta)})^\downarrow \tag{55}$$

where for a pure state $|\omega\rangle$, $\lambda_\omega^{(\beta)}$ denotes the vector of its squared Schmidt coefficients with respect to bipartition β of the n-partite system.

We are now in position to prove the following Lemma.

Lemma 23. *Let $\rho_N^\theta = |GHZ_N^\theta\rangle\langle GHZ_N^\theta|$ be defined as per Eq. (28). Then*

$$\rho_N^\theta \overset{LOSR}{\not\longrightarrow} \rho_N^\phi\,, \tag{56}$$

for any $\theta \neq \phi \in (0, {}^{\pi}\!/\!_{4}]$.

Proof. Let $\rho_N^\theta = |GHZ_N^\theta\rangle\langle GHZ_N^\theta|$ and $\rho_N^\phi = |GHZ_N^\phi\rangle\langle GHZ_N^\phi|$ with $\theta \neq \phi \in (0, {}^{\pi}\!/\!_{4}]$. All bipartitions β of ρ_N^θ and ρ_N^ϕ have the same Schmidt rank, which is 2 for all β. However, for all β, the vectors of the squared Schmidt coefficients $\lambda_\theta^{(\beta)}$ are different than $\lambda_\phi^{(\beta)}$, since

$$\lambda_\theta^{(\beta)} = \begin{bmatrix}\cos^2\theta \\ \sin^2\theta\end{bmatrix} \quad \text{and} \quad \lambda_\phi^{(\beta)} = \begin{bmatrix}\cos^2\phi \\ \sin^2\phi\end{bmatrix}.$$

By applying Corollary 9 of Ref. [21], it follows that there exists no $|\zeta\rangle$ such that the condition specified in Eq. (55) is satisfied. From Remark 22 it follows that ρ_N^θ and ρ_N^ϕ are such that neither converts to the other under LOSR operations. □

Remark 24. *The multipartite quantum assemblage $\mathbf{\Sigma}_N^\theta$ satisfies $S_\eta^{(N)}[\mathbf{\Sigma}_N^\theta] = S_\eta^{(N)\max}$ if and only if $\eta = \theta$.*

Proof. The 'if' direction is straightforward to prove, by noticing that the state and measurements used to prepare $\mathbf{\Sigma}_N^\theta$, given by Eqs. (28) and (29), together with Bob's observables from Eq. (53), produce correlations that achieve $I_\alpha^{(N)\max}$ in the corresponding Bell experiment, where $\alpha = \eta$.

The 'only if' direction follows from a proof by contradiction. Assume that there exists $\mathbf{\Sigma}_N^\theta$ with $\theta \neq \eta$, such that $S_\eta^{(N)}[\mathbf{\Sigma}_N^\theta] = S_\eta^{(N)\max}$. Then, the state and measurements used to prepare $\mathbf{\Sigma}_N^\theta$, together with Bob's observables from Eq. (53), become a quantum realisation of correlations that achieve $I_\alpha^{(N)\max}$ for the Bell functional $I_\alpha^{(N)}$, where $\alpha = \eta$. Then, it follows from Remark 19 that there exists a local isometry in Alice's lab that takes ρ_N^θ to ρ_N^η (both defined in Eq. (28)). However, such local isometry does not exist, as it would contradict Lemma 23. Hence, it must be the case that $\theta = \eta$.

□

Remark 25. *Let $\mathbf{\Sigma}_N$ be a quantum assemblage, and let ρ and $\{\{M_{a_i|x_i}\}_{a_i\in\mathbb{A}}\}_{x_i\in\mathbb{X}\,,\,i\in\{1,\ldots,N-1\}}$ be a quantum state and measurements that realise it. Then, $S_\eta^{(N)}[\mathbf{\Sigma}_N] = S_\eta^{(N)\max}$ if and only if:*

- The observables $A_0'^{(i)} = M_{0|0} - M_{1|0}$ and $A_1'^{(i)} = M_{0|1} - M_{1|1}$ are equivalent (up to local isometries) to $A_0^{(i)}$ and $A_1^{(i)}$, respectively, from Eqs. (44) and (45) for each Alice, i.e., $i \in \{1, \ldots, N-1\}$,

- The quantum state ρ is equivalent, up to local isometries, to ρ_N^η.

Proof. The 'if' direction follows trivially from Remark 24.
To prove the 'only if' direction, notice that ρ and $\{A'^{(i)}_j\}_{j=0,1\,,\,i\in\{1,\ldots,N-1\}}$, together with Bob's observables $\{B_j\}_{j=0,1}$ from Eq. (53), provide a quantum realisation of correlations that achieve the maximum value of the Bell functional $I^{(N)}_\alpha$, with $\alpha=\eta$. From Remark 19 it follows that ρ and $\{A'^{(i)}_j\}_{j=0,1\,,\,i\in\{1,\ldots,N-1\}}$ must be equal to (up to local isometries) the state and observables of Eq. (46), (44), and (45), which proves our claim. □

E Proof of Corollary 12

In this section, we give a proof of Corollary 12, which pertains to multipartite quantum assemblages.

Let us make a few remarks about the properties of the resource monotone $\mathrm{M}^{(N)}_\eta$ given by Definition 11. First, note that when $\mathbf{\Sigma}_N$ is a quantum assemblage, its LOSR processing $\widetilde{\mathbf{\Sigma}}_N$ is also a quantum assemblage. Hence, $S^{(N)}_\eta[\widetilde{\mathbf{\Sigma}}_N]$ is upperbounded by its maximum quantum violation, which is given by $S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]$:

$$\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}_N]\leq S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]\quad\forall\,\eta\in(0,{}^{\pi}/_{4}]\ . \tag{57}$$

Second, note that $\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]=S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]$. These two properties set the stage for the following theorem:

Theorem 26. *For a monotone* $\mathrm{M}^{(N)}_\eta$ *given in definition 11 and an EPR functional* $S^{(N)}_\eta$ *specified in definition 20, if* $\theta\neq\eta$*, then* $\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]<S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]$.

Proof. Let us prove this by contradiction. Our starting assumption is that there exists a pair $(\theta\,,\,\eta)$ with $\theta\neq\eta$, such that $\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]=S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]$. Then, one of the two should happen:

First case: $\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]=S^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]$.
In this case, the solution to the maximisation problem in the computation of $\mathrm{M}^{(N)}_\eta$ is achieved by $\mathbf{\Sigma}^\theta_N$ itself.
Our starting assumption then tells us that $S^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]=S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]=S^{(N)\,\max}_\eta$.
From Remark 24 it follows that necessarily $\theta=\eta$, which contradicts our initial condition.

Second case: $\mathrm{M}^{(N)}_\eta[\mathbf{\Sigma}^\theta_N]=S^{(N)}_\eta[\widetilde{\mathbf{\Sigma}}_N]$, with $\mathbf{\Sigma}^\theta_N\xrightarrow{\text{LOSR}}\widetilde{\mathbf{\Sigma}}_N$.
In this case, the solution to the maximisation problem in the computation of M_η is achieved by an LOSR processing of $\mathbf{\Sigma}^\theta_N$. Our starting assumption then tells us that $S^{(N)}_\eta[\widetilde{\mathbf{\Sigma}}_N]=S^{(N)}_\eta[\mathbf{\Sigma}^\eta_N]=S^{(N)\,\max}_\eta$. Let ρ and $\{\{M_{a_i|x_i}\}_{a_i\in A}\}_{x_i\in X\,,\,i\in\{1,\ldots,N-1\}}$ be any quantum state and measurements that realise the quantum assemblage $\widetilde{\mathbf{\Sigma}}_N$ in N-partite EPR scenario. From Remark 25, we know that ρ is equivalent to ρ^η_N up to local isometries. But since local isometries are free LOSR operations, this means that $\rho^\theta_N\xrightarrow{\text{LOSR}}\rho^\eta_N$, which contradicts Lemma 23. □

We can now prove that $\mathsf{S}^{(N)}$ is composed of pairwise unordered resources. In analogue to the bipartite case, to prove this claim it suffices to find a set of EPR LOSR monotones $\mathsf{M}=\{\mathrm{M}_j\}$ such that, for every pair (θ_1,θ_2) there exists a pair $(\mathrm{M}_1,\mathrm{M}_2)$ with $\mathrm{M}_1(\mathbf{\Sigma}^{\theta_1}_N)>\mathrm{M}_1(\mathbf{\Sigma}^{\theta_2}_N)\,,\quad\mathrm{M}_2(\mathbf{\Sigma}^{\theta_1}_N)<\mathrm{M}_2(\mathbf{\Sigma}^{\theta_2}_N)$. As we see next, this is achieved by choosing $\mathrm{M}_k=\mathrm{M}^{(N)}_{\theta_k}$.

Theorem 27. *For every pair $\theta_1 \neq \theta_2$, the monotones $M^{(N)}_{\theta_1}$ and $M^{(N)}_{\theta_2}$ given by Definition 11 satisfy*

$$\mathrm{M}^{(N)}_{\theta_1}(\mathbf{\Sigma}^{\theta_1}_N) > \mathrm{M}^{(N)}_{\theta_1}(\mathbf{\Sigma}^{\theta_2}_N)\,, \tag{58}$$

$$\mathrm{M}^{(N)}_{\theta_2}(\mathbf{\Sigma}^{\theta_1}_N) < \mathrm{M}^{(N)}_{\theta_2}(\mathbf{\Sigma}^{\theta_2}_N)\,. \tag{59}$$

Proof. Let us first prove Eq. (58). We know that $\mathrm{M}^{(N)}_{\theta_1}(\mathbf{\Sigma}^{\theta_1}_N) = S^{(N)}_{\theta_1}[\mathbf{\Sigma}^{\theta_1}_N]$. However, since $\theta_1 \neq \theta_2$, Theorem 26 implies that $\mathrm{M}^{(N)}_{\theta_1}(\mathbf{\Sigma}^{\theta_2}_N) < S^{(N)}_{\theta_1}[\mathbf{\Sigma}^{\theta_1}_N]$. Therefore, Eq. (58) follows. The proof of Eq. (59) follows similarly. □

Corollary 12 follows directly from Theorem 27.

F Proof of Proposition 13

Let us focus on the family of assemblages $\mathbf{\Sigma}^{\theta,p}_{\mathbb{A}|\mathbb{X}}$ defined by Eq. (10). In Corollary 8 we showed that under LOSR operations, all assemblages $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$ are incomparable for $\theta \in (0, \pi/4]$. We will now show that under S-1W-LOCC, the assemblage $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ is above all others in the family; hence, the pre-orders in the two different resource theories are different.

The S-1W-LOCC map that maps $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ to $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$ is very simple: the instrument has a single outcome, say 0, and the corresponding map $\mathcal{E}_0(\cdot)$ on Bob's system is $M_0 \cdot M_0^\dagger$, where $M_0 = \cos(\theta)\,|0\rangle\,\langle 0| + \sin(\theta)\,|1\rangle\,\langle 1|$ for the value of θ corresponding to the assemblage $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$. A straightforward calculation confirms that this map converts from $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ to the target assemblage, and that the outcome 0 occurs with probability 1/2. Remarkably, this map does not even require communication and is actually an example of a stochastic-LOSR operation, i.e., an LOSR map without the requirement that the map applied on Bob's subsystem is trace-preserving, but occurs with non-zero probability.

One could argue that the reason the pre-orders are different for stochastic-1W-LOCC and LOSR is because of the stochastic nature of the former. However, the pre-orders for deterministic 1W-LOCC maps and LOSR is again different; as before, there is such a deterministic 1W-LOCC map that converts $\mathbf{\Sigma}^{\pi/4,1}_{\mathbb{A}|\mathbb{X}}$ to $\mathbf{\Sigma}^{\theta,1}_{\mathbb{A}|\mathbb{X}}$. It is defined as follows: there are two outcomes $\omega \in \{0, 1\}$ where, for outcome 0, $\mathcal{E}_0(\cdot) = M_0 \cdot M_0^\dagger$ (as before), and for outcome 1, $\mathcal{E}_1 = M_1 \cdot M_1^\dagger$, where $M_1 = \sin(\theta)\,|1\rangle\,\langle 0| + \cos(\theta)\,|0\rangle\,\langle 1|$. One can readily confirm that this is trace-preserving, since $M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{I}$. Bob then communicates the outcome ω to Alice, whose input x is equal to the input x' to her measurement device. Once she gets the output a' of the measurement, the final output becomes $a = a' \oplus x\omega \oplus \omega$. To summarise, Alice will flip the output of her measurement if and only if $x = 0$ and $\omega = 1$.

The resource theory of nonclassicality of channel assemblages

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When two parties, Alice and Bob, share correlated quantum systems and Alice performs local measurements, Alice's updated description of Bob's state can provide evidence of nonclassical correlations. This simple scenario, famously introduced by Einstein, Podolsky and Rosen (EPR), can be modified by allowing Bob to also have a classical or quantum system as an input. In this case, Alice updates her knowledge of the *channel* (rather than of a state) in Bob's lab. In this paper, we provide a unified framework for studying the nonclassicality of various such generalizations of the EPR scenario. We do so using a resource theory wherein the free operations are local operations and shared randomness (LOSR). We derive a semidefinite program for studying the pre-order of EPR resources and discover possible conversions between the latter. Moreover, we study conversions between post-quantum resources both analytically and numerically.

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Contents

1 Introduction

It is a well-established fact that nature exhibits phenomena that are not explainable by classical laws of physics. Some of these phenomena, such as Bell nonclassicality [1, 2], pertain to correlational aspects of distant physical systems. For instance, Einstein-Podolsky-Rosen (EPR) 'steering' [3–5] refers to a scenario where local measurements on half of a system prepared in an entangled state can generate nonclassical correlations between two distant parties. This phenomenon is a crucial resource for various information-theoretic tasks [6, 7], such as quantum cryptography [8, 9], entanglement certification [10, 11], randomness certification [12, 13], and self-testing [14–16]. The wide applicability of EPR correlations motivates a program of characterizing their resourcefulness [5, 17–21], both in the standard EPR scenario and in multipartite generalizations thereof.

1.1 Generalizations of the EPR scenario

The standard EPR scenario consists of two parties, Alice and Bob, that share a bipartite system prepared in an entangled state. By performing measurements on her share of the system, Alice updates her knowledge about the state of Bob's subsystem[1]. Various generalizations of this standard scenario have been introduced in recent years, wherein Bob *also* may probe his system in various ways. In such cases, Alice's measurements allow her to make inferences not merely about the state of Bob's system, but also about the overall process in Bob's laboratory. Instances of such scenarios include the channel EPR scenario [25], Bob-with-input EPR scenario [26], measurement device-independent EPR scenario [27], and the famous Bell scenario [2, 22]. These scenarios are all closely related, and a unified framework for understanding them was introduced in Refs. [24, 28]. One can easily understand the basic relationship between these distinct scenarios by considering Fig. 1. The standard EPR scenario is shown in Fig. 1(a); here, Alice and Bob share a quantum system, and Alice performs measurements labeled by classical inputs to obtain classical outputs. When Bob performs measurements with classical input and output systems as well, as illustrated in Fig. 1(b), one recovers the Bell scenario [2, 22]. More generally, when Bob's input and output systems are quantum, as shown in Fig. 1(c), one obtains the channel EPR scenario [25]. If Bob's input is quantum and the output is classical, one recovers the measurement-device-independent EPR scenario [27], shown in Fig. 1(d), wherein Alice makes inferences about Bob's measurement channel. Finally, if the input is classical and the output is quantum, one recovers the Bob-with-input EPR scenario [26], illustrated in Fig. 1(e), wherein Alice makes inferences about Bob's state preparation channel. These scenarios all have a similar *common-cause* causal structure; what distinguishes them is the *type* – classical or quantum – of Bob's input and output system.

In every type of EPR scenario, Alice chooses from various incompatible methods of refining her knowledge about Bob's process. Her knowledge after performing her local measurements is represented by an *ensemble* of Bob's updated processes together with their associated probabilities of arising. In analogy to the traditional EPR scenario, the collection of these ensembles (one for each of her possible measurements) is termed an *assemblage* [17]. In other words, the standard concept of a collection of ensembles of quantum states in the standard EPR scenario

[1]The fact that Alice learns about Bob's distant subsystem is sometimes taken to be evidence for action-at-a-distance. The very term 'steering' suggests that Alice has a *causal influence* on Bob's system. In this paper, we do not endorse this view, as we take the causal structure of the EPR scenario to be a *common cause* one. For this reason, we will refer to a 'steering scenario' as an *EPR scenario* and say that Alice updates her knowledge about the state of Bob's subsystem, rather than 'steering' him. For more discussion of and motivation for this view, see Refs. [21–24]

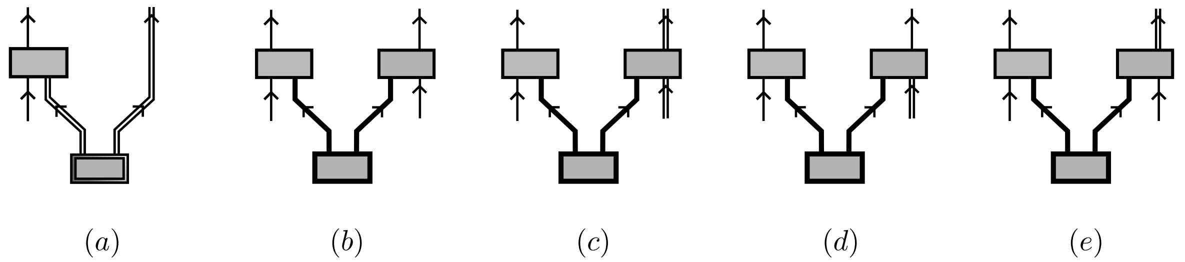

Figure 1: **Generalizations of the traditional EPR scenario:** (a) traditional bipartite EPR scenario. (b) Bell scenario. (c) Channel EPR scenario. (d) Measurement-device-independent EPR scenario. (e) Bob-with-input EPR scenario. Quantum and classical systems are depicted by double and single lines, respectively. In the special case when the common cause in each scenario is classical, these are all free resources. Thick black lines depict the possibility that the shared systems may be classical, quantum, or even post-quantum, as discussed in Section 1.2.

turns into *a collection of ensembles of processes* in these generalized EPR scenarios. The assemblage associated to a given scenario contains all the relevant information for characterizing the correlational resources shared between the parties in the scenario.

1.2 The resource theory

In every type of EPR scenario, the most basic problem is identifying which assemblages provide evidence of nonclassicality – that is, which assemblages can only be generated if Alice and Bob share a nonclassical common cause [21]. One can then ask about the relative nonclassicality of assemblages, i.e., whether a given assemblage is more or less nonclassical than another. This line of reasoning leads to the framework of resource theories, a principled approach to quantifying the value of a broad range of objects. This approach has proved to be useful in many areas of physics, including coherence [29–31], athermality [32–37], LOCC-entanglement [38–40], LOSR-entanglement [23, 41], and other common-cause processes [22, 24, 28], including some beyond what quantum theory allows (i.e., *post-quantum* processes) [42–44].

In this work, we develop a resource theory which quantifies the nonclassicality of channel assemblages in channel EPR scenarios, subsuming all of the special cases in Fig. 1. This resource theory is itself a special case (where one focuses on EPR scenarios) of the type-independent resource theory introduced in Refs. [24, 28], which are motivated by the causal modelling perspective on studying nonclassicality [45, 46]. Note that the special case of standard EPR scenarios (bipartite and multipartite, with quantum common-cause systems) is analyzed in detail in Ref. [21]; hence, we will not focus on them here. Similarly, the special case of Bell scenarios was studied in detail in Ref. [22], and so is not studied in detail here.

In this resource theory of channel assemblages, the **free resources** – that is, classical assemblages – are those that can be generated by local operations and classical common causes. This approach is motivated by previous works [21–24, 28] which argue that the resourcefulness in scenarios of this type is best characterized as *nonclassicality of the common-cause*. It follows that the operations that can be applied to the resources without increasing their nonclassicality include local operations and classical common causes. In other words, the free operations in our resource theory are *local operations and shared randomness* (LOSR).

Another important ingredient in the definition of a resource theory is its **enveloping theory**, which specifies the scope of all possible resources under consideration. Refs. [20, 21] study the case of bipartite and multipartite EPR scenarios where the enveloping theory is taken to be that of quantum resources, i.e., assemblages that Alice and Bob can prepare in the laboratory by performing classically-correlated local actions on a shared quantum system that may be prepared on an entangled state. In this work, however, we consider the enveloping theory to contain all assemblages which can be generated by Alice performing measurements on a bipartite system whose state may be described by an arbitrary theory (a possibility formalized within the framework of generalized probabilistic theories and depicted by thick lines in Fig. 1) [47]. As shown by Ref. [47], this enveloping theory contains all and only the no-signalling assemblages. Notably, not every assemblage that satisfies the no-signalling principle can be generated by Alice and Bob sharing quantum resources [26, 48]; this is in analogy to Popescu-Rohrlich boxes [49] in Bell scenarios. An exception is given by assemblages in traditional bipartite EPR scenarios, where the renowned theorem by Gisin [50] and Hughston, Josza, and Wootters [51] (GHJW) proves that the most general assemblages – the non-signalling ones – all admit a quantum realization. Our broadening of the enveloping theory of resources allows for a perspective from which to study how quantum resources emerge and how their resourcefulness is bounded relative to all logically possible resources. This is a main advantage of the resource theory presented in this work over the resource theory we developed in Ref. [21]. For this reason, we mainly focus on studying post-quantum resources in this paper.

1.3 Summary of main results

In Section 2, we specify the resource theory of channel EPR scenarios. This section includes the formal definition of the channel EPR scenario and the most general channel assemblage processing under LOSR operations. Moreover, we give a semidefinite program for testing resource conversion under free operations in the corresponding scenario. We then run this program to study the pre-order of channel assemblages.

In Sections 3 and 4, we focus on two special cases of channel EPR scenarios: Bob-with-input EPR scenarios and measurement-device-independent EPR scenarios, respectively. We show how the particular set of free operations emerges in each of these scenarios from the general case of Section 2 by specifying the necessary system types. We also present simplified semidefinite programs for testing resource conversion under free operations, and discuss properties of the pre-order of resources.

Finally, we discuss related work in Section 5. On the one hand, we note that there are significant conceptual differences between our work and Ref. [25], which first studied channel EPR scenarios. In particular, Ref. [25] seems to be interested not in nonclassicality in a common-cause scenario, but rather in scenarios with communication between Bob and Alice. While this makes no difference for the set of free resources (when one restricts to no-signaling resources), it does make a significant difference for the overall resource theory. On the other hand, we highlight the similarities and differences between our approach and the resource theory under Local Operations and Shared Entanglement from Ref. [42].

2 The channel EPR scenario

In the channel EPR scenario, two distant parties (Alice and Bob) share a physical system AB. In addition, Bob has a quantum system defined on $\mathcal{H}_{B_{in}}$ which he acts on locally by applying a quantum channel $\Lambda^{B_{in}\to B_{out}}$. As a result, he obtains a quantum system defined on $\mathcal{H}_{B_{out}}$.

It is possible that the channel $\Lambda^{B_{in}\to B_{out}}$ (or its application to system B_{in}) is being influenced by the presence of system B in Bob's lab. If this were the case, the effective channel $\Lambda^{B_{in}\to B_{out}}$ would instead be a larger process $\Gamma^{BB_{in}\to B_{out}}$ acting on both the quantum system B_{in} and the system B. If B were a quantum system, $\Gamma^{BB_{in}\to B_{out}}$ would be called the *channel extension*[2] of $\Lambda^{B_{in}\to B_{out}}$. The idea of a channel EPR scenario is to assume that indeed $\Gamma^{BB_{in}\to B_{out}}$ is the process happening at Bob's lab, and see what Alice can infer about the quantum channel $\Lambda^{B_{in}\to B_{out}}$ from the outcome statistics she locally observes on her system A. The way Alice probes her system A is by performing measurements on it. Fig. 2(a) illustrates such a channel EPR scenario, where the classical and quantum systems are depicted with single and double wires, respectively, and the thick wires depict systems which may be classical, quantum, or even post-quantum.

Let $x \in \mathbb{X}$ be the classical variable that denotes Alice's choice of measurements, and $a \in \mathbb{A}$ the classical variable that denotes her observed outcome[3]. By $p(a|x)$ we denote the probability with which Alice obtains outcome a having performed measurement x on her system A. In addition, we denote by $\mathcal{I}_{a|x}(\cdot)$ the instrument that is effectively applied on Bob's quantum system B_{in} to produce a quantum system B_{out}, given that Alice has performed measurement x on A and obtained the outcome a. It follows then that the object of study in such a channel EPR scenario is the *channel assemblage* of instruments $\mathbf{I}_{\mathbb{A}|\mathbb{X}} = \{\mathcal{I}_{a|x}(\cdot)\}_{a\in\mathbb{A}\,,x\in\mathbb{X}}$, with $\mathrm{tr}_{B_{out}}\left\{\mathcal{I}_{a|x}(\rho)\right\} = p(a|x)$ for every normalised state ρ of quantum system B_{in}, and $\sum_{a\in\mathbb{A}}\mathcal{I}_{a|x}(\cdot) = \Lambda^{B_{in}\to B_{out}}$ for all $x \in \mathbb{X}$.

Let us illustrate an example of channel assemblages in the case of Alice and Bob sharing a bipartite quantum system prepared on a state ρ_{AB} – that is, we take the common cause AB mentioned before to be quantum. In this case, Alice's most general measurements are the *generalized measurements*, i.e., positive operator-valued measures (POVMs), which we denote by $\{M_{a|x}\}_{a\in\mathbb{A},x\in\mathbb{X}}$. The elements of this channel assemblage will then be:

$$\mathcal{I}_{a|x}(\cdot) = \mathrm{tr}_A\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes\Gamma^{BB_{in}\to B_{out}})[\rho_{AB}\otimes(\cdot)]\right\}, \tag{1}$$

where $(\cdot)$ denotes the input system B_{in}. For each $x \in \mathbb{X}$, the instruments $\{\mathcal{I}_{a|x}\}_{a\in\mathbb{A}}$ form a channel which does not depend on x, i.e., $\sum_{a\in\mathbb{A}}\mathcal{I}_{a|x}$ is a completely positive and trace preserving (CPTP) map [52, 53] which does not depend on Alice's measurement choice.

In general, however, we will not take the common cause AB to necessarily be a quantum system. This common cause may be classical or even post-quantum.

2.1 LOSR-free channel assemblages

The fundamental objects that define the notion of nonclassicality in a resource theory are the free resources. The free channel assemblages, i.e., assemblages that admit a classical description, are those that can be generated by classical common-cause processes. Therefore, the free assemblages in our framework can be understood as objects that arise when Bob locally applies an instrument and Alice performs a measurement, where both actions depend on a shared classical random variable. In this situation, Alice can refine her description of Bob's process by learning the classical value of the shared variable. In other words, the classical channel assemblages are those that can be generated from LOSR operations. Formally, the elements of a free channel

[2]Ref. [25] motivates the correlation between $\Lambda^{B_{in}\to B_{out}}$ and system B by arguing that the former may be a noisy quantum channel that leaks information to the environment, which can then correlate itself with B or even get all the way to Alice's lab. From our causal perspective, such environment can be taken to be a common cause between Alice and Bob, whereas the leakage of information can be interpreted as the correlation mechanism that arises between Alice's system and Bob's quantum channel through this common cause.

[3]In principle, different measurements can have different number of outcomes. Here, for ease of the presentation, we focus on the case where all of Alice's measurements have the same number of outcomes. Our techniques generalize straightforwardly.

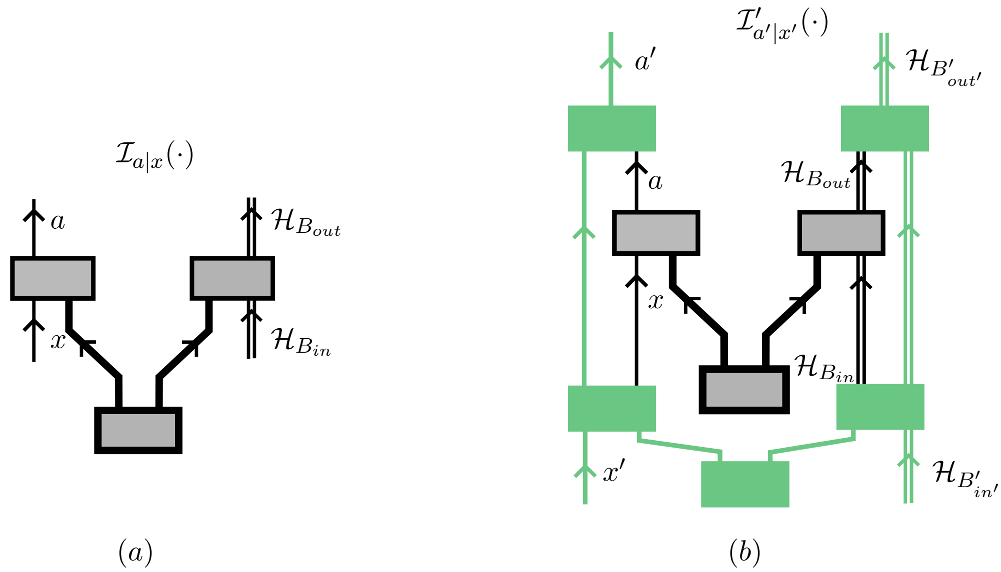

Figure 2: Depiction of a channel EPR scenario. Arbitrary systems (which may be post-quantum) are represented by thick lines, quantum systems are represented by double lines and classical systems are depicted as single lines. (a) Channel assemblage: Alice and Bob share a (possibly) post-quantum common cause; Alice's input and output systems are classical while Bob's input and output systems are quantum. (b) The most general LOSR operation on an assemblage in a channel EPR scenario.

assemblage can be written as

$$\mathcal{I}_{a|x}(\cdot) = \sum_{\lambda} p(a|x,\lambda)p(\lambda)\mathcal{I}_{\lambda}(\cdot), \tag{2}$$

where $p(\lambda)$ represents the state of the common cause, $\mathcal{I}_{\lambda}$ is a CPTP map (from system B_{in} to B_{out}) for each λ, and $p(a|x,\lambda)$ is a valid conditional probability distributions for all values of λ.

A convenient representation of Eq. (2) is given by the Choi-Jamiołkowski isomorphism [54, 55]. Recall that every CPTP map $\mathcal{E} : \mathcal{H}_B \to \mathcal{H}_{B'}$ can be associated with an operator W on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ such that $\mathcal{E}(\rho_B) = d_B \operatorname{tr}_B \left\{ W \left(\mathbb{I}_{B'} \otimes \rho_B^T \right) \right\}$, where d_B is the dimension of the system ρ_B. Conversely, the operator W can be written as $W = (\mathcal{E} \otimes \mathbb{I}_{B'}) |\Omega\rangle \langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d_B}} \sum_{i=1}^{d_B} |ii\rangle$. We will use this representation throughout the paper. Let $J_{a|x}$ and J'_{λ} represent the Choi-Jamiołkowski operators that correspond to $\mathcal{I}_{a|x}$ and $\mathcal{I}_{\lambda}$, respectively. Moreover, denote $p(\lambda)J'_{\lambda}$ as J_{λ}. If $\{\mathcal{I}_{a|x}\}_{a,x}$ form an LOSR-free channel assemblage, each $J_{a|x}$ can be expressed as

$$J_{a|x} = \sum_{\lambda} p(a|x,\lambda) J_{\lambda}. \tag{3}$$

Therefore, checking whether an assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ is free amounts to checking if the operators $J_{a|x}$ admit a decomposition of the form given in Eq. (3). Moreover, each probability distribution $p(a|x,\lambda)$ can be decomposed as $p(a|x,\lambda) = \sum_{\lambda'} p(\lambda'|\lambda) D(a|x,\lambda')$, where $D(a|x,\lambda')$ is a deterministic probability distribution. It follows that testing whether a channel assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ is free requires a single instance of a semidefinite program (SDP):

SDP 1. *The channel assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ is LOSR-free if and only if the following SDP is feasible:*

$$
\begin{aligned}
&\textit{given} \quad \{J_{a|x}\}_{a,x}\,,\ \{D(a|x,\lambda)\}_{\lambda,a,x}\,,\\
&\textit{find} \quad \{(J_\lambda)_{B_{in}B_{out}}\}_\lambda\\
&\textit{s.t.} \quad \begin{cases} J_\lambda \geq 0 \qquad \forall\lambda\,,\\ \mathrm{tr}_{B_{out}}\{J_\lambda\} \propto \mathbb{I}_{B_{in}} \qquad \forall\lambda\,,\\ \sum_\lambda \mathrm{tr}_{B_{out}}\{J_\lambda\} = \frac{1}{d_{B_{in}}}\mathbb{I}_{B_{in}}\,,\\ J_{a|x} = \sum_\lambda D(a|x,\lambda)J_\lambda \quad \forall\, a\in\mathbb{A}\,,\, x\in\mathbb{X}\,. \end{cases}
\end{aligned} \tag{4}
$$

Here $d_{B_{in}}$ is the dimension of the Hilbert space $\mathcal{H}_{B_{in}}$ associated with system B_{in}. From here on, as above, we will use the notation d_X to refer to the dimension of the Hilbert space associated with system X.

2.2 LOSR transformations between channel assemblages

The most general LOSR transformation of a channel EPR assemblage is presented in Fig. 2(b). Formally, this LOSR operation transforms elements of a channel assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ into new ones that form $\mathbf{I}'_{\mathbb{A}'|\mathbb{X}'}$ as follows:

$$
\mathcal{I}'_{a'|x'}(\cdot) = \sum_\lambda \sum_{a,x} p(\lambda)\, p(a',x|a,x',\lambda)\, \Lambda_\lambda^{B_{out}S\to B'_{out'}}\, (\mathcal{I}_{a|x}\otimes \mathbb{I}_S)\, \Lambda_\lambda^{B'_{in'}\to B_{in}S}(\cdot)\,, \tag{5}
$$

where

- $p(\lambda)$ is the probability distribution over the classical random variable λ which coordinates all the local transformations.
- $p(a',x|a,x',\lambda)$ encodes the classical pre- and post-processing of Alice's input x and output a. The output of this process is Alice's new outcome a'. The probability distribution $p(a',x|a,x',\lambda)$ satisfies the no-retrocausation condition (the variable a cannot influence the value of the variable x, i.e., $p(x|a,x',\widetilde{\lambda}) = p(x|x',\widetilde{\lambda})$).
- $\Lambda_\lambda^{B'_{in'}\to B_{in}S}$ is the map corresponding to Bob's local pre-processing. It acts on his quantum input on $\mathcal{H}_{B'_{in'}}$ and outputs a quantum system on a Hilbert space $\mathcal{H}_{B_{in}}\otimes\mathcal{H}_S$ (the index S corresponds to the quantum side channel of the local processing[4]).
- $\Lambda_\lambda^{B_{out}S\to B'_{out'}}$ is the map corresponding to Bob's post-processing stage. The output of this process is Bob's new quantum system on a Hilbert space $\mathcal{H}_{B'_{out'}}$.

It is convenient to express Bob's pre- and post-processing as a single completely positive and trace non-increasing (CPTNI) map, which we denote by $\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$, where $\sum_\lambda \xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$ forms a CPTP map. The inputs of this new map are the inputs of Bob's pre- and post-processing (defined on $\mathcal{H}_{B'_{in'}}\otimes\mathcal{H}_{B_{out}}$) and the outputs of this new map are the outputs of Bob's pre- and post-processing (defined on $\mathcal{H}_{B'_{out'}}\otimes\mathcal{H}_{B_{in}}$). The map $\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$ must be such that there is no signalling from the system defined on $\mathcal{H}_{B_{out}}$ to the system defined on $\mathcal{H}_{B_{in}}$, that is, the input of Bob's post-processing can not influence the output of his pre-processing. This condition can be expressed as follows:

$$
\forall\,\lambda \quad \exists\, F_\lambda^{B'_{in'}\to B_{in}} \qquad \text{s.t.} \qquad \mathrm{tr}_{B'_{out'}}\left\{\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}\right\} = F_\lambda^{B'_{in'}\to B_{in}} \otimes \mathbb{I}_{B_{out}}\,, \tag{6}
$$

[4]It is worth noticing that the quantum system on $\mathcal{H}_S$ is not of arbitrary dimension. By the results of Ref. [56], its dimension is restricted by the product of the dimensions of $\mathcal{H}_{B'_{in'}}$ and $\mathcal{I}'_{a'|x'}(\cdot)$.

where $\sum_\lambda F_\lambda^{B'_{in'}\to B_{in}}$ is a CPTP map. Moreover, it is convenient to use the Choi-Jamiołkowski isomorphism to express the maps of interest. Let $J'_{a'|x'}$, $J_{a|x}$ and $J_{\xi\,\lambda}$ correspond to the Choi representation of maps $\mathcal{I}'_{a'|x'}$, $\mathcal{I}_{a|x}$ and $\xi_\lambda^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$, respectively. Then, Eq. (5) can be expressed as

$$J'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,, \tag{7}$$

where the maps are composed using the link product $*$ described in Ref. [57]:

$$J_{a|x} * J_{\xi\,\lambda} = d_{B_{in}}\, d_{B_{out}}\, \mathrm{tr}_{B_{in}B_{out}} \left\{ (\mathbb{I}_{B'_{in'}B'_{out'}} \otimes J_{a|x}^{T_{B_{out}}})\, J_{\xi\,\lambda}^{T_{B_{in}}} \right\}\,. \tag{8}$$

Notice that Alice's local processing is expressed in terms of deterministic probability distributions $D(\cdot)$ – this representation was discussed in detail in previous works [21, 22, 58]; we recall this discussion in Appendix A. In this scenario, the total number of the deterministic strategies encoded in λ is equal to $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|}$.

An assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ can be converted therefore into a different assemblage $\mathbf{I}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR if and only if there exist a collection of Choi states $\{J_{\xi\,\lambda}\}$ and operators $\{F_\lambda\}$ such that the elements of $\mathbf{I}'_{\mathbb{A}'|\mathbb{X}'}$ can be decomposed as in Eq. (7) with the conditions of Eq. (6). This correspondence enables us to derive the following SDP that checks whether such a decomposition is possible.

SDP 2. *The channel assemblage $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}}$ can be converted to the channel assemblage $\boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations, denoted by $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \xrightarrow{LOSR} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$, if and only if the following SDP is feasible:*

$$\begin{aligned}
&\textit{given} \quad \{J_{a|x}\}_{a,x}\,,\ \{J'_{a'|x'}\}_{a',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
&\textit{find} \quad \{(J_{\xi\,\lambda})_{B_{in}B'_{in'}B_{out}B'_{out'}}\}_\lambda\,,\ \{(J_{F\,\lambda})_{B_{in}B'_{in'}}\}_\lambda \\
&\textit{s.t.} \quad \begin{cases}
J_{\xi\,\lambda} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} \propto \mathbb{I}_{B_{out}B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} = \frac{1}{d_{B_{out}}d_{B'_{in'}}}\mathbb{I}_{B_{out}B'_{in'}}\,, \\
J_{F\,\lambda} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} \propto \mathbb{I}_{B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} = \frac{1}{d_{B'_{in'}}}\mathbb{I}_{B'_{in'}}\,, \\
\mathrm{tr}_{B'_{out'}}\{J_{\xi\,\lambda}\} = J_{F\,\lambda} \otimes \frac{1}{d_{B_{out}}}\mathbb{I}_{B_{out}} \quad \forall\lambda\,, \\
J'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,.
\end{cases}
\end{aligned} \tag{9}$$

When the conversion is not possible, we denote it by $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\not\longrightarrow} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$.

This SDP is a feasibility problem. Let us mention that a feasibility problem can always be converted into an optimization of some linear objective function. Such formulation of the program is favourable if one wants to increase the robustness of the results. In Appendix B, we provide an alternative formulation of SDP 2 in this form.

2.3 Properties of the pre-order

One channel assemblage is claimed to be at least as nonclassical as another if a conversion under free operations from the former to the latter is possible. Therefore, testing assemblage conversion under LOSR operations provides information about relative nonclassicality of assemblages. We now proceed to study the conversions between quantum-realizable and post-quantum channel assemblages defined below.

2.3.1 Quantum-realizable channel assemblages

First, we focus on an infinite family of quantum-realizable channel assemblages. Consider a channel assemblage with $\mathbb{A} = \mathbb{X} = \{0, 1\}$. Assume that Alice and Bob have access to a two qubit system in a Bell state, denoted by $\rho_{AB} = |\phi\rangle\langle\phi|$, with $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. Define a family of channel assemblages R as follows: for an angle $\theta \in (0, \pi/2]$, Bob performs a controlled rotation around the y axis on systems BB_{in}, where the control qubit is B (it is entangled with Alice's qubit A) and θ is the rotation angle. We denote such quantum gate by CR_y^θ; it can be written as:

$$\mathrm{CR}_y^\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta/2) & -\sin(\theta/2) \\ 0 & 0 & \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{10}$$

The measurements Alice performs on her share of the system are given by $M_{a|0} = \frac{1}{2}\{\mathbb{I}+(-1)^a\sigma_z\}$ and $M_{a|1} = \frac{1}{2}\{\mathbb{I} + (-1)^a\sigma_x\}$, where σ_z and σ_x are Pauli matrices. Then, the elements of the family R are the following:

$$\mathsf{R} = \left\{ \mathbf{I}^\theta_{\mathbb{A}|\mathbb{X}} \,\middle|\, \theta \in (0, \pi/2] \right\} , \tag{11}$$

$$\text{where} \quad \mathbf{I}^\theta_{\mathbb{A}|\mathbb{X}} = \left\{ \mathcal{I}^\theta_{a|x}(\cdot) \right\}_{a\in\mathbb{A}, x\in\mathbb{X}} ,$$

$$\text{with} \quad \mathcal{I}^\theta_{a|x}(\cdot) = \mathrm{tr}_A \left\{ (M_{a|x} \otimes \mathbb{I}_{B_{out}})\, \mathrm{tr}_B \left\{ (\mathbb{I}_A \otimes \mathrm{CR}_y^\theta)\, (\rho_{AB} \otimes (\cdot)_{B_{in}})(\mathbb{I}_A \otimes \mathrm{CR}_y^\theta)^\dagger \right\} \right\} ,$$

The family of assemblages R is illustrated in Fig. 3.

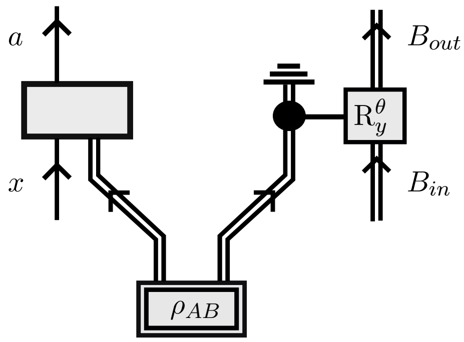

Figure 3: Depiction of an assemblage belonging to the family R. Quantum systems are represented by double lines and classical systems are depicted as single lines. Alice and Bob share an entangled common-cause ρ_{AB}. Depending on the state of the control qubit, Bob either does nothing or implements a rotation by angle θ around the y axis on his input system.

The family R has an infinite number of elements indexed by the angle $\{\theta\}$. To study the pre-order of assemblages in this family, we convert the assemblages into Choi form and run the SDP 2 (in Matlab [59], using the software CVX [60, 61], the solver SeDuMi [62] and the toolbox

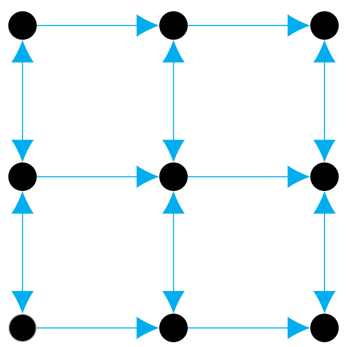

Figure 4: Possible conversions between elements of R. The black dots represent the assemblages $\mathbf{I}^{\theta,i}_{\mathbb{A}|\mathbb{X}}$, where $i \in \{x, y, z\}$. The arrows represent possible conversions.

QETLAB [63]; see the code at [64]). By checking what the possible conversions are between resources characterized by different values of θ, we observe that a conversion $\mathbf{I}^{\theta_1}_{\mathbb{A}|\mathbb{X}} \xrightarrow{\text{LOSR}} \mathbf{I}^{\theta_2}_{\mathbb{A}|\mathbb{X}}$ is possible only when $\theta_1 > \theta_2$ for every pair $\{\theta_1, \theta_2\}$ that we checked. Based on this observation, we formulate the following conjecture:

Conjecture 3. *Let the two resources $\boldsymbol{I}^{\theta_1}_{\mathbb{A}|\mathbb{X}}$ and $\boldsymbol{I}^{\theta_2}_{\mathbb{A}|\mathbb{X}}$ belong to the infinite family of assemblages* R. *Then, the conversion $\boldsymbol{I}^{\theta_1}_{\mathbb{A}|\mathbb{X}} \xrightarrow{LOSR} \boldsymbol{I}^{\theta_2}_{\mathbb{A}|\mathbb{X}}$ is possible if and only if $\theta_1 \geq \theta_2$.*

We run SDP 2 to study two more infinite families of assemblages analogous to R with the exception that rotations around the z-axis (the x-axis for the second family) instead of around the y-axis are implemented. For both cases, we observe a behaviour analogous to that stated in Conjecture 3. Finally, by checking the conversions between assemblages belonging to different families, we find assemblages that are interconvertible (See Fig. 4), which leads us to the following conjecture:

Conjecture 4. *Let $\boldsymbol{I}^{\theta,i}_{\mathbb{A}|\mathbb{X}}$ denote an assemblage generated when Bob applies CR^{θ}_{i}. Then, for a fixed value of θ, assemblages $\{\boldsymbol{I}^{\theta,x}_{\mathbb{A}|\mathbb{X}}, \boldsymbol{I}^{\theta,y}_{\mathbb{A}|\mathbb{X}}, \boldsymbol{I}^{\theta,z}_{\mathbb{A}|\mathbb{X}}\}$ are in the same LOSR equivalence class.*

Our explorations are summarized in Fig. 4.

2.3.2 Post-quantum channel assemblages

Studying possible conversions among channel assemblages may also give us insight into the pre-order of post-quantum resources. We now focus on a channel EPR scenario where $\mathbb{X} = \{0, 1, 2\}$, $\mathbb{A} = \{0, 1\}$, and $\mathcal{H}_{B_{out}} = \mathcal{H}_B$ are qubit Hilbert spaces, and we consider conversions between two post-quantum channel assemblages defined below[5].

[5]These two post-quantum channel assemblages are generalizations of Bob-with-input assemblages introduced in Ref. [26]. We elaborate on this point later in the text.

Our first example of a post-quantum channel assemblage, depicted in Fig. 5(a), can be conveniently expressed in a mathematical way as:

$$\mathbf{I}^{PTP}_{\mathbb{A}|\mathbb{X}} = \left\{\mathcal{I}^{PTP}_{a|x}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X}} , \tag{12}$$
$$\text{with} \quad \begin{cases} \mathcal{I}^{PTP}_{a|x}(\cdot) = \mathrm{tr}_A\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})\,\mathrm{tr}_B\left\{(\mathbb{I}_A\otimes \mathrm{CT}^{BB_{in}\to B_{out}})\,(\rho_{AB}\otimes(\cdot)_{B_{in}})\right\}\right\} , \\ M_{a|0} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2} , \quad M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_y}{2} , \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2} , \end{cases}$$

where $\rho_{AB} = |\phi\rangle\langle\phi|$, with $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, and σ_x, σ_y, and σ_z are the Pauli operators. The operation $\mathrm{CT}^{BB_{in}\to B_{out}}$ is a controlled-transpose operation, where the control system is B_{in}, and the transpose is applied on the system B. For simplicity, in this section we denote $\mathbf{I}^{PTP}_{\mathbb{A}|\mathbb{X}} = \mathbf{I}^{PTP}$. Here, the abbreviation PTP stands for positive trace preserving.

It is important to note that the expression given in Eq. (12) and the assemblage preparation procedure illustrated in Fig. 5(a) give a convenient representation of $\mathbf{I}^{PTP}$, but is not meant to be taken as the unique description of this assemblage. The use of the transpose map here, which is positive but not completely positive, is just one of the possible mathematical ways to represent this post-quantum assemblage.

One then may argue that the channel assemblage $\mathbf{I}^{PTP}$ is a post-quantum assemblage. To see this, one can simply see that a special case of $\mathbf{I}^{PTP}$ – that where Bob's input states are classical labels encoded into an orthonormal basis – was shown to be a post-quantum assemblage [26]. It follows hence that $\mathbf{I}^{PTP}$ is post-quantum as well.

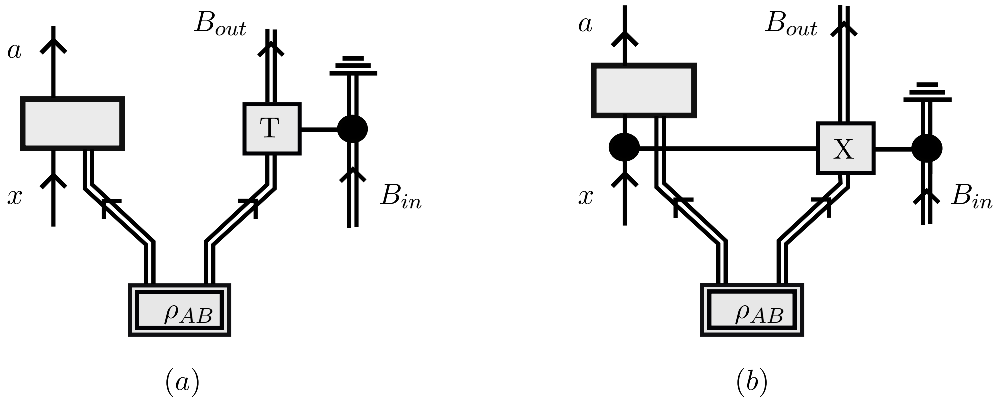

Figure 5: Mathematical depiction of two post-quantum channel assemblages. (a) Channel assemblage $\mathbf{I}^{PTP}$: Alice and Bob share a Bell state; Alice performs measurements on her system, while Bob performs a controlled-transpose operation. (b) Channel assemblage $\mathbf{I}^{PR}$: Alice and Bob share a Bell state; Alice performs measurements on her system, while Bob performs a CNOT operation controlled by both Alice's and Bob's inputs.

The second example of a post-quantum channel assemblage that we consider may be mathematically expressed as:

$$\mathbf{I}^{PR}_{\mathbb{A}|\mathbb{X}} = \left\{\mathcal{I}^{PR}_{a|x}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X}} , \tag{13}$$
$$\text{with} \quad \mathcal{I}^{PR}_{a|x}(\cdot) = \begin{cases} \mathrm{tr}_A\left\{(M_a\otimes \mathbb{I}_{B_{out}})\,\mathrm{tr}_B\left\{(\mathbb{I}_A\otimes \mathrm{CX}^{BB_{in}\to B_{out}})\,(\rho_{AB}\otimes(\cdot)_{B_{in}})\right\}\right\} & \text{if}\, x\in\{0,1\}, \\ \mathcal{I}^{PR}_{a|x}(\cdot) = \frac{\mathbb{I}}{2}a & \text{if}\, x=2\,, \end{cases}$$
$$\text{and} \quad M_0 = |0\rangle\langle 0| , \quad M_1 = |1\rangle\langle 1| ,$$

where ρ_{AB} is the same as in the previous example. This assemblage is illustrated in Fig. 5(b). If $x\in\{0,1\}$, Bob applies a controlled-X operation, where x and B_{in} are the control systems.

More precisely, Bob applies a quantum instrument to his input system B_{in} – he measures B_{in} in the computational basis, registers the classical output, which we here denote by y, and a quantum system containing the post-measurement state. Then, Bob transforms the system B depending on the values of x and y. If $xy = 0$, he applies the identity map. If $xy = 1$, Bob flips the system B. Finally, if $x = 2$, Bob prepares the system $\frac{\mathbb{I}}{2}a$. For simplicity, hereon we denote $\mathbf{I}^{PR}_{\mathbb{A}|\mathbb{X}} = \mathbf{I}^{PR}$.

Note that Eq. (13) is merely meant as a mathematical description of the channel assemblage $\mathbf{I}^{PR}$, and is not meant to be taken as its experimental implementation. In particular, notice that the channel assemblage $\mathbf{I}^{PR}$ is no-signalling between Alice and Bob, contrary to what our chosen mathematical description may suggest.

One may now argue that $\mathbf{I}^{PR}$ is a post-quantum assemblage. To to this, imagine that B_{in} contains just a set of classical labels $\{|0\rangle, |1\rangle\}$, and Alice and Bob generate $\mathbf{I}^{PR}$. Then, notice that for $x \in \{0, 1\}$, if Bob performs a measurement on his subsystem on the $\{|0\rangle, |1\rangle\}$ basis and observes a classical outcome b, Alice and Bob obtain correlations $p(ab|xy)$ that correspond to Popescu-Rohrlich (PR) box correlations [49], which are known to be post-quantum. This shows that the assemblage $\mathbf{I}^{PR}$ is a post-quantum channel assemblage.

To study the relative order of $\mathbf{I}^{PTP}$ and $\mathbf{I}^{PR}$, we convert the assemblages into Choi form and run the SDP 2 (in Matlab [59], using the software CVX [60, 61], the solver SeDuMi [62] and the toolbox QETLAB [63]; see the code at [64]). We find that the two assemblages are incomparable, which is summarized by the following observation:

Observation 5. *The two post-quantum channel assemblages $\mathbf{I}^{PR}$ and $\mathbf{I}^{PTP}$ are unordered resources in the LOSR resource theory of common-cause assemblages.*

3 The Bob-with-input EPR scenario

The Bob-with-input EPR scenario, first introduced in Ref. [26], is a special case of the channel EPR scenario. In this setting, Bob can locally influence the state preparation of his system. This scenario is illustrated in Fig. 6(a). On the one hand, Alice acts on her share of the system by performing measurements and registering the obtained outcome – this is identical to the role she plays in the channel EPR scenario. On the other hand, Bob chooses the value of a classical variable y, referred to as 'Bob's input', which influences the state preparation of a quantum system in his laboratory. Operationally, one may think of Bob as holding a device that accepts the classical input y (together with a physical system), and produces a quantum system prepared on some specified state. Bob's device's inner-workings can be thought of as a transformation of his subsystem (not necessarily a quantum one) into a new quantum system, where the transformation depends on the value of y. Notice that the system shared by Alice and Bob is depicted with a thick line in Fig. 6(a) – indeed, as mentioned regarding the channel EPR scenario, the GHJW theorem does not apply when Bob has an input; hence, one can find instances of *post-quantum* Bob-with-input assemblages that do not admit a quantum realization [48].

The relevant *Bob-with-input assemblage* is now given by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \{\sigma_{a|xy}\}_{a,x,y}$, with $\mathrm{tr}\left\{\sigma_{a|xy}\right\} = p(a|x)$ and $\mathrm{tr}\left\{\sum_a \sigma_{a|xy}\right\} = 1$ for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. Notice that if y takes only one value, this scenario coincides with the traditional EPR scenario for which an LOSR-based resource theory was developed in Ref. [21].

If the common-cause Alice and Bob share is a quantum state, which we denote ρ_{AB}, the most general local operation Bob's device can implement is a collection of CPTP maps $\{\xi_y\}_{y\in\mathbb{Y}}$. When Alice implements a POVM from the set $\{M_{a|x}\}_{a\in\mathbb{A},x\in\mathbb{X}}$, we say that the elements of the

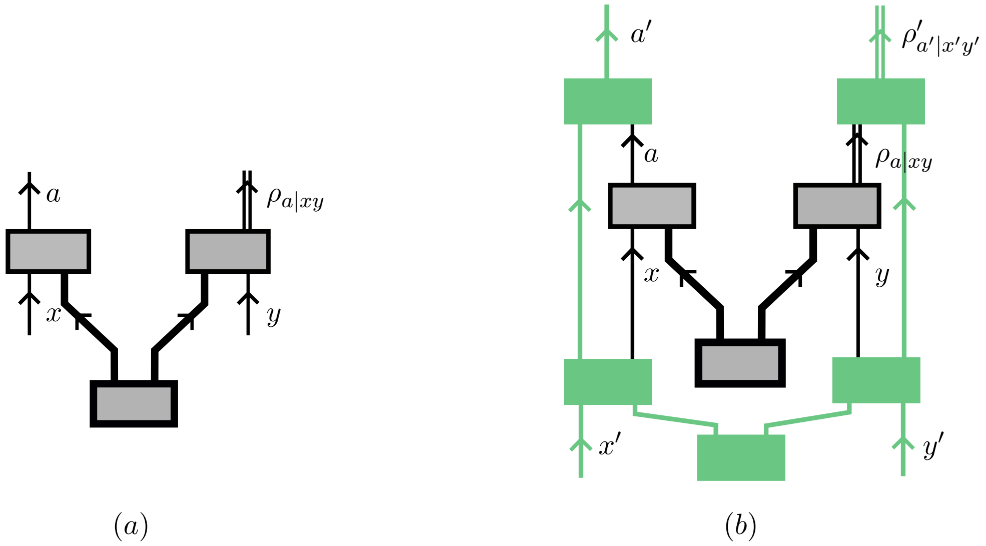

Figure 6: Depiction of a Bob-with-input EPR scenario. Systems that may be classical, quantum, or even post-quantum, are represented by thick lines. Quantum systems are represented by double lines and classical systems are depicted as single lines. (a) Bob-with-input assemblage: Alice and Bob share a (possibly post-quantum) common cause; Alice's input and output systems are classical while Bob's input is classical and output is quantum. (b) The most general LOSR operation on an assemblage in a Bob-with-input EPR scenario.

assemblage admit a *quantum realization* of the form

$$\sigma_{a|xy} = \xi_y[\mathrm{tr}_A\{(M_{a|x} \otimes \mathbb{I}_B)\rho_{AB}\}] \tag{14}$$

for all $a \in \mathbb{A}, x \in \mathbb{X}$, and $y \in \mathbb{Y}$.

Similarly to the channel EPR scenario, nonclassical Bob-with-input assemblages are those that require a nonclassical common-cause. The classical Bob-with-input assemblages, i.e., the LOSR-free ones, can always be viewed as generated by local operations applied by each party that depend on the value of a shared classical random variable. Formally, a free assemblage in this scenario can be expressed as $\sigma_{a|xy} = \sum_\lambda p(\lambda)p(a|x\lambda)\rho_{\lambda y}$. Here, λ is the shared classical variable that is sampled according to $p(\lambda)$, $p(a|x\lambda)$ is a well-defined conditional probability distribution for all values of λ, and the quantum states $\rho_{\lambda y}$ are locally generated by Bob depending on the values of the classical variables y and λ. This set of classical assemblages was first defined in Ref. [26], where it is referred to as the set of 'unsteerable' assemblages. Determining whether a given Bob-with-input assemblage is LOSR-free is possible with a single instance of an SDP, which is analogous both to SDP 1 and (when $|\mathbb{Y}| = 1$) to the SDP for checking 'steerability' of a standard EPR assemblage given in Ref. [65].

3.1 LOSR transformations between Bob-with-input assemblages

The most general LOSR transformation of a Bob-with-input assemblage is illustrated in Fig. 6(b), where a local processing known as *comb* (which locally pre- and post-processes the relevant systems in each wing) [57] with appropriate input/output system types is applied to each party. Notice how these are a special case of the processes of Fig. 2(b) for the specific type of Bob's input system. This set of operations transforms one assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ into a new assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ as follows:

$$\sigma'_{a'|x'y'} = \sum_\lambda \sum_{a,x,y} p(a',x|a,x',\lambda)p(y|y',\lambda)p(\lambda)\xi_{\lambda,y'}(\sigma_{a|xy}). \tag{15}$$

Here, $p(a',x|a,x',\lambda)$ encodes Alice's variables pre- and post-processing (this process is the same as in the channel EPR scenario), $p(y|y',\lambda)$ encodes a classical pre-processing of Bob's classical input y as a function of y' and λ, and $\xi_{\lambda,y'}(\cdot)$ is the map corresponding to Bob's local post-processing of his quantum system as a function of λ and y'. Notice that, just like in the case of channel EPR scenarios, $p(a',x|a,x',\lambda)$ satisfies the no-retrocausation condition.

A simplified characterisation of a generic Bob-with-input LOSR transformation in terms of deterministic probability distributions $D(\cdot)$ is given by

$$\sigma'_{a'|x'y'} = \sum_{\lambda}\sum_{a,x,y} D(x|x',\lambda)D(a'|a,x',\lambda)D(y|y',\lambda)\widetilde{\xi}_{\lambda,y'}(\sigma_{a|xy})\,, \tag{16}$$

where $\widetilde{\xi}_{\lambda,y'}(\cdot) = p(\lambda)\xi_{\lambda,y'}(\cdot)$. We will use this representation throughout this section; its detailed derivation is given in Appendix A.2. In the Bob-with-Input scenario, the total number of the deterministic strategies encoded in λ is equal to $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|} \times |\mathbb{X}|^{|\mathbb{X}'|} \times |\mathbb{Y}|^{|\mathbb{Y}'|}$.

Given two assemblages, $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$, deciding whether $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be converted into $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ under LOSR operations is equivalent to checking whether the elements of $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ admit a decomposition as per Eq. (16). The Bob-with-input EPR scenario is simply a special case of the channel EPR scenario wherein Bob's input is a classical system. However, as the Bob-with-input EPR scenario exhibits a simpler characterization of an LOSR transformation of an assemblage, the SDP for testing resource conversion in this scenario can be simplified compared to SDP 2, as we show below.

Notice that the CPTNI map $\widetilde{\xi}_{\lambda,y'}(\sigma_{a|xy})$ can be represented in terms of its (possibly subnormalized) Choi state $J_{\xi\,\lambda\,y'}$ as follows:

$$\widetilde{\xi}_{\lambda,y'}(\sigma_{a|xy}) = d_B \operatorname{tr}_B\left\{J_{\xi\,\lambda\,y'}\left(\mathbb{I}_{B'}\otimes\sigma^T_{a|xy}\right)\right\}\,, \tag{17}$$

where Bob's output system is defined on $\mathcal{H}_{B'}$. Therefore, for $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ to admit a decomposition as per Eq. (16), each $\sigma'_{a'|x'y'}$ must decompose as

$$\sigma'_{a'|x'y'} = \sum_{\lambda}\sum_{a,x,y} D(x|x',\lambda)D(a'|a,x',\lambda)D(y|y',\lambda)\, d_B \operatorname{tr}_B\left\{J_{\xi\,\lambda\,y'}\left(\mathbb{I}_{B'}\otimes\sigma^T_{a|xy}\right)\right\}\,. \tag{18}$$

We implement this condition in the following SDP:

SDP 6. $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} \overset{LOSR}{\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$.
The assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ can be converted into the assemblage $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ under LOSR operations if and only if the following SDP is feasible:

$$\begin{aligned}
&\textit{given} \quad \{\sigma_{a|xy}\}_{a,x,y}\,,\ \{\sigma'_{a'|x'y'}\}_{a',x',y'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(y|y',\lambda)\}_{\lambda,y,y'}\\
&\textit{find} \quad \{(J_{\xi\,\lambda\,y'})_{BB'}\}_{\lambda,y'}\\
&\textit{s.t.} \quad \begin{cases}
J_{\xi\,\lambda\,y'} \geq 0 \quad \forall\,\lambda, y'\,,\\
\operatorname{tr}_{B'}\left\{J_{\xi\,\lambda\,y'}\right\} \propto \mathbb{I}_B \quad \forall\,\lambda, y'\,,\\
\sum_\lambda \operatorname{tr}_{B'}\left\{J_{\xi\,\lambda\,y'}\right\} = \frac{1}{d}\mathbb{I}_B \quad \forall\, y'\,,\\
\operatorname{tr}_{B'}\left\{J_{\xi\,\lambda\,y'_1}\right\} = \operatorname{tr}_{B'}\left\{J_{\xi\,\lambda\,y'_2}\right\} \quad \forall\,\lambda\,, y'_1, y'_2\,,\\
\sigma'_{a'|x'y'} = \sum_\lambda\sum_{a,x,y} D(x|x',\lambda)\, D(a'|a,x',\lambda)\, D(y|y',\lambda)\, d_B \operatorname{tr}_B\left\{J_{\xi\,\lambda\,y'}\left(\mathbb{I}_{B'}\otimes\sigma^T_{a|xy}\right)\right\}\,.
\end{cases}
\end{aligned} \tag{19}$$

When the conversion is not possible, we denote it by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} \overset{LOSR}{\not\longrightarrow} \mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$.

For the robust formulation of this SDP, see Appendix B.

3.2 Properties of the pre-order

In analogy to the channel EPR scenario, studying post-quantum Bob-with-input assemblages gives us insight into the pre-order of resources. In this section, we introduce four Bob-with-input assemblages and study the possible conversions between them, both analytically and using SDP 6.

In Section 2.3, we focused on accessing the pre-order of channel assemblages using SDP 2. Due to the simpler nature of the Bob-with-input EPR scenario compared to the channel EPR scenario, here we first focus on studying the pre-order analytically. We start with a Bob-with-input EPR scenario where $\mathbb{X} = \{0, 1, 2\}$, $\mathbb{A} = \{0, 1\}$ and $\mathbb{Y} = \{0, 1\}$, and we consider conversions between two post-quantum Bob-with-input assemblages introduced in Ref. [26]. These two assemblages are special cases of channel assemblages $\mathbf{I}^{PTP}$ and $\mathbf{I}^{PR}$, where Bob's input states are just elements of an orthonormal basis $\{|0\rangle, |1\rangle\}$.

For our first post-quantum Bob-with-input assemblage, consider the Bob-with-input assemblage studied in Ref. [26, Eq. (6)], which is a special case of $\mathbf{I}^{PTP}$. This assemblage can be mathematically expressed as follows:

$$\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{PTP}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}}, \tag{20}$$
$$\text{with} \quad \begin{cases} \sigma^{PTP}_{a|xy} = \xi_y \left\{\mathrm{tr}_{\mathrm{A}} \left\{(M_{a|x} \otimes \mathbb{I}_B)\, |\phi\rangle\langle\phi|\right\}\right\}, \\ M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2}, \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a\sigma_y}{2}, \quad M_{a|3} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2}, \end{cases}$$

where $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and σ_x, σ_y, and σ_z are the Pauli operators. The map ξ_y is the following: the identity quantum channel for $y = 0$, and the transpose operation for $y = 1$. For simplicity, in this section we denote $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \mathbf{\Sigma}^{PTP}$. In Ref. [26, Appendix D], it was shown that $\mathbf{\Sigma}^{PTP}$ is a post-quantum assemblage. There it was moreover shown that, if Bob decides to measure his subsystem, the correlations that arise between him and Alice always admit a quantum explanation. This will prove relevant for the resource-conversion statements in this manuscript.

As a second example of a post-quantum Bob-with-input assemblage, we consider a special case of $\mathbf{I}^{PR}$. We follow the construction introduced in Ref. [26, Eq. (5)], and define:

$$\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{PR}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}}, \tag{21}$$
$$\text{with} \quad \sigma^{PR}_{a|xy} = \begin{cases} |a \oplus xy\rangle\langle a \oplus xy| & \text{if } x \in \{0, 1\} \\ \frac{\mathbb{I}}{2}a & \text{if } x = 2. \end{cases}$$

For simplicity, hereon we denote $\mathbf{\Sigma}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \mathbf{\Sigma}^{PR}$. As we already pointed out in the channel EPR scenario, for $x \in \{0, 1\}$, if Bob decides to measure his subsystem in the computational basis and registers a classical outcome b, Alice and Bob obtain PR box correlations, what certifies that the assemblage $\mathbf{\Sigma}^{PR}$ is post-quantum.

We will now show the the two post-quantum assemblages $\mathbf{\Sigma}^{PR}$ and $\mathbf{\Sigma}^{PTP}$ are unordered in our LOSR resource theory.

Theorem 7. *$\mathbf{\Sigma}^{PTP}$ cannot be converted into $\mathbf{\Sigma}^{PR}$ with LOSR operations.*

Proof. Let us prove this by contradiction. Assume that an LOSR-processing of $\mathbf{\Sigma}^{PTP}$ yields $\mathbf{\Sigma}^{PR}$. Then, since $\mathbf{\Sigma}^{PR}$ can generate post-quantum correlations in a Bell-type experiment, it follows that an LOSR-processing of $\mathbf{\Sigma}^{PTP}$ can also generate post-quantum correlations. However, we know that $\mathbf{\Sigma}^{PTP}$ can only generate quantum correlations [26, Appendix D]. As LOSR operations cannot create post-quantum Bell non-locality, this contradicts the initial assumption and hence proves the claim. □

Theorem 8. *$\mathbf{\Sigma}^{PR}$ cannot be converted into $\mathbf{\Sigma}^{PTP}$ with LOSR operations.*

The proof of this theorem is given in Appendix E.1. In the proof we use the 'steering' functional constructed in Ref. [26, Eq. (D3)] which achieves its minimum value when evaluated on $\mathbf{\Sigma}^{PTP}$. We show that neither $\mathbf{\Sigma}^{PR}$ or any LOSR-processing of $\mathbf{\Sigma}^{PR}$ can achieve the minimum value of this 'steering' functional, which completes the proof.

Corollary 9. *The two post-quantum assemblages $\mathbf{\Sigma}^{PR}$ and $\mathbf{\Sigma}^{PTP}$ are unordered resources in the LOSR resource theory of common-cause assemblages.*

This result can be verified with SDP 6 (we verified it in Matlab [59], using the software CVX [60, 61], the solver SeDuMi [62] and the toolbox QETLAB [63]; see the code at [64]).

It is usually the case that the pre-order in a given resource theory is studied using *resource monotones*. Interestingly, Corollary 9 does not rely on a construction of resource monotones, but it arises from specific considerations of LOSR transformations.

We now move on to introducing two more examples of Bob-with-input assemblages. We start by introducing an assemblage that, as far as we are aware, has not been studied in the literature before. For this purpose, let us briefly recall the meaning of almost-quantum correlations.

The set of almost-quantum correlations $\widetilde{Q}$ [66] was originally proposed as a set of correlations that satisfies numerous principles of a reasonable physical theory, including information causality [67], macroscopic locality [68] and local orthogonality [69]. It was first defined for Bell-type scenarios and it was showed to be larger than the set of quantum correlations and to strictly contain them. The concept of almost-quantum correlations was later generalized to other physical set-ups [70, 71], including EPR scenarios [26, 48]. Within this generalization, a particular relaxation of the definition of quantum assemblages allows one to construct almost-quantum assemblages.

From now on let us focus on a Bob-with-input scenario where $\mathbb{X} = \{0,1\}$, $\mathbb{A} = \{0,1\}$ and $\mathbb{Y} = \{0,1\}$. To construct an example of a post-quantum assemblage, consider the probability distribution generated in a bipartite Bell scenario introduced in Ref. [66], which we denote $\vec{p}_{AQ}$. We recall the exact form of $\vec{p}_{AQ}$ in Appendix C; for now, the important property of $\vec{p}_{AQ}$ to note is that it is post-quantum and it lives in $\widetilde{Q}$. Consider the following Bob-with-input assemblage:

$$\begin{aligned} \mathbf{\Sigma}^{AQ}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} &= \left\{ \sigma^{AQ}_{a|xy} \right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}} \,, \\ \text{with} \quad \sigma^{AQ}_{a|xy} &= \sum_b p_{AQ}(ab|xy) \left|b\right\rangle\left\langle b\right| \,, \end{aligned} \tag{22}$$

where the elements $p_{AQ}(ab|xy)$ can be read from the vector $\vec{p}_{AQ}$. For simplicity, we denote $\mathbf{\Sigma}^{AQ}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \mathbf{\Sigma}^{AQ}$. This assemblage is clearly not quantum-realizable. Indeed, if Bob chooses to measure his system in the computational basis, i.e., $N_b = |b\rangle\langle b|$, the correlations that Alice and Bob obtain are given by $p(ab|xy) = \mathrm{tr}\left\{N_b \sigma_{a|xy}\right\}$, which gives exactly $p_{AQ}(ab|xy)$. This shows that $\mathbf{\Sigma}^{AQ}$ is post-quantum[6], since for any quantum-realizable assemblage, Alice and Bob can only generate quantum correlations if Bob decides to measure his system.

As the next example, consider the assemblage $\mathbf{\Sigma}'^{PR}$ built from $\mathbf{\Sigma}^{PR}$ by considering the assemblage elements of the latter that correspond to $x = 0, 1$. To study the relative order of $\mathbf{\Sigma}^{AQ}$ and $\mathbf{\Sigma}'^{PR}$, we run the SDP 6 (see the code at [64]). We find that the two assemblages are strictly ordered, which is summarized by the following observation:

[6] It is important to note that the assemblage $\mathbf{\Sigma}^{AQ}$ is not necessarily an almost-quantum assemblage. The set of almost-quantum assemblages is a strict subset of the set of post-quantum assemblages. If Bob decides to measure his subsystem with a measurement different than $N_b = |b\rangle\langle b|$, it might be possible that post-quantum correlations that are not almost-quantum are generated.

Observation 10. *The post-quantum Bob-with-input assemblage $\mathbf{\Sigma}'^{PR}$ is strictly above $\mathbf{\Sigma}^{AQ}$ in the pre-order of resources in the LOSR resource theory of common-cause assemblages.*

In Appendix E.2, we analytically show that $\mathbf{\Sigma}'^{PR} \xrightarrow{\text{LOSR}} \mathbf{\Sigma}^{AQ}$.

For our final example in a Bob-with-input scenario, we consider the following quantum-realizable assemblage:

$$\mathbf{\Sigma}^{CHSH}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \left\{\sigma^{CHSH}_{a|xy}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}}, \tag{23}$$

$$\text{with} \quad \begin{cases} \sigma^{CHSH}_{a|xy} = \text{tr}_{\text{A}}\left\{(M_{a|x}\otimes \mathbb{I}_B)\,|\phi\rangle\langle\phi|\right\}, \\ |\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}, \\ M_{a|0} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2}, \quad M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2}, \end{cases}$$

where σ_x, σ_y, and σ_z are the Pauli operators. For simplicity, we denote $\mathbf{\Sigma}^{CHSH}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \mathbf{\Sigma}^{CHSH}$. Notice that if Alice and Bob generate $\mathbf{\Sigma}^{CHSH}$, and then Bob decides to measure his system with suitable measurements, they will obtain correlations that violate the $CHSH$ Bell inequality maximally [72].

The relation between $\mathbf{\Sigma}^{CHSH}$ and $\mathbf{\Sigma}^{AQ}$, obtained using SDP 6, is the following:

Observation 11. *The two Bob-with-input assemblages $\mathbf{\Sigma}^{CHSH}$ and $\mathbf{\Sigma}^{AQ}$ are unordered resources in the LOSR resource theory of common-cause assemblages.*

Notice that the direction $\mathbf{\Sigma}^{CHSH} \overset{\text{LOSR}}{\not\longrightarrow} \mathbf{\Sigma}^{AQ}$ is straight-forward to prove by noting that LOSR operations cannot create post-quantumness.

4 The measurement-device-independent EPR scenario

We will now consider a special case of a channel EPR scenario in which Bob has a measurement channel rather than a general quantum channel. This measurement-device-independent (MDI) EPR scenario is illustrated in Fig. 7(a). Alice and Bob share a system AB. Alice's role is still the same as in the channel and Bob-with-input scenarios: she performs measurements $\{M_{a|x}\}_{a\in\mathbb{A},x\in\mathbb{X}}$ to obtain a classical output a. Now, Bob holds a collection of measurement channels[7], which we denote by $\{\Omega_b^{B_{in}\to B_{out}}\}_{b\in\mathbb{B}}$, where the output system B_{out} is just a classical variable that may take values within $\mathbb{B}$. In a way one can think of these measurement channels as a measuring device that implements a single generalised measurement (as given by a POVM) and keeps a record of the obtained outcome.

Similarly to the channel EPR scenario, we are here interested in the case where the measurement channels that Bob has access to may in addition be correlated with some physical system in Alice's lab. This situation is formalised by the premise that Bob has instead access to a processing $\Theta_b^{BB_{in}\to B_{out}}$ that takes as input system his own B_{in} together with the half of the system AB he shares with Alice (denoted by B). The marginal measurement channel $\{\Omega_b^{B_{in}\to B_{out}}\}_{b\in\mathbb{B}}$ that was introduced at the beginning is therefore a function of these superseding measurement apparatus $\Theta_b^{BB_{in}\to B_{out}}$ and the state of the system AB shared by Alice and Bob (the specific dependence is specified further below). The idea is then to see what Alice can infer about the measurement channel $\Theta_b^{BB_{in}\to B_{out}}$ from the local measurements she is performing on her share of AB.

[7]Here, by *measurement channel* is meant a *quantum instrument* with a trivial output Hilbert space.

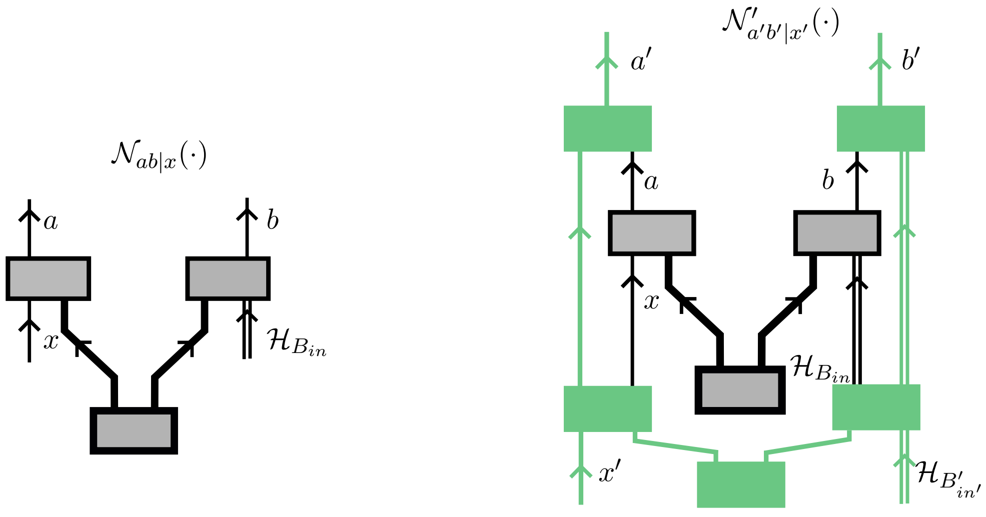

Figure 7: Depiction of a measurement-device-independent EPR scenario. Systems that may be classical, quantum, or even post-quantum, are represented by thick lines. Quantum systems are represented by double lines and classical systems are depicted as single lines. (a) MDI assemblage: Alice and Bob share a (possibly post-quantum) common cause; Alice's input and output systems are both classical, while Bob's input and output systems are quantum and classical, respectively. (b) The most general LOSR operation on an assemblage in a MDI EPR scenario.

Formally, the relevant *MDI assemblages* in this scenario are given by $\mathbf{N}_{\mathbb{AB}|\mathbb{X}} = \{\mathcal{N}_{ab|x}(\cdot)\}$ with $a \in \mathbb{A}, b \in \mathbb{B}$ and $x \in \mathbb{X}$, where $\mathcal{N}_{ab|x}(\cdot)$ is a channel – with trivial output space – associated to the POVM element corresponding to outcome b of Bob's measurement device (when Alice's measurement event is $a|x$). The elements of a valid MDI assemblage satisfy the conditions $\sum_b \mathcal{N}_{ab|x}(\cdot) = p(a|x)$ and $\sum_a \mathcal{N}_{ab|x}(\cdot) = \Omega_b^{B_{in}\to B_{out}}$. These conditions, equivalent to no-signalling requirements between Alice and Bob, define hence the enveloping theory of resources. A natural question then is when a given MDI assemblage has a classical, quantum, or post-quantum realisation, as we briefly recall next.

Let us begin with quantumly-realisable MDI assemblages, that is, those compatible with Alice and Bob sharing a quantum common cause. When Alice and Bob share a bipartite quantum system prepared on state ρ_{AB}, the elements of the MDI assemblage are given by

$$\mathcal{N}_{ab|x}(\cdot) = \mathrm{tr}\left\{(M_{a|x} \otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A \otimes \Theta_b^{BB_{in}\to B_{out}})[\rho_{AB} \otimes (\cdot)]\right\}. \tag{24}$$

Notably, MDI assemblages are not always compatible with a quantum common cause, i.e., there exist post-quantum MDI assemblages[8] [73].

Now, let us specify the free sub-theory of resources: those that may arise by the parties implementing local operations and shared randomness. Free MDI assemblages in our resource theory are those that can be generated with a classical common cause. They can be thought of as objects generated by Alice's local measurements and Bob's local measurement channel, where both parties are correlated by a classical variable. Formally, the elements of an LOSR-free MDI assemblage can be written as $\mathcal{N}_{ab|x}(\cdot) = \sum_\lambda p(\lambda)p(a|x\lambda)\mathcal{N}_{b,\lambda}(\cdot)$, where the measurement channels $\mathcal{N}_{b,\lambda}$ depend on the value of the classical common cause λ.

[8]Although Ref. [73] does not discuss measurement-device-independent assemblages directly, this scenario naturally appears if one wants to compare Buscemi non-locality and EPR scenarios.

4.1 LOSR transformations between measurement-device-independent assemblages

The most general LOSR transformation of an MDI assemblage is presented in Fig. 7(b). On Alice's side, the LOSR map is exactly the same as in the channel and Bob-with-input scenarios. On Bob's side, however, the processing has appropriate input and output system types adjusted to the MDI scenario. Let us express Bob's processing as a single CPTNI map $\zeta_{b\,b'\,\lambda}^{B'_{in'}\to B_{in}}$. This map has two inputs (one classical, b, and one quantum, $B'_{in'}$) and two outputs (again, one is classical, b', and one is quantum, B_{in}). Additionally, the map $\zeta_{b\,b'\,\lambda}^{B'_{in'}\to B_{in}}$ must satisfy the no-retrocausation condition, i.e., it must be such that b does not influence B_{in}. This condition means that $\sum_{\lambda,b'}\zeta_{b\,b'\,\lambda}^{B'_{in'}\to B_{in}}$ must be a CPTP map.

Similarly to the channel and Bob-with-input scenarios, it is convenient to use the Choi-Jamiołkowski isomorphism here. Let $J_{a'b'|x'}$, $J_{ab|x}$ and $J_{\zeta\,b\,b'\,\lambda}$ correspond to the Choi representation of maps $\mathcal{N}'_{a'b'|x'}$, $\mathcal{N}_{ab|x}$ and $\zeta_{b\,b'\,\lambda}$, respectively. Then, the most general LOSR processing of a MDI assemblage can be written as

$$J_{a'b'|x'} = \sum_{\lambda}\sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{ab|x} * J_{\zeta\,b\,b'\,\lambda}\,. \tag{25}$$

Here, $D(\cdot)$ are deterministic probability distributions (see previous sections and Appendix A.1 for details) and the total number of the deterministic strategies encoded in λ is equal to $|\mathbb{A}'|^{|\mathbb{A}|\times|\mathbb{X}'|}\times|\mathbb{X}|^{|\mathbb{X}'|}$. As mentioned before, the link product is given by

$$J_{ab|x} * J_{\zeta\,b\,b'\,\lambda} = d_{B_{in}}\,\mathrm{tr}_{B_{in}}\left\{(\mathbb{I}_{B'_{in'}}\otimes J_{ab|x})\, J_{\zeta\,b\,b'\,\lambda}^{T_{B_{in}}}\right\}\,. \tag{26}$$

Deciding if an assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ can be converted into a different assemblage $\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$ with LOSR operations comes down to checking whether there exist a collection of Choi states $\{J_{\zeta\,b\,b'\,\lambda}\}_{b,b',\lambda}$ such that the Choi form of $\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$ can be decomposed as in Eq. (25). This can be decided with a single instance of the following semidefinite program:

SDP 12. *The MDI assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ can be converted to the MDI assemblage $\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$ under LOSR operations, denoted by $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}\xrightarrow{LOSR}\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$, if and only if the following SDP is feasible:*

$$\begin{aligned}
\textit{given}\quad & \{J_{ab|x}\}_{a,b,x}\,,\ \{J'_{a'b'|x'}\}_{a',b',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'}\\
\textit{find}\quad & \{J_{\zeta\,b\,b'\,\lambda}\}_{b,b',\lambda}\\
\textit{s.t.}\quad & \begin{cases}
J_{\zeta\,b\,b'\,\lambda}\geq 0 \quad \forall\, b,b',\lambda\,,\\
\sum_{b'}\mathrm{tr}_{B_{in}}\{J_{\zeta\,b\,b'\,\lambda}\}\propto\mathbb{I}_{B'_{in'}}\quad \forall\, b,\lambda\,,\\
\sum_{\lambda,b'}\mathrm{tr}_{B_{in}}\{J_{\zeta\,b\,b'\,\lambda}\} = \frac{1}{d_{B'_{in'}}}\mathbb{I}_{B'_{in'}}\quad \forall\, b\,,\\
\sum_{b'} J_{\zeta\,b\,b'\,\lambda}\geq 0\quad \forall\, b,\lambda\,,\\
J_{a'b'|x'} = \sum_{\lambda}\sum_{a,b,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{ab|x} * J_{\zeta\,b\,b'\,\lambda}\,.
\end{cases}
\end{aligned} \tag{27}$$

When the conversion is not possible, we denote it by $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}\overset{LOSR}{\not\longrightarrow}\mathbf{N}'_{\mathbb{A}'\mathbb{B}'|\mathbb{X}'}$.

For the discussion of the robustness of this SDP, see Appendix B.

4.2 Properties of the pre-order

Similarly to the scenarios studied in the previous sections, we can use semidefinite programming to study possible conversions among post-quantum measurement-device-independent assemblages. In this section, we focus on an MDI scenario where $\mathbb{X}=\{0,1,2\}$, $\mathbb{A}=\{0,1\}$ and

$\mathbb{B} = \{0, 1\}$. We consider conversions between two post-quantum MDI assemblages that are special cases of channel assemblages $\mathbf{I}^{PTP}$ and $\mathbf{I}^{PR}$.

Let us start with a post-quantum MDI assemblage that can be mathematically expressed as if arising from a controlled-transpose operation. Such assemblage can be expressed as:

$$\mathbf{N}^{PTP}_{\mathbb{A}\mathbb{B}|\mathbb{X}} = \left\{\mathcal{N}^{PTP}_{ab|x}\right\}_{a\in\mathbb{A},\, b\in\mathbb{B},\, x\in\mathbb{X}} , \qquad (28)$$
$$\text{with} \quad \begin{cases} \mathcal{N}^{PTP}_{ab|x} = \mathrm{tr}\left\{(M_{a|x} \otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A \otimes \mathrm{CT}_b^{BB_{in}\to B_{out}})[\rho_{AB} \otimes (\cdot)]\right\} , \\ M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2} , \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a\sigma_y}{2} , \quad M_{a|3} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2} , \end{cases}$$

where $\rho_{AB} = |\phi\rangle\langle\phi|$ with $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and σ_x, σ_y, and σ_z are the Pauli operators. Here, the processing $\mathrm{CT}_b^{BB_{in}\to B_{out}}$ is the following. First, a controlled transpose operation is applied on BB_{in}, where B_{in} is the control qubit and B is the system that is being transposed. Second, the system B_{in} is traced-out and the system B is measured by Bob. The measurement elements are $N_0 = \frac{1}{3}\mathbb{I} + \frac{1}{3}\sigma_y$ and $N_1 = \frac{2}{3}\mathbb{I} - \frac{1}{3}\sigma_y$. The outcome of Bob's measurement is defined on B_{out}. This process is illustrated in Fig. 8(a). For simplicity, we hereon denote $\mathbf{N}^{PTP}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ as $\mathbf{N}^{PTP}$. In Appendix D, we prove that $\mathbf{N}^{PTP}$ is postquantum: we construct an SDP that tests a membership of a measurement-device-independent assemblage to the relaxation of a quantum set.

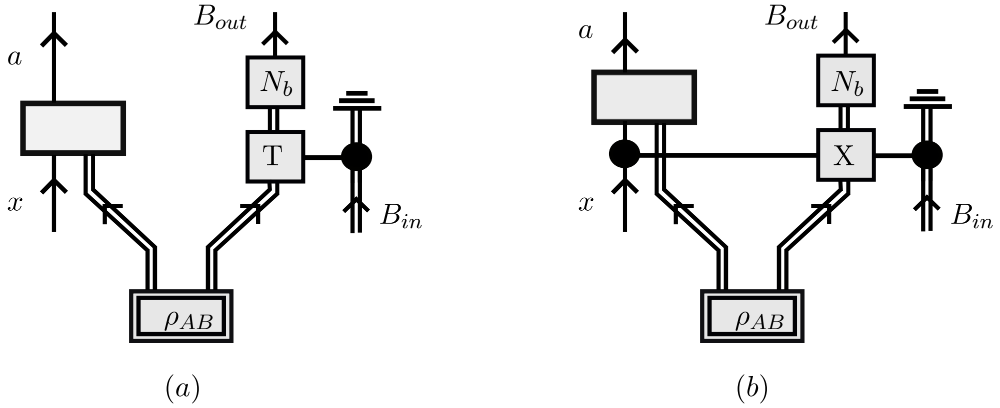

Figure 8: Mathematical depiction of two post-quantum MDI assemblages. (a) MDI assemblage $\mathbf{N}^{PTP}$: Alice and Bob share a Bell state; Alice performs measurements on her system, while Bob performs a controlled-transpose operation. (b) MDI assemblage $\mathbf{N}^{PR}$: Alice and Bob share a Bell state; Alice performs measurements on her system, while Bob performs a CNOT operation controlled by both Alice's and Bob's inputs.

For our second example we define a post-quantum assemblage that generates PR box correlations between Alice and Bob. Mathematically it can be expressed as follows:

$$\mathbf{N}^{PR}_{\mathbb{A}\mathbb{B}|\mathbb{X}} = \left\{\mathcal{N}^{PR}_{ab|x}\right\}_{a\in\mathbb{A},\, x\in\mathbb{X},\, b\in\mathbb{B}} , \qquad (29)$$
$$\text{with} \quad \begin{cases} \mathcal{N}^{PR}_{ab|x}(\cdot) = \mathrm{tr}\left\{(M_a \otimes \mathbb{I}_{B_{out}})\,(\mathbb{I}_A \otimes \mathrm{CX}_b^{BB_{in}\to B_{out}})\,(\rho_{AB} \otimes (\cdot)_{B_{in}})\right\} & \text{if } x \in \{0,1\}, \\ \mathcal{N}^{PR}_{ab|x}(\cdot) = \frac{1}{4} & \text{if } x = 2 , \\ M_0 = |0\rangle\langle 0| , \quad M_1 = |1\rangle\langle 1| . \end{cases}$$

Here, the mapping $\mathrm{CX}_b^{BB_{in}\to B_{out}}$ is the same as in the channel EPR scenario, followed by a measurement in a computational basis. It is illustrated in Fig. 8(b). Notice that for $x \in \{0, 1\}$, if Bob's input states are classical labels $\{|y\rangle\}_{y\in\{0,1\}}$, Alice and Bob obtain correlations $p(ab|xy)$ that correspond to the PR box. For simplicity, hereon we denote $\mathbf{N}^{PR}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \mathbf{N}^{PR}$.

To study the pre-order of $\mathbf{N}^{PR}$ and $\mathbf{N}^{PTP}$, we convert the assemblages into Choi form and run the SDP 12 (in Matlab [59], using the software CVX [60, 61], the solver SeDuMi [62] and the toolbox QETLAB [63]; see the code at [64]). We observe the following:

Observation 13. *The post-quantum measurement-device-independent assemblages* $\boldsymbol{N}^{PR}$ *and* $\boldsymbol{N}^{PTP}$ *are incomparable in the LOSR resource theory of common-cause assemblages.*

5 Related prior work

5.1 Channel EPR scenarios

Ref. [25], which first introduced channel EPR scenarios, takes 'channel steering' to be an instance of the resource theory of local operations and classical communication (LOCC), rather than the LOSR resource theory we have developed here. Both of these approaches lead to a meaningful resource theory: the LOSR approach is suited to the study of nonclassicality in scenarios where there are no cause-effect influences between parties, whereas the LOCC approach is suited to the study of scenarios in which there *are* cause-effect influences between parties. In our view, EPR scenarios do not involve causal influences from one party to the other, as argued in detail in Ref. [21]; hence, we consider the LOSR approach to be better suited to it.

The biggest distinction between the LOSR and LOCC approaches is that all resources in the LOSR approach are no-signalling by construction, while in the LOCC approach even free resources might be signalling from Bob to Alice. In Ref. [25], the classical channel assemblages are defined as the ones that admit a decomposition $\widetilde{\mathcal{I}}_{a|x}(\cdot) = \sum_\lambda p(a|x,\lambda)\widetilde{\mathcal{I}}_\lambda(\cdot)$, where $\widetilde{\mathcal{I}}_\lambda(\cdot)$ is a CP map that does not need to be trace preserving. In general, this definition of a free set of channel assemblages does not coincide with our definition specified in Eq. (2). However, when one restricts the scenario to non-signalling channel assemblages, these two approaches coincide, as we show in Appendix F.

When restricting the study of channel assemblages to that of no-signalling channel assemblages, the relationship between the LOSR and LOCC resource-theoretic approaches is analogous to the relationship between the traditional notion of LOCC-entanglement and the notion of LOSR-entanglement introduced in Refs. [23, 41]. Indeed, just as for the LOCC and LOSR resource theories of entanglement (where the definitions of free resources—i.e., separable states—coincide), the sets of free assemblages for the LOCC and LOSR approaches also coincide. However, the different choice of free operations—LOCC or LOSR—*does* impact the relative ordering of assemblages.

This leads to significant differences between the LOCC and LOSR approaches. For example, Ref. [74] shows that when one allows signalling from Bob to Alice, then all bipartite channel assemblages admit a quantum realization. In the LOSR approach this is not true - throughout the paper we studied many examples of bipartite post-quantum assemblages in channel, Bob-with-input and measurement-device-independent scenarios.

Ref. [25] attempts to give a few arguments in favor of the LOCC approach over the LOSR approach to channel EPR scenarios; e.g., by noting that the LOCC set of free channel assemblages is elegantly characterized by properties of their Choi-Jamiołkowski representation, and by noting that it recovers the separable states in the appropriate special case. However, all of the arguments given are equally true in the LOSR approach, and so do not discriminate between the two approaches.

Lastly, let us comment that neither LOCC or LOSR is the largest set of transformations leaving the free objects invariant in the resource theory of channel assemblages. We leave the question of what this set is for future work, focusing only on physically motivated free operations in this paper [9].

5.2 The resource theory of Local Operations and Shared Entanglement

Another relevant resource theory with which we can compare our approach to is that of Local Operations and Shared Entanglement (LOSE) [42]. Ref. [42] considers various common-cause non-signalling resources, including assemblages. The free operations of this resource theory differ from LOSR only in that the arbitrary shared randomness (classical common cause) is replaced with arbitrary shared entanglement (quantum common cause). Just as the set of LOSR operations is a subset of LOCC operations, LOSR is also clearly a subset of LOSE.

Rather than nonclassicality of common causes, the LOSE resource theory studies *postquantumness* of a common cause as a resource, and it allows some nonclassical common causes (the quantum ones) for free. Unsurprisingly, this implies that the pre-orders under LOSR and LOSE operations are different; e.g., in the latter, all quantum-realisable assemblages are freely interconvertible. The interesting differences between the pre-orders, then, will arise for post-quantum channel assemblages. Although we are far from a complete understanding of these differences, we can here already comment on one. Recall that here we showed that under LOSR operations the post-quantum assemblages $\mathbf{\Sigma}^{PTP}$ and $\mathbf{\Sigma}^{PR}$ are incomparable resources in a Bob-with-input EPR scenario. From the viewpoint of LOSE, the situation however changes. In Ref. [42] it was shown that the Bob-with-input assemblage $\mathbf{\Sigma}^{PTP}$ can be generated from the PR box resource using LOSE operations. This highlights the power of quantum entanglement even when considering resources that are post-quantum.

Outlook

In this work we have fleshed out the details of a resource theory of channel assemblages in channel EPR scenarios under local operations and shared randomness as free operations. We have explored the general case of channel EPR scenarios, and also the particular ones that come from changing the system type of Bob's input and output systems, denoted by Bob-with-input EPR scenarios and measurement-device-independent EPR scenarios. In all cases we specified a semidefinite program that tests resource conversion under LOSR operations, and found curious properties of the resource pre-order.

Looking forward, there are plenty of questions one may tackle. In this paper, we approached the problem of characterizing the preorder of assemblages using SDPs. Indeed, the SDPs we derive can in principle give *complete* information about the preorder of resources. One relevant question from a quantum information perspective is how to leverage this framework to define resource monotones that can quantify how useful an assemblage is as a resource for specific quantum information processing and communication tasks. From a foundational perspective, there are also various curiosities one may pursue. One pertains to how the pre-order of post-quantum resources changes from one scenario to another. For example, one can map a Bob-with-input assemblage to a correlation in a Bell scenario by performing a measurement on Bob's system: could there exist a map (given by a fixed measurement) that takes two incomparable assemblages into the same post-quantum correlation? How can we phrase and pursue the

[9] One example of an operation outside the LOSR set that leaves the free resources free is the following. Consider a Bob-with-input assemblage. First, swap Alice and Bob's subsystems. Next, take what is now Alice's quantum system and convert it to a classical output (by measuring in a preferred basis). On Bob's side, convert his classical system to a quantum one by conditioning a quantum state preparation on the value of the classical variable. This operation will not create nonclassicality, hence it could be considered a free operation. However, it does not seem to have any meaningful physical motivation.

question for the cases of channel EPR scenarios and MDI EPR scenarios? What valuable insights about nature can such EPR scenarios give us?

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References

[1] John S Bell. "On the Einstein Podolsky Rosen paradox". Physics Physique Fizika **1**, 195 (1964).
[2] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. "Bell nonlocality". Reviews of Modern Physics **86**, 419 (2014).
[3] Albert Einstein, Boris Podolsky, and Nathan Rosen. "Can quantum-mechanical description of physical reality be considered complete?". Physical review **47**, 777 (1935).
[4] Erwin Schrödinger. "Discussion of probability relations between separated systems". Mathematical Proceedings of the Cambridge Philosophical Society **31**, 555–563 (1935).
[5] Eric Gama Cavalcanti, Steve J Jones, Howard M Wiseman, and Margaret D Reid. "Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox". Physical Review A **80**, 032112 (2009).
[6] Howard M Wiseman, Steve James Jones, and Andrew C Doherty. "Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox". Physical review letters **98**, 140402 (2007).
[7] Roope Uola, Ana CS Costa, H Chau Nguyen, and Otfried Gühne. "Quantum steering". Reviews of Modern Physics **92**, 015001 (2020).
[8] Cyril Branciard, Eric G Cavalcanti, Stephen P Walborn, Valerio Scarani, and Howard M Wiseman. "One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering". Physical Review A **85**, 010301 (2012).
[9] Yu Xiang, Ioannis Kogias, Gerardo Adesso, and Qiongyi He. "Multipartite gaussian steering: Monogamy constraints and quantum cryptography applications". Phys. Rev. A **95**, 010101 (2017).
[10] Daniel Cavalcanti, Paul Skrzypczyk, GH Aguilar, RV Nery, PH Souto Ribeiro, and SP Walborn. "Detection of entanglement in asymmetric quantum networks and multipartite quantum steering". Nature communications **6**, 1–6 (2015).
[11] Alejandro Máttar, Paul Skrzypczyk, GH Aguilar, RV Nery, PH Souto Ribeiro, SP Walborn, and Daniel Cavalcanti. "Experimental multipartite entanglement and randomness certifi-

cation of the w state in the quantum steering scenario". Quantum Science and Technology **2**, 015011 (2017).
[12] Elsa Passaro, Daniel Cavalcanti, Paul Skrzypczyk, and Antonio Acín. "Optimal randomness certification in the quantum steering and prepare-and-measure scenarios". New Journal of Physics **17**, 113010 (2015).
[13] Yun Zhi Law, Jean-Daniel Bancal, Valerio Scarani, et al. "Quantum randomness extraction for various levels of characterization of the devices". Journal of Physics A: Mathematical and Theoretical **47**, 424028 (2014).
[14] Ivan Šupić and Matty J Hoban. "Self-testing through EPR-steering". New Journal of Physics **18**, 075006 (2016).
[15] Suchetana Goswami, Bihalan Bhattacharya, Debarshi Das, Souradeep Sasmal, C Jebaratnam, and AS Majumdar. "One-sided device-independent self-testing of any pure two-qubit entangled state". Physical Review A **98**, 022311 (2018).
[16] Shin-Liang Chen, Huan-Yu Ku, Wenbin Zhou, Jordi Tura, and Yueh-Nan Chen. "Robust self-testing of steerable quantum assemblages and its applications on device-independent quantum certification". Quantum **5**, 552 (2021).
[17] Matthew F Pusey. "Negativity and steering: A stronger Peres conjecture". Physical Review A **88**, 032313 (2013).
[18] Paul Skrzypczyk, Miguel Navascués, and Daniel Cavalcanti. "Quantifying Einstein-Podolsky-Rosen steering". Physical review letters **112**, 180404 (2014).
[19] Marco Piani and John Watrous. "Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering". Physical review letters **114**, 060404 (2015).
[20] Rodrigo Gallego and Leandro Aolita. "Resource theory of steering". Physical Review X **5**, 041008 (2015).
[21] Beata Zjawin, David Schmid, Matty J Hoban, and Ana Belén Sainz. "Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages". Quantum **7**, 926 (2023).
[22] Elie Wolfe, David Schmid, Ana Belén Sainz, Ravi Kunjwal, and Robert W Spekkens. "Quantifying Bell: The resource theory of nonclassicality of common-cause boxes". Quantum **4**, 280 (2020).
[23] David Schmid, Thomas C Fraser, Ravi Kunjwal, Ana Belén Sainz, Elie Wolfe, and Robert W Spekkens. "Understanding the interplay of entanglement and nonlocality: motivating and developing a new branch of entanglement theory" (2020). url: `https://arxiv.org/abs/2004.09194`.
[24] David Schmid, Denis Rosset, and Francesco Buscemi. "The type-independent resource theory of local operations and shared randomness". Quantum **4**, 262 (2020).
[25] Marco Piani. "Channel steering". JOSA B **32**, A1–A7 (2015).
[26] Ana Belén Sainz, Matty J Hoban, Paul Skrzypczyk, and Leandro Aolita. "Bipartite postquantum steering in generalized scenarios". Physical Review Letters **125**, 050404 (2020).
[27] Eric G Cavalcanti, Michael JW Hall, and Howard M Wiseman. "Entanglement verification and steering when alice and bob cannot be trusted". Physical Review A **87**, 032306 (2013).
[28] Denis Rosset, David Schmid, and Francesco Buscemi. "Type-independent characterization of spacelike separated resources". Physical Review Letters **125**, 210402 (2020).
[29] Iman Marvian and Robert W Spekkens. "How to quantify coherence: Distinguishing speakable and unspeakable notions". Physical Review A **94**, 052324 (2016).
[30] Iman Marvian, Robert W Spekkens, and Paolo Zanardi. "Quantum speed limits, coherence, and asymmetry". Physical Review A **93**, 052331 (2016).
[31] Andreas Winter and Dong Yang. "Operational resource theory of coherence". Physical review letters **116**, 120404 (2016).

[32] Fernando GSL Brandao, Michał Horodecki, Jonathan Oppenheim, Joseph M Renes, and Robert W Spekkens. "Resource theory of quantum states out of thermal equilibrium". Physical review letters **111**, 250404 (2013).
[33] Paul Skrzypczyk, Anthony J Short, and Sandu Popescu. "Work extraction and thermodynamics for individual quantum systems". Nature communications **5**, 1–8 (2014).
[34] Dominik Janzing, Pawel Wocjan, Robert Zeier, Rubino Geiss, and Th Beth. "Thermodynamic cost of reliability and low temperatures: tightening landauer's principle and the second law". International Journal of Theoretical Physics **39**, 2717–2753 (2000).
[35] Michał Horodecki and Jonathan Oppenheim. "Fundamental limitations for quantum and nanoscale thermodynamics". Nature communications **4**, 1–6 (2013).
[36] Gilad Gour, Markus P Müller, Varun Narasimhachar, Robert W Spekkens, and Nicole Yunger Halpern. "The resource theory of informational nonequilibrium in thermodynamics". Physics Reports **583**, 1–58 (2015).
[37] Zoë Holmes, Erick Hinds Mingo, Calvin Y-R Chen, and Florian Mintert. "Quantifying athermality and quantum induced deviations from classical fluctuation relations". Entropy **22**, 111 (2020).
[38] Michael A Nielsen. "Conditions for a class of entanglement transformations". Physical Review Letters **83**, 436 (1999).
[39] Charles H Bennett, Herbert J Bernstein, Sandu Popescu, and Benjamin Schumacher. "Concentrating partial entanglement by local operations". Physical Review A **53**, 2046 (1996).
[40] Yuval Rishu Sanders and Gilad Gour. "Necessary conditions for entanglement catalysts". Physical Review A **79**, 054302 (2009).
[41] Francesco Buscemi. "All entangled quantum states are nonlocal". Physical review letters **108**, 200401 (2012).
[42] David Schmid, Haoxing Du, Maryam Mudassar, Ghi Coulter-de Wit, Denis Rosset, and Matty J Hoban. "Postquantum common-cause channels: the resource theory of local operations and shared entanglement". Quantum **5**, 419 (2021).
[43] Jonathan Barrett, Noah Linden, Serge Massar, Stefano Pironio, Sandu Popescu, and David Roberts. "Nonlocal correlations as an information-theoretic resource". Physical review A **71**, 022101 (2005).
[44] Nicolas Brunner and Paul Skrzypczyk. "Nonlocality distillation and postquantum theories with trivial communication complexity". Physical review letters **102**, 160403 (2009).
[45] Judea Pearl. "Causality". Cambridge University Press. (2009).
[46] Christopher J Wood and Robert W Spekkens. "The lesson of causal discovery algorithms for quantum correlations: causal explanations of bell-inequality violations require fine-tuning". New Journal of Physics **17**, 033002 (2015).
[47] Paulo J Cavalcanti, John H Selby, Jamie Sikora, Thomas D Galley, and Ana Belén Sainz. "Post-quantum steering is a stronger-than-quantum resource for information processing". npj Quantum Information **8**, 1–10 (2022).
[48] Ana Belén Sainz, Nicolas Brunner, Daniel Cavalcanti, Paul Skrzypczyk, and Tamás Vértesi. "Postquantum steering". Physical review letters **115**, 190403 (2015).
[49] Sandu Popescu and Daniel Rohrlich. "Quantum nonlocality as an axiom". Foundations of Physics **24**, 379–385 (1994).
[50] Nicolas Gisin. "Stochastic quantum dynamics and relativity". Helvetica Physica Acta **62**, 363–371 (1989).
[51] Lane P Hughston, Richard Jozsa, and William K Wootters. "A complete classification of quantum ensembles having a given density matrix". Physics Letters A **183**, 14–18 (1993).
[52] Michael A. Nielsen and Isaac L. Chuang. "Quantum computation and quantum information: 10th anniversary edition". Cambridge University Press. (2011).

[53] David Schmid, Katja Ried, and Robert W. Spekkens. “Why initial system-environment correlations do not imply the failure of complete positivity: A causal perspective”. Phys. Rev. A **100**, 022112 (2019).
[54] Man-Duen Choi. “Completely positive linear maps on complex matrices”. Linear algebra and its applications **10**, 285–290 (1975).
[55] Andrzej Jamiołkowski. “Linear transformations which preserve trace and positive semidefiniteness of operators”. Reports on Mathematical Physics **3**, 275–278 (1972).
[56] Gus Gutoski and John Watrous. “Toward a general theory of quantum games”. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing. Page 565. (2007).
[57] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. “Theoretical framework for quantum networks”. Physical Review A **80**, 022339 (2009).
[58] Arthur Fine. “Hidden Variables, Joint Probability, and the Bell Inequalities”. Phys. Rev. Lett. **48**, 291–295 (1982).
[59] “Matlab”. url: `https://www.mathworks.com/`.
[60] Michael Grant and Stephen Boyd. “CVX: MATLAB software for disciplined convex programming”. url: `http://cvxr.com/cvx`.
[61] Michael Grant and Stephen Boyd. “Graph implementations for nonsmooth convex programs”. In V. Blondel, S. Boyd, and H. Kimura, editors, Recent Advances in Learning and Control. Pages 95–110. Lecture Notes in Control and Information Sciences. Springer-Verlag Limited (2008).
[62] Jos F Sturm. “Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones”. Optimization methods and software **11**, 625–653 (1999).
[63] Nathaniel Johnston. “QETLAB: a MATLAB toolbox for quantum entanglement”. url: `http://qetlab.com`.
[64] Beata Zjawin, David Schmid, Matty J. Hoban, and Ana Belén Sainz. code: beatazjawin/Quantifying-EPR.
[65] Daniel Cavalcanti and Paul Skrzypczyk. “Quantum steering: a review with focus on semidefinite programming”. Reports on Progress in Physics **80**, 024001 (2016).
[66] Miguel Navascués, Yelena Guryanova, Matty J Hoban, and Antonio Acín. “Almost quantum correlations”. Nature communications **6**, 1 (2015).
[67] Marcin Pawłowski, Tomasz Paterek, Dagomir Kaszlikowski, Valerio Scarani, Andreas Winter, and Marek Żukowski. “Information causality as a physical principle”. Nature **461**, 1101 (2009).
[68] Miguel Navascués and Harald Wunderlich. “A glance beyond the quantum model”. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **466**, 881 (2010).
[69] Ana Belén Sainz, Tobias Fritz, Remigiusz Augusiak, J Bohr Brask, Rafael Chaves, Anthony Leverrier, and Antonio Acín. “Exploring the local orthogonality principle”. Physical Review A **89**, 032117 (2014).
[70] Antonio Acín, Tobias Fritz, Anthony Leverrier, and Ana Belén Sainz. “A combinatorial approach to nonlocality and contextuality”. Communications in Mathematical Physics **334**, 533–628 (2015).
[71] Joe Henson and Ana Belén Sainz. “Macroscopic noncontextuality as a principle for almost-quantum correlations”. Physical Review A **91**, 042114 (2015).
[72] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. “Proposed experiment to test local hidden-variable theories”. Physical review letters **23**, 880 (1969).
[73] Matty J Hoban and Ana Belén Sainz. “A channel-based framework for steering, non-locality and beyond”. New Journal of Physics **20**, 053048 (2018).

[74] Michał Banacki, Ravishankar Ramanathan, and Paweł Horodecki. "Multipartite channel assemblages" (2022). url: https://arxiv.org/pdf/2205.05033.pdf.
[75] Miguel Navascués, Stefano Pironio, and Antonio Acín. "Bounding the set of quantum correlations". Physical Review Letters **98**, 010401 (2007).
[76] Miguel Navascués, Stefano Pironio, and Antonio Acín. "A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations". New Journal of Physics **10**, 073013 (2008).
[77] Tilo Eggeling, Dirk Schlingemann, and Reinhard F Werner. "Semicausal operations are semilocalizable". EPL (Europhysics Letters) **57**, 782 (2002).

A LOSR transformations in terms of deterministic combs

A.1 The channel EPR scenario

The most general LOSR operation on a channel assemblage transforms $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ into $\mathbf{I}'_{\mathbb{A}'|\mathbb{X}'}$ as specified in Eq. (5), recalled below:

$$\mathcal{I}'_{a'|x'}(\cdot) = \sum_{\lambda}\sum_{a,x} p(a',x|a,x',\lambda)\, p(\lambda)\, \Lambda_{\lambda}^{B_{out}S\to B'_{out'}}\left(\mathcal{I}_{a|x}\otimes \mathbb{I}_S\right)\Lambda_{\lambda}^{B'_{in'}\to B_{in}S}(\cdot)\,.$$

It was shown in Refs. [22, 58] that any indeterminism in Alice's local comb can be absorbed into the shared common cause λ. That is, the indeterministic probability distribution $p(a',x|a,x',\lambda)$ is as general as a deterministic one is. In this subsection we recall the proof of this statement.

First, decompose Alice's local comb as a convex combination of deterministic combs:

$$p(a',x|a,x',\lambda) = \sum_{\widetilde{\lambda}} p(\widetilde{\lambda}|\lambda) D(a',x|a,x',\widetilde{\lambda}), \tag{30}$$

with $D(a',x|a,x',\widetilde{\lambda})$ being a deterministic probability distribution. We can always express $D(a',x|a,x',\widetilde{\lambda}) = D(a'|a,x',\widetilde{\lambda})D(x|a,x',\widetilde{\lambda})$. Moreover, since $D(a',x|a,x',\widetilde{\lambda})$ satisfies the condition of no-retrocausation (the variable a is the causal future of the variable x, therefore a cannot influence the value of x), without loss of generality $D(x|a,x',\widetilde{\lambda}) = D(x|x',\widetilde{\lambda})$. Hence

$$D(a',x|a,x',\widetilde{\lambda}) = D(a'|a,x',\widetilde{\lambda})\, D(x|x',\widetilde{\lambda}). \tag{31}$$

Here, $D(a'|a,x',\widetilde{\lambda})$ assigns a fixed outcome a' for each possible choice of a, x', and $\widetilde{\lambda}$, and $D(x|x',\widetilde{\lambda})$ assigns a fixed outcome x for each measurement x' and value of $\widetilde{\lambda}$.

Putting this together in the Choi form with the definition of the map $J_{\xi\,\lambda}^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$ (after Eq. (5)) one obtains:

$$J'_{a'|x'}(\cdot) = \sum_{\lambda,\widetilde{\lambda}}\sum_{a,x} p(\widetilde{\lambda}|\lambda)\, D(a'|a,x',\widetilde{\lambda})\, D(x|x',\widetilde{\lambda})\, J_{a|x} * J_{\xi\,\lambda}.$$

From here, one can then define

$$\widetilde{J}_{\xi\,\widetilde{\lambda}}^{B'_{in'}B_{out}\to B_{in}B'_{out'}}(\cdot) = \sum_{\lambda} p(\widetilde{\lambda}|\lambda)\, J_{\xi\,\lambda}^{B'_{in'}B_{out}\to B_{in}B'_{out'}}(\cdot)\,, \tag{32}$$

where $\widetilde{J}_{\xi\,\lambda}^{B'_{in'}B_{out}\to B_{in}B'_{out'}}$ is by definition a map that is non-signalling from the system defined on $\mathcal{H}_{B_{out}}$ to the system defined on $\mathcal{H}_{B_{in}}$. From here follows that:

$$J'_{a'|x'}(\cdot) = \sum_{\widetilde{\lambda}}\sum_{a,x} D(a'|a,x',\widetilde{\lambda})\, D(x|x',\widetilde{\lambda})\, J_{a|x} * \widetilde{J}_{\xi\,\widetilde{\lambda}}\,, \tag{33}$$

which gives Eq. (7) from the main text.

A.2 The Bob-with-input EPR scenario

The arguments given above also apply to the Bob-with-input EPR scenario. The most general LOSR transformation on a Bob-with-input assemblage transforms $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ into $\mathbf{\Sigma}'_{\mathbb{A}'|\mathbb{X}'\mathbb{Y}'}$ as follows:

$$\sigma'_{a'|x'y'} = \sum_{\lambda} \sum_{a,x,y} p(a', x|a, x', \lambda) p(y|y', \lambda) p(\lambda) \xi_{\lambda,y'}(\sigma_{a|xy}). \tag{34}$$

We already discussed that Alice's local comb can be expressed in terms of deterministic probability distributions as in Eq.(30). The same technique can be applied to Bob's pre-processing to express it as

$$p(y|y', \lambda) = \sum_{\widetilde{\lambda}} p(\widetilde{\lambda}|\lambda) D(y|y', \widetilde{\lambda}). \tag{35}$$

Define the following CPTP map:

$$\widetilde{\xi}_{\widetilde{\lambda},y'}(\sigma_{a|xy}) = \sum_{\lambda} p(\widetilde{\lambda}|\lambda) p(\lambda) \xi_{\lambda,y'}(\sigma_{a|xy}). \tag{36}$$

We can now rewrite the Bob-with-input LOSR transformation as

$$\sigma'_{a'|x'y'} = \sum_{\widetilde{\lambda}} \sum_{a,x,y} D(x|x', \widetilde{\lambda}) D(a'|a, x', \widetilde{\lambda}) D(y|y', \widetilde{\lambda}) \widetilde{\xi}_{\widetilde{\lambda},y'}(\sigma_{a|xy}), \tag{37}$$

which is exactly Eq. (16) from the main text if one labels $\widetilde{\lambda}$ as λ.

B Robust formulation of the SDPs

All SDPs derived in this paper are feasibility problems, i.e., they are written in a form where the objective function vanishes. One can relax the constraints on the Choi matricies and instead of requiring their positive semi-definiteness, require them to be *close* to a positive semi-definite matrix. Such reformulation makes the SDPs more robust. In SDP 2, this can be implemented by relaxing the constraints $J_{\xi\,\lambda} \geq 0$ and $J_{F\,\lambda} \geq 0$ to $J_{\xi\,\lambda} + \mu\mathbb{I} \geq 0$ and $J_{F\,\lambda} + \mu\mathbb{I} \geq 0$, respectively. Then, the SDP 2 can be written as a minimization of the new parameter μ:

SDP 14. *The channel assemblage $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}}$ can be converted to the channel assemblage $\boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$ under LOSR operations, denoted by $\boldsymbol{I}_{\mathbb{A}|\mathbb{X}} \overset{LOSR}{\longrightarrow} \boldsymbol{I}'_{\mathbb{A}'|\mathbb{X}'}$, if and only the solution of the following SDP satisfies $\mu < 10^{-10}$:*

$$\begin{aligned}
&\textit{given} \quad \{J_{a|x}\}_{a,x}\,,\ \{J'_{a'|x'}\}_{a',x'}\,,\ \{D(a'|a,x',\lambda)\}_{\lambda,a',a,x'}\,,\ \{D(x|x',\lambda)\}_{\lambda,x,x'} \\
&\underset{(J_{\xi\,\lambda})_\lambda\,,\,(J_{F\,\lambda})_\lambda}{\textit{min}} \quad \mu \\
&\textit{s.t.} \begin{cases}
\mu \geq 0\,, \\
J_{\xi\,\lambda} + \mu\mathbb{I} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} \propto \mathbb{I}_{B_{out}B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B'_{out'}B_{in}}\{J_{\xi\,\lambda}\} = \frac{1}{d_{B_{out}}d_{B'_{in'}}}\mathbb{I}_{B_{out}B'_{in'}}\,, \\
J_{F\,\lambda} + \mu\mathbb{I} \geq 0 \quad \forall\lambda\,, \\
\mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} \propto \mathbb{I}_{B'_{in'}} \quad \forall\lambda\,, \\
\sum_\lambda \mathrm{tr}_{B_{in}}\{J_{F\,\lambda}\} = \frac{1}{d_{B'_{in'}}}\mathbb{I}_{B'_{in'}}\,, \\
\mathrm{tr}_{B'_{out'}}\{J_{\xi\,\lambda}\} = J_{F\,\lambda} \otimes \frac{1}{d_{B_{out}}}\mathbb{I}_{B_{out}} \quad \forall\lambda\,, \\
J'_{a'|x'} = \sum_\lambda \sum_{a,x} D(a'|a,x',\lambda)\, D(x|x',\lambda)\, J_{a|x} * J_{\xi\,\lambda}\,.
\end{cases}
\end{aligned} \tag{38}$$

The threshold for the value of μ can be selected depending on the application of the program. Above, we decide on $\mu < 10^{-10}$ solely for the purpose of the presentation.

This method can be applied to the SDP 6 and SDP 12 in an analogous way, hence we do not present the modified SDPs here.

We run the SDP 14 (and analogous robust SDPs for the Bob-with-input and measurement-device-independent scenarios) to confirm results stated in the paper. For conversions which are possible, the result of the optimization is $\mu \approx 10^{-10}$. For impossible conversions, it is $\mu > 10^{-4}$. The code is available at [64].

C Almost quantum correlations

In this Appendix, we recall an example from Ref. [66] of a probability distribution that is almost-quantum yet not quantum. Consider a bipartite Bell scenario where a, b, x, and y are binary classical variables. In this scenario, the probability distribution is fully specified by an 8-dimensional vector $\vec{p}$ with the following entries:

$$\vec{p} = \Big\{p_A(1|0), p_A(1|1), p_B(1|0), p_B(1|1), p(1,1|0,0), p(1,1|1,0), p(1,1|0,1), p(1,1|1,1)\Big\}. \quad (39)$$

Here, $p_A(a|x)$ and $p_B(b|y)$ are the marginal probabilities corresponding to Alice's and Bob's individual measurements, respectively, and $p(ab|xy)$ is the conditional joint probability distribution. To calculate the probabilities that are not explicitly contained in $\vec{p}$, one must simply use the normalization and no-signalling constraints:

$$\sum_{a\in\mathbb{A}, b\in B} p(ab|xy) = 1 \quad \forall\, x, y, \quad (40)$$

$$\sum_{b\in B} p(ab|xy) = p_A(a|x) \quad \forall\, x, y, a, \quad (41)$$

$$\sum_{a\in A} p(ab|xy) = p_B(b|y) \quad \forall\, x, y, b. \quad (42)$$

In Ref. [66], it was shown that the following probability vector belongs to the almost quantum set and it does not admit a quantum realization:

$$\vec{p}_{AQ} = \Big\{\frac{9}{20}, \frac{2}{11}, \frac{2}{11}, \frac{9}{20}, \frac{22}{125}, \frac{\sqrt{2}}{9}, \frac{37}{700}, \frac{22}{125}\Big\}. \quad (43)$$

In the main text, we use the probability $\vec{p}_{AQ}$ to construct the elements of the Bob-with-input assemblage $\mathbf{\Sigma}^{AQ}$.

D Membership problem for the measurement-device-independent EPR scenario

Deciding whether a measurement-device-independent assemblage has a quantum realization is a highly non-trivial problem. In this section, we introduce the first level of a hierarchy of semidefinite programs for MDI EPR scenarios which tests a membership of the set of quantum assemblages. The hierarchy of programs will be presented in future work; for the purpose of this paper, we only specify its first level. This hierarchy is analogous to the Navascués-Pironio-Acín (NPA) hierarchy for quantum correlations [75, 76].

The elements of a quantum MDI assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}} = \{\mathcal{N}_{ab|x}(\cdot)\}$ can be expressed as

$$\mathcal{N}_{ab|x}(\cdot) = \mathrm{tr}\left\{(M^A_{a|x} \otimes \Theta_b^{BB_{in}\to B_{out}})[\rho_{AB} \otimes (\cdot)_{B_{in}}]\right\}.$$

Without loss of generality, we can take the shared state to be pure, which we denote by $|\psi\rangle\langle\psi|_{AB}$, and the measurement Θ_b to be projective, which we denote by $F_b^{BB_{in}}$. Thus, each element $\mathcal{N}_{ab|x}$ of the assemblage can be written as $\mathcal{N}_{ab|x}(\cdot) = \mathrm{tr}\left(\widetilde{\mathcal{N}}_{ab|x}(\cdot)\right)$, with

$$\begin{aligned} \widetilde{\mathcal{N}}_{ab|x}(\cdot) &= \mathrm{tr}_{AB}\left(M_{a|x}^A \otimes F_b^{BB_{in}}\,|\psi\rangle\langle\psi|_{AB} \otimes (\cdot)\right) \\ &= \langle\psi|M_{a|x}^A \otimes F_b^{BB_{in}}|\psi\rangle(\cdot). \end{aligned}$$

Notice that $\widetilde{\mathcal{N}}_{ab|x}$ is a matrix and we can write its elements as

$$\begin{aligned} \langle i|\widetilde{\mathcal{N}}_{ab|x}|j\rangle &= \langle i|\langle\psi|M_{a|x} \otimes F_b|\psi\rangle|j\rangle \\ &= \langle\psi|M_{a|x} \otimes \langle i|F_b|j\rangle|\psi\rangle \\ &= \langle\psi|M_{a|x} \otimes F_b^{ij}|\psi\rangle. \end{aligned}$$

Here, the operator $F_b^{ij} := \langle i|F_b|j\rangle$ must satisfy the following properties:

$$\sum_b F_b^{ij} = \delta_{i,j}\mathbb{I}_B\,, \qquad \sum_i (F_b^{ij})^\dagger F_{b'}^{ij'} = \delta_{b,b'}F_b^{jj'}.$$

Therefore, all elements of the matrix $\widetilde{\mathcal{N}}_{ab|x}$ can be written as linear combinations of inner products of the form $\langle\psi|O_k^\dagger O_m|\psi\rangle$, where $O_k \in \{\mathbb{I}, \{M_{a|x}\}, \{F_b^{ij}\}\}$. We can now write a moment matrix with columns and rows indexed by the choice of operator O_k. Denote $\langle\psi|O_k^\dagger O_m|\psi\rangle$ as $\langle O_k^\dagger O_m\rangle$. The moment matrix for $a, b \in \{0,1\}$, $x \in \{0,1,2\}$ and B_{in} being two-dimensional is the following:

$$\begin{bmatrix}
\langle\psi|\psi\rangle & \langle M_{0|0}\rangle & \langle M_{0|1}\rangle & \langle M_{0|2}\rangle & \langle F_0^{00}\rangle & \langle F_0^{01}\rangle & \langle F_0^{10}\rangle & \langle F_0^{11}\rangle \\
\langle M_{0|0}\rangle & \langle M_{0|0}M_{0|0}\rangle & \langle M_{0|0}M_{0|1}\rangle & \langle M_{0|0}M_{0|2}\rangle & \langle M_{0|0}F_0^{00}\rangle & \langle M_{0|0}F_0^{01}\rangle & \langle M_{0|0}F_0^{10}\rangle & \langle M_{0|0}F_0^{11}\rangle \\
\langle M_{0|1}\rangle & \langle M_{0|1}M_{0|0}\rangle & \langle M_{0|1}M_{0|1}\rangle & \langle M_{0|1}M_{0|2}\rangle & \langle M_{0|1}F_0^{00}\rangle & \langle M_{0|1}F_0^{01}\rangle & \langle M_{0|1}F_0^{10}\rangle & \langle M_{0|1}F_0^{11}\rangle \\
\langle M_{0|2}\rangle & \langle M_{0|2}M_{0|0}\rangle & \langle M_{0|2}M_{0|1}\rangle & \langle M_{0|2}M_{0|2}\rangle & \langle M_{0|2}F_0^{00}\rangle & \langle M_{0|2}F_0^{01}\rangle & \langle M_{0|2}F_0^{10}\rangle & \langle M_{0|2}F_0^{11}\rangle \\
\langle F_0^{00}\rangle & \langle F_0^{00}M_{0|0}\rangle & \langle F_0^{00}M_{0|1}\rangle & \langle F_0^{00}M_{0|2}\rangle & \langle F_0^{00}F_0^{00}\rangle & \langle F_0^{00}F_0^{01}\rangle & \langle F_0^{00}F_0^{10}\rangle & \langle F_0^{00}F_0^{11}\rangle \\
\langle F_0^{10}\rangle & \langle F_0^{10}M_{0|0}\rangle & \langle F_0^{10}M_{0|1}\rangle & \langle F_0^{10}M_{0|2}\rangle & \langle F_0^{10}F_0^{00}\rangle & \langle F_0^{10}F_0^{01}\rangle & \langle F_0^{10}F_0^{10}\rangle & \langle F_0^{10}F_0^{11}\rangle \\
\langle F_0^{01}\rangle & \langle F_0^{01}M_{0|0}\rangle & \langle F_0^{01}M_{0|1}\rangle & \langle F_0^{01}M_{0|2}\rangle & \langle F_0^{01}F_0^{00}\rangle & \langle F_0^{01}F_0^{01}\rangle & \langle F_0^{01}F_0^{10}\rangle & \langle F_0^{01}F_0^{11}\rangle \\
\langle F_0^{11}\rangle & \langle F_0^{11}M_{0|0}\rangle & \langle F_0^{11}M_{0|1}\rangle & \langle F_0^{11}M_{0|2}\rangle & \langle F_0^{11}F_0^{00}\rangle & \langle F_0^{11}F_0^{01}\rangle & \langle F_0^{11}F_0^{10}\rangle & \langle F_0^{11}F_0^{11}\rangle
\end{bmatrix}.$$

This moment matrix can be seen to specify the first level of a hierarchy of semidefinite programs that test membership of an MDI assemblage in the quantum set. Higher levels of the hierarchy can be generated by considering various sequences of products of operators $M_{a|x}$ and F_b^{ij}, as we will specify in a follow up manuscript.

We used the first level of the hierarchy to check if the MDI assemblage specified in Eq. (28) is quantum. We run the SDP (in Matlab [59], using the software CVX [60, 61]; see the code at [64]) to check if any moment matrix of the form above is positive semidefinite, and it is not. We conclude that the assemblage is postquantum.

E Proofs

E.1 Proof of Theorem 8

In this section, we give a proof of Theorem 8 stated in the main text. To show that $\mathbf{\Sigma}^{PR}$ cannot be converted into $\mathbf{\Sigma}^{PTP}$ with LOSR operations, we will make use of the 'steering' functional constructed in Ref. [26, Eq. (D3)], which we denote S_{PTP}. A generic functional in a Bob-with-input scenario is defined as:

$$S[\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}] = \text{tr}\left\{ \sum_{a\in\mathbb{A},\, x\in\mathbb{X},\, y\in\mathbb{Y}} F_{axy}\, \sigma_{a|xy} \right\}, \tag{44}$$

where $\{F_{axy}\}$ is a set of Hermitian operators. S_{PTP} is specified by the following operators

$$F^{PTP}_{axy} = \frac{1}{2}(\mathbb{I} - (-1)^a \sigma_x)^{T^y}, \tag{45}$$

where T^y denotes that the transpose operation is applied when $y = 1$, and the identity operation is applied when $y = 0$. Here, however, the operator σ_x should not be confused with the Pauli-X operator, since $x \in \mathbb{X}$ here denotes the choice of Alice's measurement. Indeed, the operators σ_0, σ_1, and σ_2 (since $x \in \{0, 1, 2\}$) should be thought of as the Pauli X, Y, and Z operators respectively. This alternative way of denoting the Pauli operators will also be used later on in this section.

It was shown in Ref. [26, Appendix D] that for any non-signalling assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$, the minimum value S^{min}_{PTP} of $S_{PTP}[\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}]$ is given by $S^{\text{min}}_{PTP} = 0$. It was also proved that $\mathbf{\Sigma}^{PTP}$ achieves the minimum value, that is, $S_{PTP}[\mathbf{\Sigma}^{PTP}] = S^{\text{min}}_{PTP}$.

It can be shown by direct calculation that $\mathbf{\Sigma}^{PR}$ does not achieve the value of 0 for S_{PTP}. In order to show, hence, that $\mathbf{\Sigma}^{PR}$ cannot be converted to $\mathbf{\Sigma}^{PTP}$, it then suffices to show that any LOSR-processing of $\mathbf{\Sigma}^{PR}$ will also not give this minimum value of 0 for S_{PTP}. To show this we will use the following observation.

Remark 15. *An assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ satisfies $S_{PTP}[\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}] = S^{\text{min}}_{PTP} = 0$ iff each element satisfies $\sigma_{a|xy} = \alpha_{a,x,y}\, \frac{1}{2}\left(\mathbb{I} + (-1)^a \sigma_x\right)^{T^y}$ for all $a \in \mathbb{A}, x \in \mathbb{X}, y \in \mathbb{Y}$, where $\alpha_{a,x,y}$ is a real number such that $0 \leq \alpha_{a,x,y} \leq 1$ for all $a \in \mathbb{A}, x \in \mathbb{X}, y \in \mathbb{Y}$.*

Proof. First note that the operators $F^{PTP}_{axy} = \frac{1}{2}(\mathbb{I} - (-1)^a \sigma_x)^{T^y}$ are rank-1 projectors. Hence, $\text{tr}\left\{ F^{PTP}_{axy}\, \sigma_{a|xy} \right\} \geq 0$ for all $a \in \mathbb{A},\ x \in \mathbb{X},\ y \in \mathbb{Y}$. Since by assumption $S_{PTP}[\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}] = 0$, it follows that $\text{tr}\left\{ F^{PTP}_{axy}\, \sigma_{a|xy} \right\} = 0$ for all $a \in \mathbb{A},\ x \in \mathbb{X},\ y \in \mathbb{Y}$. Since each F^{PTP}_{axy} is a projector onto a one-dimensional subspace, this implies that $\sigma_{a|xy}$ cannot have any support on this one-dimensional subspace, and thus the assemblage element $\sigma_{a|xy}$ only has support on the one-dimensional subspace orthogonal to F^{PTP}_{axy}. In other words, if $F^{PTP}_{axy} = \frac{1}{2}(\mathbb{I} - (-1)^a \sigma_x)^{T^y}$, then the assemblage element has support on the one-dimensional subspace spanned by the projector $\frac{1}{2}(\mathbb{I} + (-1)^a \sigma_x)^{T^y}$, which concludes the proof. □

Remark 15 is crucial for proving Theorem 8, which we recall here:

Theorem 8. *$\mathbf{\Sigma}^{PR}$ cannot be converted into $\mathbf{\Sigma}^{PTP}$ with LOSR operations.*

Proof. Let us prove this by contradiction. Assume that $\mathbf{\Sigma}^{PTP}$ is given by an LOSR-processing of $\mathbf{\Sigma}^{PR}$, denoted as $\mathbf{\Sigma}^{PR}_{\Lambda} = \{\sigma^{PR,\Lambda}_{a|xy}\}$. Then, it follows that this LOSR-processing of $\mathbf{\Sigma}^{PR}$ saturates the non-signalling bound of the functional S_{PTP}. From Remark 15, it can only be the case that each $\sigma^{PR,\Lambda}_{a|xy}$ is proportional to a particular Pauli eigenstate, where x denotes its corresponding Pauli basis. In other words, each element of $\mathbf{\Sigma}^{PR}_{\Lambda}$ is diagonal in a particular Pauli basis, and has one null eigenvalue.

Now note that every element of $\mathbf{\Sigma}^{PR}$ is diagonal in the computational (Pauli Z) basis. Therefore, the original assemblage $\mathbf{\Sigma}^{PR}$ is invariant under the dephasing map in the computational basis, i.e., for all $a \in \mathbb{A}, x \in \mathbb{X}, y \in \mathbb{Y}$, $D[\sigma^{PR}_{a|xy}] := \sum_{i\in\{0,1\}} |i\rangle\langle i|\, \sigma^{PR}_{a|xy}\, |i\rangle\langle i| = \sigma^{PR}_{a|xy}$. Therefore, without loss of generality, any LOSR-processing of $\mathbf{\Sigma}^{PR}$ can be pre-composed with the dephasing map $D[\cdot]$ without changing the assemblage that results from the processing. More formally, for any LOSR transformation Λ, $\Lambda[D[\mathbf{\Sigma}^{PR}]] = \Lambda[\mathbf{\Sigma}^{PR}] = \mathbf{\Sigma}^{PR}_{\Lambda}$. Equivalently, *any* LOSR-processing of $\mathbf{\Sigma}^{PR}$ can be first described as an application of the dephasing map on Bob's qubit, following by another LOSR transformation.

Therefore, for the elements of $\mathbf{\Sigma}^{PR}_{\Lambda}$ that are diagonal in Pauli bases other than the Pauli Z basis, the LOSR-processing needs to transform Bob's qubit so that the assemblage elements are diagonal in another basis. Furthermore, the elements of $\mathbf{\Sigma}^{PR}_{\Lambda}$ only have support in (at most) one-dimensional subspace. Such a situation for $\mathbf{\Sigma}^{PR}_{\Lambda}$ can only happen if, in the LOSR-processing on $\mathbf{\Sigma}^{PR}$, after the dephasing map, Bob applies another CPTP map $\mathcal{F}_{y,\lambda}$, which can depend on y and the shared randomness λ. Because of the dephasing map, to produce assemblage elements with support on a one-dimensional subspace, the map $\mathcal{F}_{y,\lambda}$ is such that it can be simulated by a measure-and-prepare channel, where the preparation step in the channel should prepare the state corresponding to the one-dimensional subspace. However, the Pauli bases for the assemblage elements are necessarily determined by x, and, since $\mathcal{F}_{y,\lambda}$ cannot depend on x, it is impossible for an LOSR-processing to produce the desired assemblage $\mathbf{\Sigma}^{PR}_{\Lambda}$. That is, if the state preparation step of the channel $\mathcal{F}_{y,\lambda}$ could prepare states in the correct basis, then the measurement in the channel would reveal what the input x of Alice is, which is incompatible with the no-signalling condition. □

E.2 Proof of Observation 10

In Section 3.2, we used SDP 6 to certify that $\mathbf{\Sigma}'^{PR}$ can be converted into $\mathbf{\Sigma}^{AQ}$ with LOSR operations, but not vice versa. We now prove the first part of this observation analytically.

Theorem 15. *The post-quantum Bob-with-input assemblage* $\mathbf{\Sigma}'^{PR}$ *can be converted into* $\mathbf{\Sigma}^{AQ}$ *with LOSR operations.*

Proof. We prove the theorem by providing an explicit LOSR protocol that preforms the desired transformation. This protocol is depicted in Fig. 9.

1. $\mathbf{\Sigma'^{PR}} \overset{\text{LOSR}}{\longrightarrow} \vec{\mathbf{p}}_{\mathbf{PR}}$**:** Bob measures his share of the system in the computational basis. This local operation maps the assemblage $\mathbf{\Sigma}'^{PR}$ into a conditional probability distribution $\vec{p}_{PR} = \{p_{PR}(ab|xy)\}$ corresponding to PR-box correlations.
2. $\vec{\mathbf{p}}_{\mathbf{PR}} \overset{\text{LOSR}}{\longrightarrow} \vec{\mathbf{p}}_{\mathbf{AQ}}$**:** In the resource theory of common-cause boxes under LOSR operations, the equivalence class of PR-boxes is at the top of the pre-order of bipartite correlations with $|\mathbb{A}| = |\mathbb{B}| = |\mathbb{X}| = |\mathbb{Y}| = 2$ [22]. Hence there exits an LOSR process that maps $\vec{p}_{PR}$ into $\vec{p}_{AQ}$. Let Alice and Bob apply this LOSR operation to $\vec{p}_{PR}$. They are then left sharing the correlations $\vec{p}_{AQ} = \{p_{AQ}(a'b'|x'y')\}$.
3. $\vec{\mathbf{p}}_{\mathbf{AQ}} \overset{\text{LOSR}}{\longrightarrow} \mathbf{\Sigma^{AQ}}$**:** in this step, Bob implements a measure-and-prepare channel which reads out the value of the classical output b' and prepares a qubit on state $|b'\rangle$. This state will be prepared with probability $p_{AQ}(a'b'|x'y')$ since Alice and Bob share $\vec{p}_{AQ}$. This hence effectively prepares $\mathbf{\Sigma}^{AQ}$.

□

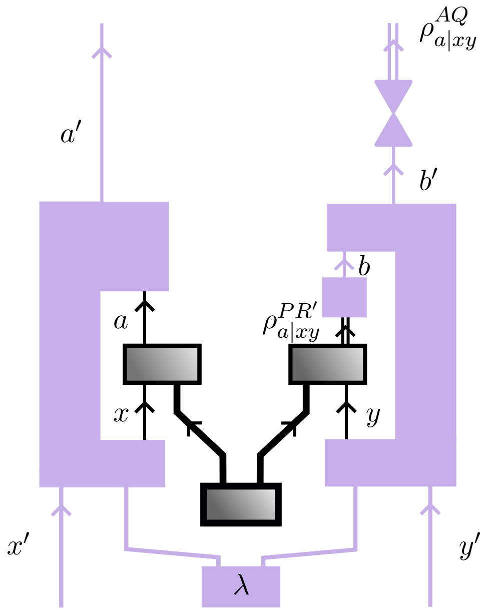

Figure 9: The LOSR protocol (in purple) that converts $\mathbf{\Sigma}'^{PR}$ into $\mathbf{\Sigma}^{AQ}$.

F The sets of free non-signalling channel assemblages under LOCC and LOSR coincide

In our resource theory, we require the elements of a free channel assemblage to decompose as per Eq. (2), i.e.

$$\mathcal{I}_{a|x}(\cdot) = \sum_{\lambda} p(a|x,\lambda)p(\lambda)\mathcal{I}_{\lambda}(\cdot),$$

where $\mathcal{I}_{\lambda}(\cdot)$ is a CPTP map. It is straightforward to see that for *any* state ρ, $\operatorname{tr}\left\{\mathcal{I}_{a|x}(\rho)\right\} = p(a|x)$, i.e., Alice's probability distribution $p(a|x)$ is independent of Bob's input state. This is the no-signalling condition from Bob to Alice, which is therefore satisfied in our LOSR approach.

In Refs. [25, 74], a free assemblage (see Fig. 10(a)) is defined as one that decomposes as

$$\widetilde{\mathcal{I}}_{a|x}(\cdot) = \sum_{\lambda} p(a|x,\lambda)\widetilde{\mathcal{I}}_{\lambda}(\cdot),$$

with $\widetilde{\mathcal{I}}_{\lambda}$ being a CPTNI map. One can think about this construction as a channel that is semi-causal, i.e., non-signalling from Alice to Bob. In general, however, $\operatorname{tr}\left\{\widetilde{\mathcal{I}}_{a|x}(\rho)\right\}$ can depend on the particular choice of ρ, and so it may allow signalling from Bob to Alice.

If one wants the free assemblage to be non-signalling (from Alice to Bob *and* from Bob to Alice), an additional condition must be imposed on $\widetilde{\mathcal{I}}_{a|x}(\cdot)$. For this channel to be causal, it must also be semi-causal from Bob to Alice. Let us denote such assemblage by $\widetilde{\mathcal{I}}^{NS}_{a|x}(\cdot)$. Then, from the main result of Ref. [77], if follows that $\widetilde{\mathcal{I}}^{NS}_{a|x}(\cdot)$ must be semi-localizable from Bob to Alice. Therefore, $\widetilde{\mathcal{I}}^{NS}_{a|x}(\cdot)$ can be graphically represented as in Fig. 10(b). Here, Bob's local operation is a CPTP map. The state that Alice and Bob now share is a classical-quantum separable state. For this reason, we can treat Bob's subsystem of the shared state, ρ_λ, as a quantum state that is being prepared conditioned on the value of the classical variable λ. This means that we can treat this preparation as a local operation on Bob's side, and the only system that he shares with Alice is the classical variable λ as illustrated in Fig. 10(c). It is clear to see that the assemblage illustrated in Fig. 10(b) is LOSR-free, i.e., it can be expressed as $\mathcal{I}_{a|x}(\cdot) = \sum_\lambda p(a|x,\lambda)p(\lambda)\mathcal{I}_\lambda(\cdot)$, with $\mathcal{I}_\lambda(\cdot)$ being a CPTP map. Therefore, we showed that the set of LOSR-free channel assemblages coincides with the set of free channel assemblages introduced in Refs. [25, 74] if no-signalling from Bob to Alice is imposed.

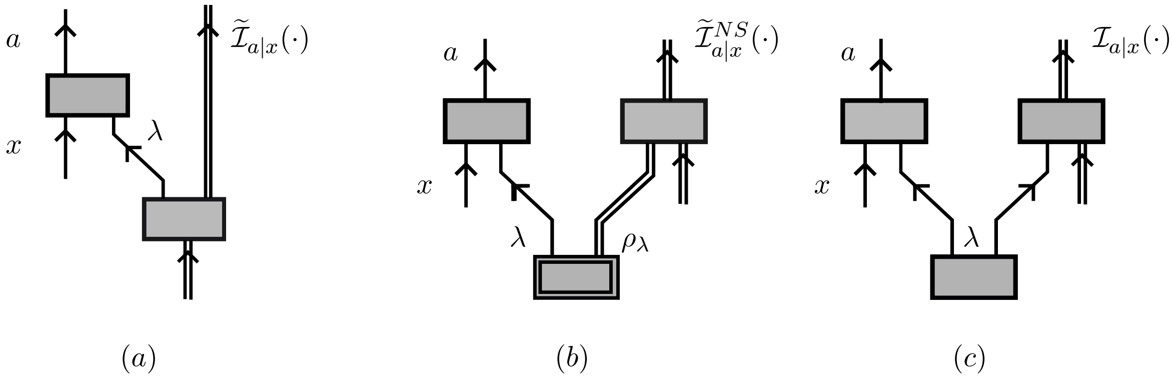

Figure 10: (a) An LOCC-free channel assemblage that allows signalling from Bob to Alice. (b) An LOCC-free channel assemblage that is moreover non-signalling. (c) An LOSR-free channel assemblage. We prove here that the class of resources defined by (b) is identical to that defined by (c).

Activation of post-quantumness in bipartite generalised EPR scenarios

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Correlations generated in generalised Einstein-Podolsky-Rosen (EPR) scenarios are examples of non-signalling bipartite resources that can exhibit post-quantum behavior. There exist assemblages that, despite being post-quantum resources, can only generate quantum correlations in bipartite Bell-type scenarios. Here, we present a protocol for activation of post-quantumness in bipartite Bob-with-input, measurement-device-independent and channel EPR scenarios. By designing a protocol that involves distributing the assemblages in a larger network, we derive tailored Bell inequalities which can be violated beyond their quantum bound in this new set-up.

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I. INTRODUCTION

In the bipartite Einstein-Podolsky-Rosen (EPR) scenario [1–3], nonclassical correlations are generated between two distant parties, Alice and Bob, upon local measurements on half of a system prepared in an entangled state. Along with Bell-nonclassicality [4–6] and entanglement [7], it is one of the notable resources studied in quantum theory. Correlations generated in the EPR scenario are crucial for various information-theoretic tasks [8, 9], such as quantum cryptography [10, 11], entanglement certification [12, 13] and randomness certification [14, 15]. From the quantum foundations perspective, the EPR scenario allows us to study the structure and limitations of quantum theory [16–20].

The standard EPR scenario [1–3] is illustrated in Fig. 1(a), where the classical and quantum systems are depicted with single and double lines, respectively. Conventionally, Alice and Bob share a quantum system denoted by a density operator ρ_{AB}. Alice performs a local measurement on her subsystem (associated with a generalised measurement, i.e., a positive operator-valued measure (POVM) $\{M_{a|x}\}_{a,x}$): she chooses a measurement setting labeled by $x \in \mathbb{X}$ and registers a classical outcome $a \in \mathbb{A}$ with probability $p(a|x)$. Upon this measurement, the subsystem in Bob's laboratory can be described by a conditional state $\rho_{a|x}$. The relevant object of study in the EPR scenario is an *assemblage* [21] defined as $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}} = \{\sigma_{a|x}\}_{a,x}$, where each unnormalised state $\sigma_{a|x} = \mathrm{tr}_A\left\{(M_{a|x} \otimes \mathbb{I})\rho_{AB}\right\}$ is given by $\sigma_{a|x} := p(a|x)\rho_{a|x}$. In this paper, we do not limit Alice and Bob's shared resources to be quantum systems nor Alice to perform quantum measurements, but we do limit Bob to having a quantum system. That is, we take the space of all resources to be the non-signalling ones (meaning that the parties cannot utilise the shared system for instantaneous communication). Notably, in the standard bipartite EPR scenario, this does not make a difference. Due to the GHJW theorem, proven by Gisin [22] and Hughston, Jozsa, and Wootters [23], all possible non-signalling assemblages can be reproduced by Alice making a measurement on a quantum system. However, this is not true for non-bipartite scenarios, as shown in Ref. [24]; in this work we will not consider such multipartite settings.

Different generalizations of this standard EPR scenario have been introduced in recent years, wherein Bob may also probe his system in various ways. Then, it is possible to relax the condition of the shared physical system being quantum, and consider more general theories in the set-up of this experiment (*post-quantum* systems). Distinction between these generalizations is made by different processing types on Bob's side, which are characterised by different input and output types in Fig. 1. In every type of EPR scenario, the relevant assemblage is then given by a collection of ensembles of Bob's conditional processes labeled by Alice's measurement input and output variables. EPR scenarios considered in this paper are the following:

- (b) Bob-with-input EPR scenario [25]: a bipartite EPR scenario, where Bob has a classical input which allows him to locally influence the state preparation of his quantum output system.
- (c) Measurement-device-independent EPR scenario [26, 27]: a bipartite EPR scenario where Bob has a measurement channel with a quantum input and a classical output, i.e., a quantum instrument with a trivial output Hilbert space.
- (d) Channel EPR scenario [27, 28]: a bipartite EPR scenario where Bob has quantum input and output systems.

These scenarios are all closely related, and a unified framework for understanding them was introduced in Refs. [27, 29, 30]. In particular, all these scenarios have a *common-cause* structure (Alice and Bob share a physical system) and Alice is measuring her local subsystem. Therefore, the correlations generated between Alice and Bob depend on the setting and output of Alice's measurement.

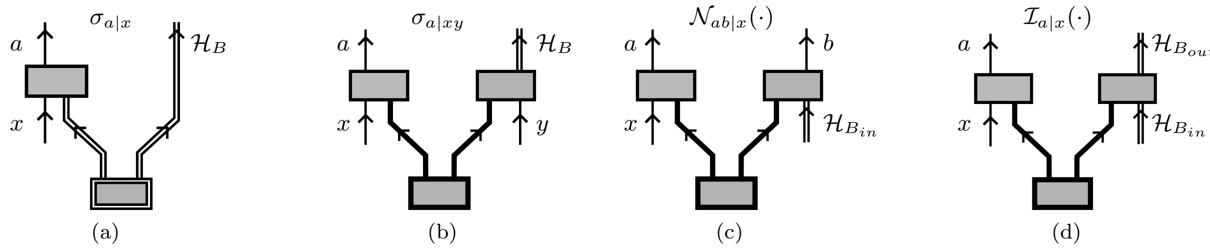

FIG. 1. **Generalizations of the standard EPR scenario:** (a) Standard bipartite EPR scenario. (b) Bob-with-input EPR scenario. (c) Measurement-device-independent EPR scenario. (d) Channel EPR scenario. Quantum and classical systems are depicted by double and single lines, respectively. Thick black lines depict the possibility that the shared systems may be classical, quantum, or even post-quantum. Formal definitions of the assemblage elements for each EPR scenario ($\sigma_{a|x}$ for the Bob-with-input scenario, $\mathcal{N}_{ab|x}(\cdot)$ for the measurement-device-independent scenario and $\mathcal{I}_{a|x}(\cdot)$ for the channel scenario) are introduced in the following sections.

In contrast to the standard EPR scenario, its generalizations can exhibit post-quantum resources. Moreover, for each of the generalizations introduced above, there exist assemblages such that their post-quantumness cannot be tested in a bipartite device-independent set-up[1]. For such assemblages, testing their post-quantumness requires either evaluating EPR functionals or using hierarchies of semi-definite programs [25, 27, 31].

In this paper, we design a protocol for activating post-quantumness of bipartite assemblages in device-independent multipartite settings. We show how to distribute and process a post-quantum assemblage such that a device-independent test in this new setting can certify post-quantumness of the assemblage. The protocol has a similar structure for all types of generalised EPR scenarios. First, an additional quantum resource (a standard bipartite EPR assemblage) shared between Bob and an additional third party is self-tested. In this step of the protocol, Bob measures his share of this additional resource to learn about its structure in a device-independent way. Second, this additional quantum resource is used to promote EPR functionals to Bell functionals on correlations generated in this new larger network (with Alice, Bob, and the additional third party). Then, these Bell functionals can be used to certify post-quantumness of the correlations generated in the protocol set-up.

Given that the protocols slightly differ for various types of generalised EPR scenarios, below we dedicate a section to each scenario to introduce and analyze its protocol. In the main body of the paper, we restrict our considerations to qubit assemblages; generalization to higher-dimensional assemblages is discussed in the Appendix F. The method that we use to show activation of post-quantumness is analogous to the one introduced in Ref. [32] for multipartite assemblages, which adapts the results of Refs. [33, 34] for the self-testing stage and the general idea for the protocol. The most significant difference between our set-ups is that we focus on bipartite scenarios in which now Bob can apply a local processing on the system shared with Alice.

II. BOB-WITH-INPUT SCENARIO

A. Description of the scenario

In the Bob-with-input (BwI) EPR scenario, Bob can choose the value of a classical input $y \in \mathbb{Y}$ which influences the local transformation of his subsystem (not necessarily a quantum one) into a new quantum system. Alice's local operations are the same as in the standard EPR scenario. Let $p(a|x)$ denote the

[1] Here, the concept of a bipartite device-independent set-up is quite expansive. Essentially, device-independent set-ups are configurations involving devices with classical inputs and outputs whose characteristics are not specified. Further elaboration on this will be provided for each EPR scenario later on.

probability that Alice obtains outcome a when performing measurement x. Then, if in addition Bob chooses an input y, the normalised state in his laboratory is given by $\rho_{a|xy}$. The relevant object of study in this scenario is a *Bob-with-input assemblage* which is given by $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}} = \{\sigma_{a|xy}\}_{a,x,y}$, with $\sigma_{a|xy} = p(a|x)\rho_{a|xy}$. The most general constraints on the possible unnormalised states $\sigma_{a|xy}$ are the following.

Definition 1 (Non-signalling Bob-with-input assemblages)**.** *An assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ with elements $\{\sigma_{a|xy}\}_{a,x,y}$ is a valid non-signalling assemblage in a Bob-with-input EPR scenario if the following constraints are satisfied*

$$\sigma_{a|xy} \geq 0 \quad \forall\, a, x, y\,, \tag{1}$$

$$\mathrm{tr}\left\{\sum_a \sigma_{a|xy}\right\} = 1 \quad \forall\, x, y\,, \tag{2}$$

$$\mathrm{tr}\left\{\sigma_{a|xy}\right\} = p(a|x) \quad \forall\, a, x, y\,, \tag{3}$$

$$\sum_a \sigma_{a|xy} = \sum_a \sigma_{a|x'y} \quad \forall\, x, y, x'. \tag{4}$$

Conditions (3) and (4) are imposed by relativistic causality. They simply imply that Alice and Bob cannot signal to each other.

It was shown in Ref. [25] that there exist non-signalling BwI assemblages that do not admit a quantum realization of the form $\sigma_{a|xy} = \mathrm{tr}_A\left\{(M_{a|x} \otimes \mathcal{E}_y)\rho_{AB}\right\}$, i.e., cannot be generated by Alice and Bob sharing a quantum state ρ_{AB}, Bob applying channels (completely positive and trace preserving (CPTP) maps [35, 36]) $\{\mathcal{E}_y\}_y$ on his subsystem and Alice measuring with POVMs $\{M_{a|x}\}_{a,x}$. We will refer to such assemblages as *post-quantum* assemblages.

To certify post-quantumness of a BwI assemblage, one can define an EPR inequality and show that its quantum bound can be violated by this assemblage. Take an arbitrary fixed post-quantum BwI assemblage $\mathbf{\Sigma}^*_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ with elements $\{\sigma^*_{a|xy}\}_{a,x,y}$. There always exist Hermitian operators $\{F_{axy}\}_{a,x,y}$ such that the EPR functional

$$\mathrm{tr}\left\{\sum_{a,x,y} F_{axy}\sigma^*_{a|xy}\right\} < 0, \tag{5}$$

while $\mathrm{tr}\left\{\sum_{a,x,y} F_{axy}\sigma_{a|xy}\right\} \geq 0$ when evaluated on any quantum assemblage $\{\sigma_{a|xy}\}_{a,x,y}$.

One particularly interesting set of assemblages in the BwI scenario considered in Ref. [24] consists of post-quantum assemblages that can only lead to quantum correlations if Bob decides to measure his quantum state. Formally, such assemblages with elements $\{\sigma_{a|xy}\}_{a,x,y}$ are such that when Bob performs a measurement $\{N_b\}_b$, the observed correlations $p(ab|xy) = \mathrm{tr}\left\{N_b\sigma_{a|xy}\right\}$ belong to the quantum set, i.e., admit a quantum realisation of the form $p(ab|xy) = \mathrm{tr}\left\{N_{a|x} \otimes N_{b|y}\rho\right\}$ with $\{N_{a|x}\}_{a,x}$ and $\{N_{b|y}\}_{b,y}$ representing POVMs and ρ being a valid quantum state. This set of assemblages is interesting, because it shows that post-quantum assemblages are a fundamentally different resource than post-quantum correlations in Bell-type scenarios. We denote this set $\mathbf{\Sigma}^{QC}$.

In this paper, we propose a protocol in which a bipartite Bob-with-input post-quantum assemblage is distributed in a tripartite network such that post-quantum correlations can be generated in this new network. If the input assemblage belongs to the set $\mathbf{\Sigma}^{QC}$, we refer to this phenomenon as activation of post-quantumness, as it enables one to reveal the post-quantum nature of the BwI assemblage in a tripartite Bell-type setting. This result shows that all assemblages from the set $\mathbf{\Sigma}^{QC}$ can be utilised to generate post-quantum correlations. As there is no known example of a post-quantum assemblage which does not produce post-quantum correlations, it raises the question of whether there is a fundamental difference between these two resources.

B. The activation protocol

The activation protocol is illustrated in Fig. 2. The experiment consists of three parties: Alice, Bob and Charlie. Alice and Bob share a bipartite Bob-with-input assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ whose post-quantumness we want to certify. Moreover, Bob and Charlie share a standard quantum assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$, with $\mathbb{C} \coloneqq \{0,1\}$ and $\mathbb{W} \coloneqq \{1,2,3\}$. Bob's quantum subsystem arising in this bipartition is defined on $\mathcal{H}_{B'}$; here we consider $\mathcal{H}_{B'}$ of dimension 2 – generalization of this case is discussed in the Appendix F. Finally, Bob has a measurement device $\{\mathrm{M}_{b|z}^{BB'}\}_{b,z}$ which measures the system defined on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$. The input and output sets of this device are the following: $b \in \mathbb{B} \coloneqq \{0,1\}$ and $z \in \mathbb{Z} \coloneqq \{1,2,3,4,\star\}$. Hereon, we assume that the assemblages $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ and $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ are independent. Additionally, Alice, Bob and Charlie must be space-like separated.

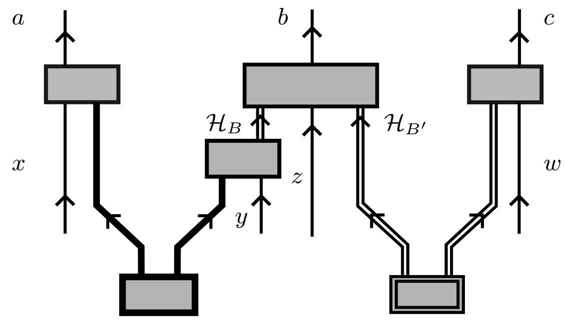

FIG. 2. Depiction of the activation protocol for the Bob-with-input scenario. Arbitrary systems (which may be post-quantum) are represented by thick lines, quantum systems are represented by double lines and classical systems are depicted as single lines

If the BwI assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{X}\mathbb{Y}}$ belongs to the set $\mathbf{\Sigma}^{QC}$, this protocol can activate its post-quantumness by utilizing the additional quantum resource $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ to produce post-quantum correlations in the new tripartite network. In the set-up described above, the protocol consists of the following two steps.

Step 1: *Self-testing of* $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$.

The first step of the protocol is to certify that the state of Bob's quantum system $\mathcal{H}_{B'}$ is prepared in the assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}^{(r)}$ with elements $\sigma_{c|w}^{(r)} = r\widetilde{\sigma}_{c|w} + (1-r)(\widetilde{\sigma}_{c|w})^T$. Here, $0 \leq r \leq 1$ is an unknown parameter, and $\widetilde{\sigma}_{c|1} = (\mathbb{I} + (-1)^c Z)/4$, $\widetilde{\sigma}_{c|2} = (\mathbb{I} + (-1)^c X)/4$, $\widetilde{\sigma}_{c|3} = (\mathbb{I} - (-1)^c Y)/4$, with X, Y, Z being the Pauli operators. Self-testing[2] of this assemblage is possible due to the result proven in Refs. [34, 37]. We recall the relevant Bell functional and the choice of measurements $\{M_{b|z}^{B'}\}_{b,z}$ for $z \in \{1,2,3,4\}$ which maximises it in Appendix A.

Step 2: *Certification of post-quantumness.*

The second step of the protocol is to measure the system on $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ and evaluate an appropriate Bell functional to certify post-quantumness of the observed correlations. For now, we only consider the case when Bob's measurement setting and outcome is fixed such that $M_{b=0|z=\star}^{BB'} = |\phi^+\rangle\langle\phi^+|$, with $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Moreover, for simplicity we assume that the elements of the assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ are given by $\{\widetilde{\sigma}_{c|w}\}$, not by the mixture $\{\sigma_{c|w}^{(r)}\}$. We denote such assemblage by $\widetilde{\mathbf{\Sigma}}_{\mathbb{C}|\mathbb{W}}$. In the Appendix B and C, we show that these assumptions do not limit our result and the protocol for activation of post-quantumness works when they are relaxed. Under these assumptions, the observed correlations are given by

$$p(a, b=0, c|x, y, z=\star, w) = \operatorname{tr}\left\{M_{b=0|z=\star}^{BB'}(\sigma_{a|xy} \otimes \widetilde{\sigma}_{c|w})\right\}. \tag{6}$$

[2] Self-testing is a well-defined concept if the states and possible operations are specified within quantum theory. One may wonder if self-testing results are still relevant in our set-up, where we consider more general theories (post-quantum resources). In the specific case of self-testing of a standard bipartite EPR assemblage, post-quantum non-signalling resources are not more powerful than the quantum ones. Due to GHJW theorem [22, 23], all non-signalling assemblages in this standard bipartite scenario admit of a quantum realization. Hence, using self-testing in this scenario is justified.

Using the fact that $\mathrm{tr}_1\left\{A_1 \otimes \mathbb{I}_2 \left|\phi^+\right\rangle\left\langle\phi^+\right|\right\} = \frac{1}{2}A^T$ for any operator A, we can write Eq. (6) as

$$p(a, b=0, c|x, y, z=\star, w) = \frac{1}{2}\mathrm{tr}\left\{(\widetilde{\sigma}_{c|w})^T \sigma_{a|xy}\right\}. \tag{7}$$

Given observed correlations, one way to certify their post-quantumness is to evaluate a suitable Bell functional, which we will now derive from the EPR functional specified in Eq. (5). For $w \in \{1,2,3\}$, let $\pi_{c|w}$ denote the projector onto the eigenspace of the eigenstates of Pauli Z, X and Y operators with eigenvalue $(-1)^c$. Notice that $\{\pi_{c|w}\}_{c,w}$ form a basis of the set of Hermitian matrices in a two-dimensional Hilbert space. Therefore, the operators $\{F_{axy}\}_{a,x,y}$ can be decomposed as $F_{axy} = \sum_{c,w} \xi^{axy}_{cw}\pi_{c|w}$ for some numbers $\{\xi^{axy}_{cw}\}_{a,x,y,c,w}$. We can use these coefficients to construct the following Bell functional on the correlation $\mathbf{p} = \{p(a,b,c|x,y,z,w)\}$:

$$I^*_{BwI}[\mathbf{p}] \equiv \sum_{a,x,y,c,w} \xi^{axy}_{cw} p(a, b=0, c|x, y, z=\star, w). \tag{8}$$

Here, the coefficients corresponding to (b,z) different than $(0,\star)$ are equal to zero. To calculate the value of this functional, we use the form of correlations given in Eq. (7):

$$\begin{aligned} I^*_{BwI}[\mathbf{p}] &= \sum_{a,x,y,c,w} \xi^{axy}_{cw}\frac{1}{2}\mathrm{tr}\left\{(\widetilde{\sigma}_{c|w})^T\sigma_{a|xy}\right\}, \\ &= \frac{1}{4}\sum_{a,x,y}\mathrm{tr}\left\{\left(\sum_{c,w}\xi^{axy}_{cw}\pi_{c|w}\right)\sigma_{a|xy}\right\}, \\ &= \frac{1}{4}\mathrm{tr}\left\{\sum_{a,x,y}F_{axy}\sigma_{a|xy}\right\}. \end{aligned} \tag{9}$$

Here, we used the fact that the elements $(\widetilde{\sigma}_{c|w})^T$ are proportional to the projectors $\pi_{c|w}$. Therefore, $I^*_{BwI}[\mathbf{p}] \geq 0$ for any correlation $\mathbf{p}$ arising from a quantum assemblage $\{\sigma_{a|xy}\}_{a,x,y}$[3]. For the particular choice of measurement $M^{BB'}_{b=0|z=\star} = \left|\phi^+\right\rangle\left\langle\phi^+\right|$, the Bell functional evaluated on the fixed assemblage $\mathbf{\Sigma}^*_{\mathbb{A}|\mathbb{XY}} = \{\sigma^*_{a|xy}\}_{a,x,y}$ gives $I^*_{BwI}[\mathbf{p^*}] < 0$. In the Appendix C, we give a detailed analysis of the case when $M^{BB'}_{b=0|z=\star} \neq \left|\phi^+\right\rangle\left\langle\phi^+\right|$ and I^*_{BwI} needs to be optimised over all possible measurements. We show that the quantum bound is also given by 0 in this case; hence, if one observes $I^*_{BwI}[\mathbf{p}] < 0$, it necessarily means that the correlations $\mathbf{p}$ were generated from a post-quantum BwI assemblage.

If the first step of the protocol is successful, we can use the construction presented above to certify post-quantumness of the correlations in the tripartite set-up. For each BwI assemblage $\mathbf{\Sigma}^*_{\mathbb{A}|\mathbb{XY}}$, Theorem 2 can be used to derive a tailored Bell functional to be used in the activation protocol.

Theorem 2. *There always exists a Bell functional of the form*

$$I^*_{BwI}[\boldsymbol{p}] = \frac{1}{2}\sum_{a,x,y,c,w}\xi^{axy}_{cw}p(a, b=0, c|x, y, z=\star, w), \tag{10}$$

for which $I^*_{BwI}[\boldsymbol{p}] \geq 0$ *when evaluated on any correlations* $\boldsymbol{p} = \{p(a,b,c|x,y,z,w)\}$ *arising from quantum Bob-with-input assemblage with elements* $\{\sigma_{a|xy}\}_{a,x,y}$ *as* $p(a, b=0, c|x, y, z=\star, w) = \mathrm{tr}\left\{M^{BB'}_{b=0|z=\star}(\sigma_{a|xy}\otimes\widetilde{\sigma}_{c|w})\right\}$.

---

[3] Although in principle it is difficult to find operators $\{F_{axy}\}_{a,x,y}$ such that the quantum bound of the EPR functional is exactly $\beta^Q = 0$, one can use numerical calculations to find a lower bound on β^Q. One possible approximation is an almost-quantum bound, which we denote β^{AQ} [25]. We show how to use this approximation for the activation protocol in the Appendix D, where we present how the protocol works for a specific fixed post-quantum assemblage.

As an explicit example of this method, in the Appendix D, we use the construction from Theorem 2 to derive a Bell functional that activates post-quantumness of the assemblage $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ introduced in Ref. [25, Eq. (6)], which belongs to the set $\mathbf{\Sigma}^{QC}$.

The protocol designed in this section enables one to generate post-quantum Bell-type correlations from bipartite BwI assemblages. In particular, it can be used to activate post-quantumness of assemblages from the set $\mathbf{\Sigma}^{QC}$. Therefore, although the assemblages from the set $\mathbf{\Sigma}^{QC}$ do not exhibit bipartite Bell-type nonclassicality, they can generate post-quantum correlations in a larger network. This result suggests that the fundamental differences between the post-quantumness embedded in Bob-with-input EPR assemblages and in correlations generated in Bell scenarios are subtle, as the former can always generate latter.

III. MEASUREMENT-DEVICE-INDEPENDENT SCENARIO

A. Description of the scenario

We will now consider a measurement-device-independent (MDI) EPR scenario in which Bob holds a collection of measurement channels that may be correlated with the physical system in Alice's lab [26, 27]. Bob's processing, which we denote by $\Theta_b^{BB_{in}\to B_{out}}$, takes as inputs a quantum system $\mathcal{H}_{B_{in}}$ and the subsystem B. Bob's output system B_{out} is just a classical variable that may take values within $\mathbb{B}$. Formally, the relevant *MDI assemblages* in this scenario are given by $\mathbf{N}_{\mathbb{AB}|\mathbb{X}} = \{\mathcal{N}_{ab|x}(\cdot)\}_{a,b,x}$, where $\mathcal{N}_{ab|x}(\cdot)$ is a measurement channel – a quantum instrument with a trivial output Hilbert space – associated to the POVM element corresponding to outcome b of Bob's measurement device (when Alice's measurement event is $a|x$).

Definition 3 (Non-signalling measurement-device-independent assemblages)**.** *An assemblage $\mathbf{N}_{\mathbb{AB}|\mathbb{X}}$ with elements $\{\mathcal{N}_{ab|x}(\cdot)\}_{a,b,x}$ is a valid non-signalling assemblage in a measurement-device-independent EPR scenario if the following constraints are satisfied*

$$\sum_b \mathcal{N}_{ab|x}(\rho) = p(a|x) \quad \forall\, a, x, \rho, \tag{11}$$

$$\sum_a \mathcal{N}_{ab|x}(\cdot) = \Omega_b^{B_{in}\to B_{out}}(\cdot) \quad \forall\, b, x, \tag{12}$$

where ρ represents any normalised state of the quantum system B_{in} and $\{\Omega_b^{B_{in}\to B_{out}}(\cdot)\}_b$ is a collection of measurement channels with $(\cdot)$ representing the input space B_{in}.

The no-signalling conditions (11) and (12) imply that Alice's output does not depend on Bob's input state and Alice's classical input cannot influence Bob's output. The elements of a quantumly-realizable MDI assemblage can be written as

$$\mathcal{N}_{ab|x}(\cdot) = \mathrm{tr}_{AB_{out}}\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes\Theta_b^{BB_{in}\to B_{out}})[\rho_{AB}\otimes(\cdot)]\right\}. \tag{13}$$

Notably, there exist non-signalling MDI assemblages that do not admit of a decomposition of the form (13) [27, 38], which we refer to as *post-quantum* MDI assemblages.

One method for determining whether a measurement-device-independent EPR assemblage is post-quantum involves defining an EPR functional that it violates with respect to a quantum bound. Take an arbitrary fixed post-quantum MDI assemblage $\mathbf{N}^*_{\mathbb{AB}|\mathbb{X}}$ with elements $\{\mathcal{N}^*_{ab|x}(\cdot)\}_{a,b,x}$. There always exist Hermitian operators $\{F_{abx}\}_{a,b,x}$ such that the EPR functional

$$\mathrm{tr}\left\{\sum_{a,b,x} F_{abx}\, J(\mathcal{N}^*_{ab|x})\right\} < 0, \tag{14}$$

while $\mathrm{tr}\left\{\sum_{a,b,x} F_{abx}\, J(\mathcal{N}_{ab|x})\right\} \geq 0$ when evaluated on any quantum assemblage $\mathbf{N}_{\mathbb{AB}|\mathbb{X}}$.

Moreover, there exist MDI assemblages which can only generate quantum correlations $p(ab|xy) = \mathcal{N}_{ab|x}(\rho_y)$ given a set of fixed input states $\{\rho_y\}_y$. We denote the set of such assemblages by $\mathbf{N}^{QC}$. We give an example of such behaviour in the Appendix E, where we show that a post-quantum assemblage introduced in Ref. [27] cannot generate post-quantum correlations in a bipartite Bell-type scenario.

We now propose a protocol which certifies post-quantumness of MDI assemblages. Similarly to the previous section, we show that although existence of the set $\mathbf{N}^{QC}$ seems to imply that post-quantum assemblages are fundamentally different resources than post-quantum Bell nonclassicality, it is not the case. If the MDI assemblage in the protocol belongs to the set $\mathbf{N}^{QC}$, its post-quantumness can be activated, as it can generate post-quantum correlations in a tripartite Bell-like network.

B. The activation protocol

The experiment for the activation protocol consists of three parties – Alice, Bob and Charlie, as illustrated in Fig. 3. Alice and Bob share a bipartite measurement-device-independent assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$. Bob's input state, which is on the Hilbert space $\mathcal{H}_{B_{in}}$, is prepared by Charlie through a standard quantum assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{Z}}$, with $\mathbb{C} := \{0,1\}$ and $\mathbb{W} := \{1,2,3\}$. That is, Bob and Charlie share a system $B_{in}C$ and Charlie performs measurements $\{M^C_{c|z}\}_{c,z}$, which result in assemblage elements $\{\sigma_{c|z}\}_{c,z}$. Finally, Bob has a measurement device $\{\mathrm{M}^{BB_{in}}_{b|y}\}_{b,y}$ with input and output sets $b \in \mathbb{B}$ and $y \in \mathbb{Y} := \{1,2,3,4,\star\}$.

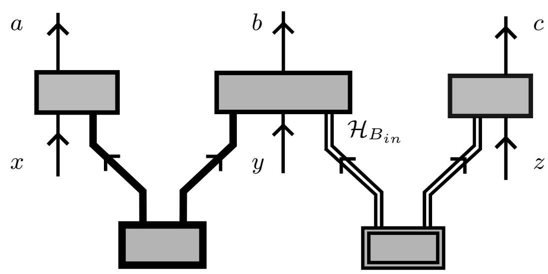

FIG. 3. Depiction of the activation protocol for the measurement-device-independent scenario. Quantum systems are represented by double lines, while classical systems are depicted as single lines. Thick lines depict the possibility that the shared systems may be classical, quantum, or post-quantum.

In the set-up of Fig. 3, $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ and $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{Z}}$ are two independent assemblages. This can be guaranteed by preparing the systems AB and $B_{in}C$ separately. Additionally, Alice, Bob and Charlie must be space-like separated.

The protocol for the MDI scenario has the same structure as the protocol for the BwI scenario introduced in the previous section. In the set-up described above, the protocol consists of the following two steps.

Step 1: *Self-testing of* $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{Z}}$.

For $y \in \{1,2,3,4\}$, Bob measures the quantum system $\mathcal{H}_{B'}$ for the self-testing procedure that is equivalent to Step 1 in the Bob-with-input activation protocol. The goal of this step is to certify that Bob and Charlie share an assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{Z}}$ with elements $\sigma^{(r)}_{c|z} = r\widetilde{\sigma}_{c|z} + (1-r)(\widetilde{\sigma}_{c|z})^T$. Details of the self-testing procedure are specified in the Appendix A.

Step 2: *Certification of post-quantumness.*

In the second step of the protocol, for $y = \star$, Bob applies the process $\Theta_b^{BB_{in}\to B_{out}}$ that defines the assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$. Then, it is possible to certify the post-quantumness of the correlations $p(a,b,c|x,y=\star,z)$ via an appropriate Bell functional.

Assume that the first step of the protocol was successful. Here, we again assume that the elements of the assemblage Bob and Charlie share are $\{\widetilde{\sigma}_{c|z}\}_{c,z}$, not $\{\sigma^{(r)}_{c|z}\}_{c,z}$ (see the Appendix B for discussion about this assumption). Then, the observed correlations can be written as

$$p(a,b,c|x,y=\star,z) = \mathcal{N}_{ab|x}(\widetilde{\sigma}_{c|z}). \tag{15}$$

Now, let us use Choi-Jamiołkowski isomorphism to transform the channel $\mathcal{N}_{ab|x}(\cdot)$ to an operator. Define $J(\mathcal{N}_{ab|x}) = (\mathcal{N}_{ab|x} \otimes \mathbb{I}^C)(|\phi^+\rangle\langle\phi^+|)_{B_{in}C}$. Moreover, notice that the elements $\widetilde{\sigma}_{c|z}$ can be written as $\widetilde{\sigma}_{c|z} = \mathrm{tr}_C\left\{(\mathbb{I}^{B_{in}} \otimes \widetilde{M}^C_{c|z})\,|\phi^+\rangle\langle\phi^+|_{B_{in}C}\right\}$, with $\widetilde{M}^C_{c|1} = (\mathbb{I} + (-1)^c Z)/2$, $\widetilde{M}^C_{c|2} = (\mathbb{I} + (-1)^c X)/2$ and $\widetilde{M}^C_{c|3} = (\mathbb{I} + (-1)^c Y)/2$. Then, correlation (15) can be written as

$$p(a,b,c|x,y=\star,z) = \mathrm{tr}\left\{\widetilde{M}^C_{c|z}\, J(\mathcal{N}_{ab|x})\right\}. \tag{16}$$

To derive a suitable Bell functional for the considered set-up, we will use the EPR functional defined in Eq. (14). Let $\pi_{c|z}$ denote the projector onto the eigenspace of the eigenstates of Pauli Z,X and Y operators with eigenvalue $(-1)^c$. Then, the operators $\{F_{abx}\}_{a,b,x}$ can be decomposed as $F_{abx} = \sum_{c,z}\xi^{abx}_{cz}\pi_{c|z}$ for some real numbers $\{\xi^{abx}_{cz}\}_{a,b,x,c,z}$. To certify post-quantum correlations in this protocol, it suffices to consider the following Bell functional on the correlation $\mathbf{p} = \{p(a,b,c|x,y=\star,z)\}$:

$$I^*_{MDI}[\mathbf{p}] \equiv \sum_{a,b,x,c,z} \xi^{abx}_{cz}\, p(a,b,c|x,y=\star,z). \tag{17}$$

To calculate the value of this functional, we use the form of correlations given in Eq. (16):

$$\begin{aligned} I^*_{MDI}[\mathbf{p}] &= \sum_{a,b,x,c,z} \xi^{abx}_{cz}\,\mathrm{tr}\left\{\widetilde{M}^C_{c|z}\, J(\mathcal{N}_{ab|x})\right\}, \\ &= \sum_{a,b,x} \mathrm{tr}\left\{\left(\sum_{c,z}\xi^{abx}_{cz}\pi_{c|z}\right) J(\mathcal{N}_{ab|x})\right\}, \\ &= \mathrm{tr}\left\{\sum_{a,b,x} F_{abx}\, J(\mathcal{N}_{ab|x})\right\}. \end{aligned} \tag{18}$$

Here, we used the fact that the measurement elements $\widetilde{M}^C_{c|z}$ correspond to the projectors $\pi_{c|z}$. If $\mathbf{p}$ was generated from a quantum MDI assemblage, then $I^*_{MDI}[\mathbf{p}] \geq 0$. For the fixed post-quantum assemblage $\{\mathcal{N}^*_{ab|x}(\cdot)\}_{a,b,x}$, the functional evaluates to $I^*_{MDI}[\mathbf{p^*}] < 0$. This result is summarised in the theorem below.

Theorem 4. *There always exists a Bell functional of the form*

$$I^*_{MDI}[\boldsymbol{p}] = \sum_{a,b,x,c,z} \xi^{abx}_{cz}\, p(a,b,c|x,y=\star,z), \tag{19}$$

for which $I^*_{MDI}[\boldsymbol{p}] \geq 0$ when evaluated on any correlations $\boldsymbol{p} = \{p(a,b,c|x,y,z)\}$ arising from quantum measurement-device-independent assemblage with elements $\{\mathcal{N}_{ab|x}(\cdot)\}_{a,b,x}$ as $p(a,b,c|x,y=\star,z) = \mathcal{N}_{ab|x}(\widetilde{\sigma}_{c|z})$.

To summarise, here we introduced a protocol in which a post-quantum MDI assemblage is processed to generate post-quantum Bell-type correlations. In analogy to the protocol for the BwI scenario, even assemblages from the set $\mathbf{N}^{QC}$ can be activated to generate post-quantum correlations when distributed in a larger network. It shows that there exists a relation between post-quantumness in measurement-device-independent EPR scenarios and device-independent post-quantum correlations.

IV. CHANNEL EPR SCENARIO

A. Description of the scenario

In the channel EPR scenario [27, 28], Bob has access to a channel with a quantum input defined on $\mathcal{H}_{B_{in}}$ and a quantum output defined on $\mathcal{H}_{B_{out}}$. Bob's local process, which we denote by $\Gamma^{BB_{in}\to B_{out}}$, might be correlated with Alice's system through the system B. Denote by $\mathcal{I}_{a|x}(\cdot)$ the instrument that is effectively applied on Bob's quantum system B_{in} to output a quantum system B_{out}, given that Alice has performed a measurement x on A and obtained the outcome a. Then, the object of study in a channel EPR scenario is the *channel assemblage* of instruments $\mathbf{I}_{\mathbb{A}|\mathbb{X}} = \{\mathcal{I}_{a|x}(\cdot)\}_{a,x}$.

Definition 5 (Non-signalling channel assemblages)**.** *An assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ with elements $\{\mathcal{I}_{a|x}(\cdot)\}_{a,x}$ is a valid non-signalling assemblage in a channel EPR scenario if the following constraints are satisfied*

$$\operatorname{tr}\left\{\mathcal{I}_{a|x}(\rho)\right\} = p(a|x) \quad \forall\, a, x, \rho, \tag{20}$$

$$\sum_a \mathcal{I}_{a|x}(\cdot) = \Lambda^{B_{in}\to B_{out}}(\cdot) \quad \forall\, x, \tag{21}$$

where ρ represents any normalised state of the quantum system B_{in} and $\Lambda^{B_{in}\to B_{out}}(\cdot)$ is a quantum channel with $(\cdot)$ representing an input defined on B_{in}.

The no-signalling conditions (20) and (21) ensure that Alice's output does not depend on Bob's input state and Alice's classical input cannot influence Bob's output state.

When Alice and Bob share a bipartite quantum system prepared on a state ρ_{AB} and Alice performs POVM measurements $\{M_{a|x}\}_{a,x}$, the elements of a channel assemblage admit quantum realization of the form:

$$\mathcal{I}_{a|x}(\cdot) = \operatorname{tr}_A\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes\Gamma^{BB_{in}\to B_{out}})[\rho_{AB}\otimes(\cdot)_{B_{in}}]\right\}. \tag{22}$$

For each measurement input x, the instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ form a channel which does not depend on Alice's measurement choice, i.e., $\sum_{a\in\mathbb{A}}\mathcal{I}_{a|x}$ is a CPTP map. In general, however, there exists post-quantum non-signalling channel assemblages.

To certify post-quantumness of an arbitrary but fixed post-quantum channel assemblage $\mathbf{I}^*_{\mathbb{A}|\mathbb{X}}$ with elements $\{\mathcal{I}^*_{a|x}\}$ one can use a channel EPR functional similar to the functional introduced for the BwI and MDI scenarios. The channel EPR functional is given by

$$\operatorname{tr}\left\{\sum_{a,x} F_{ax} J(\mathcal{I}^*_{a|x})\right\} < 0, \tag{23}$$

wheretr $\left\{\sum_{a,x,y} F_{ax} J(\mathcal{I}_{a|x})\right\} \geq 0$ when evaluated on any quantum assemblage with elements $\{\mathcal{I}_{a|x}(\cdot)\}_{a,x}$.

Moreover, there exist post-quantum assemblages $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$ that only generate quantum correlations $\{p(ab|xy)\}$ when the input systems are fixed to be $\{\rho_y\}_y$ and Bob measures his output with a measurement $\{N_b\}_b$, i.e., $p(ab|xy) = \operatorname{tr}\left\{N_b(\mathcal{I}_{a|x}(\rho_y))\right\}$ has a quantum realization. We denote this set of assemblages $\mathbf{I}^{QC}$.

Here we show how to embed a bipartite channel assemblage belonging to the set $\mathbf{I}^{QC}$ in a larger network such that post-quantum correlations are generated in this new set-up. The protocol combines the techniques introduced for the BwI and MDI scenarios in the previous sections. In particular, Bob's quantum input is generated in the same way as Bob's input in the protocol for the MDI scenario and Bob's quantum output is measured in the same manner as Bob's output in the protocol for the BwI scenario.

B. The activation protocol

The experiment consists of four parties: Alice, Bob, Charlie and Dani, as illustrated in Fig 4. Alice and Bob share a channel assemblage $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$. Bob's input to his local channel is defined on the Hilbert space $\mathcal{H}_{B_{in}}$.

It is prepared by Charlie through a standard quantum assemblage $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{Z}}$, with $\mathbb{C} \coloneqq \{0,1\}$ and $\mathbb{W} \coloneqq \{1,2,3\}$. Additionally, Bob has a classical input $y \in \mathbb{Y} \coloneqq \{1,2,3,4,\star,\lozenge,\blacklozenge\}$ which controls this stage of the protocol. Moreover, Bob and Dani share a standard quantum assemblage $\mathbf{\Sigma}_{\mathbb{D}|\mathbb{U}}$, with $\mathbb{D} \coloneqq \{0,1\}$ and $\mathbb{U} \coloneqq \{1,2,3\}$, where Bob's system is defined on $\mathcal{H}_{B'}$. Finally, Bob has a measurement device $\{\mathrm{M}_{b|z}^{B_{out}B'}\}_{b,z}$ which measures the system defined on $\mathcal{H}_{B_{out}} \otimes \mathcal{H}_{B'}$. The input and output sets of this device are the following: $b \in \mathbb{B} \coloneqq \{0,1\}$ and $z \in \mathbb{Z} \coloneqq \{1,2,3,4,\star,\lozenge,\blacklozenge\}$. Throughout this section, we assume that the assemblages $\mathbf{I}_{\mathbb{A}|\mathbb{X}}$, $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{Z}}$ and $\mathbf{\Sigma}_{\mathbb{D}|\mathbb{U}}$ are independent, i.e., the systems AB, $B_{in}C$ and $B'D$ are prepared separately and the four parties are space-like separated. Then, the activation protocol is the following.

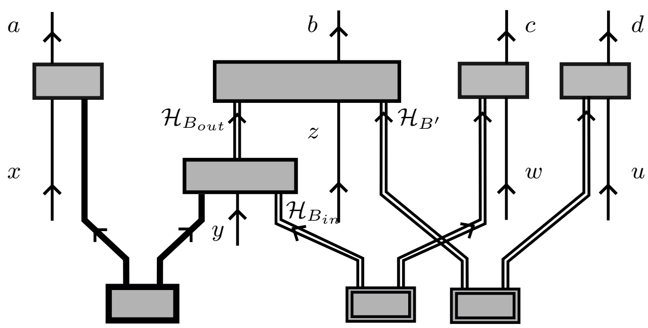

FIG. 4. Depiction of the activation protocol for the channel EPR scenario. Systems that may be classical, quantum, or even post-quantum, are represented by thick lines. Quantum systems are represented by double lines and classical systems are depicted as single lines.

Step 1: *Self-testing of* $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ *and* $\mathbf{\Sigma}_{\mathbb{D}|\mathbb{U}}$.

The first step of the protocol is to certify that the state of Bob's quantum systems $\mathcal{H}_{B_{in}}$ and $\mathcal{H}_{B'}$ are both independently prepared in the assemblages $\mathbf{\Sigma}^{(r_1)}_{\mathbb{C}|\mathbb{W}}$ and $\mathbf{\Sigma}^{(r_2)}_{\mathbb{D}|\mathbb{U}}$. Using Bob's inputs $y \in \{1,2,3,4\}$ and $z \in \{1,2,3,4\}$, the elements of the assemblages need to be self-tested to be $\sigma^{(r_1)}_{c|w} = r_1\widetilde{\sigma}_{c|w} + (1-r_1)(\widetilde{\sigma}_{c|w})^T$ and $\sigma^{(r_2)}_{d|u} = r_2\widetilde{\sigma}_{d|u} + (1-r_2)(\widetilde{\sigma}_{d|u})^T$. Then, these assemblages must be aligned such that their tensor product can be written as $\{r(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u}) + (1-r)(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u})^T\}$. The alignment procedure, which relies on the measurement settings $y \in \{\lozenge,\blacklozenge\}, z \in \{\lozenge,\blacklozenge\}$, is discussed in the Appendix A 3.

Step 2: *Certification of post-quantumness.*

In the second step of the protocol, when $y = z = \star$, Bob applies the process $\Gamma^{BB_{in}\to B_{out}}$ and measures the system on $\mathcal{H}_{B_{out}} \otimes \mathcal{H}_{B'}$. Then, the post-quantumness of the correlations $p(a,b=0,c,d|x,y=\star,z=\star,w,u)$ can be tested with a tailored Bell functional. For now, we only consider the case when Bob's measurement is given by $M^{B_{out}B'}_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$ and the elements of the assemblages $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ and $\mathbf{\Sigma}_{\mathbb{D}|\mathbb{U}}$ are given by $\{\widetilde{\sigma}_{c|w}\}$ and $\{\widetilde{\sigma}_{d|u}\}$, respectively. The activation protocol also works without these assumptions, as we discuss in the Appendix B and C. Under these assumptions, the observed correlations are given by

$$p(a,b=0,c,d|x,y=\star,z=\star,w,u) = \mathrm{tr}\left\{M^{B_{out}B'}_{b=0|z=\star}(\mathcal{I}_{a|x}(\widetilde{\sigma}_{c|w}) \otimes \widetilde{\sigma}_{d|u})\right\}. \tag{24}$$

Due to Choi-Jamiołkowski isomorphism, the instrument $\mathcal{I}_{a|x}(\cdot)$ can be expressed as an operator $J(\mathcal{I}_{a|x}) = (\mathcal{I}_{a|x} \otimes \mathbb{I}^C)(|\phi^+\rangle\langle\phi^+|)_{B_{in}C}$. Using the techniques introduced for the BwI and MDI protocols, we can rewrite Eq. (24) as

$$p(a,b=0,c,d|x,y=\star,z=\star,w,u) = \frac{1}{2}\mathrm{tr}\left\{((\widetilde{\sigma}_{d|u})^T \otimes \widetilde{M}_{c|w})J(\mathcal{I}_{a|x})\right\}. \tag{25}$$

We will now transform the EPR functional given in Eq. (23) into a Bell functional. For $u \in \{1,2,3\}$, let $\pi_{d|u}$ denote the projector onto the eigenspace of the eigenstates of Pauli Z, X and Y operators with eigenvalue $(-1)^d$. The projectors $\{\pi_{c|w} \otimes \pi_{d|u}\}$ form a basis of the set of Hermitian matrices in a four-dimensional Hilbert space. Then, the operators $\{F_{ax}\}_{a,x}$ can be decomposed as $F_{ax} = \sum_{c,d,w,u} \xi^{ax}_{cdwu}(\pi_{c|w} \otimes \pi_{d|u})$ for some real coefficients $\{\xi^{ax}_{cdwu}\}_{a,x,c,d,w,u}$. Construct the following Bell functional on the correlation $\mathbf{p} = \{p(a,b,c,d|x,y,z,w,u)\}$:

$$I^*[\mathbf{p}] \equiv \sum_{a,x,c,d,w,u} \xi^{ax}_{cdwu} p(a, b=0, c, d|x, y=\star, z=\star, w, u). \tag{26}$$

To calculate the value of this functional, take the correlations given in Eq. (25):

$$\begin{aligned} I^*[\mathbf{p}] &= \sum_{a,x,c,d,w,u} \xi^{ax}_{cdwu} \frac{1}{2} \mathrm{tr}\left\{((\widetilde{\sigma}_{d|u})^T \otimes \widetilde{M}_{c|w}) J(\mathcal{I}_{a|x})\right\}, \\ &= \frac{1}{4}\sum_{a,x} \mathrm{tr}\left\{\left(\sum_{c,d,w,u} \xi^{ax}_{cdwu}(\pi_{c|w} \otimes \pi_{d|u})\right) J(\mathcal{I}_{a|x})\right\}, \\ &= \frac{1}{4}\mathrm{tr}\left\{\sum_{a,x} F_{ax} J(\mathcal{I}_{a|x})\right\}. \end{aligned} \tag{27}$$

Similarly to the BwI and MDI scenarios, $I^*[\mathbf{p}] \geq 0$ for any $\mathbf{p}$ arising from a quantum channel assemblage. For the measurement $M^{BB'}_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$ and the fixed assemblage $\mathbf{I}^*_{\mathbb{A}|\mathbb{X}} = \{\mathcal{I}^*_{a|x}(\cdot)\}$, the Bell functional evaluates to $I^*[\mathbf{p^*}] < 0$. The analysis of the case $M^{BB'}_{b=0|z=\star} \neq |\phi^+\rangle\langle\phi^+|$ is the same as in the Bob-with-input scenario. Therefore, if one observes $I^*[\mathbf{p}] < 0$, it certifies that the correlations $\mathbf{p}$ were generated from a post-quantum channel assemblage.

Theorem 6. *There always exists a Bell functional of the form*

$$I^*[\boldsymbol{p}] = \sum_{a,x,c,d,w,u} \xi^{ax}_{cdwu} p(a, b=0, c, d|x, y=\star, z=\star, w, u) \tag{28}$$

for which $I^*[\boldsymbol{p}] \geq 0$ when evaluated on any correlations $\boldsymbol{p} = \{p(a,b,c,d|x,y,z,w,u)\}$ arising from quantum channel assemblage with elements $\{\mathcal{I}_{a|x}(\cdot)\}$ as $p(a,b=0,c,d|x,y=\star,z=\star,w,u) = \mathrm{tr}\left\{M^{B_{out}B'}_{b=0|z=\star}(\mathcal{I}_{a|x}(\widetilde{\sigma}_{c|w}) \otimes \widetilde{\sigma}_{d|u})\right\}$.

In summary, based on the protocols introduced for the BwI and MDI scenarios, we showed that all channel assemblages that belong to the set $\mathbf{I}^{QC}$ can generate post-quantum Bell-type correlations when embedded in a larger network. This result completes our analysis and motivates the following question: is it possible to construct a generalised post-quantum assemblage that cannot generate post-quantum correlations?

V. DISCUSSION

In this paper we have shown how to embed a bipartite assemblage in a multipartite set-up such that post-quantum correlations can be generated in this larger network. For post-quantum assemblages that can only generate quantum correlations in a bipartite device-independent setting, this protocol activates their post-quantumness, i.e., it allows one to witness the post-quantum nature of the assemblage in a Bell-type scenario. We introduced and analyzed protocols for the Bob-with-input, measurement-device-independent and channel EPR scenarios.

The existence of post-quantum bipartite generalized EPR assemblages, which generate only quantum correlations when Bob's processing is somehow transformed to be device-independent, may suggest that there is a fundamental difference between post-quantum assemblages and post-quantum Bell-type correlations. However, our results show a relation between these two resources. For such sets of bipartite assemblages introduced in this work, we showed that they can generate post-quantum correlations when embedded in a larger network. As of now, there are no examples of post-quantum assemblages in EPR scenarios that cannot generate post-quantum correlations in a device-independent setting.

The construction of the protocols for all scenarios relies on the same idea. In each protocol, an additional quantum resource is self-tested and later used to transform EPR functionals to Bell functionals. This idea was first introduced as a method of device-independent entanglement certification [33, 34]. However, it appears to have broad and powerful applications beyond this problem. Exploring other research areas where this concept could be applied seems interesting.

Looking forward, it would be interesting to see how the activation protocol changes for multipartite generalised EPR scenarios. The activation protocol for assemblages generated in experiments with multiple Alice's or Bob's should be similar to the one we introduced in this paper; however, a detailed study of this problem could reveal some differences.

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Appendix A: Self-testing stage of the protocol

1. Bob-with-input EPR scenario: self-testing of $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$

The first step of the activation protocol for the Bob-with-input EPR scenario relies on the self-testing of the assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$. The Bell inequality used to self-tests the assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$ was analysed in Refs. [34, 37]. In this appendix, we follow the result of Ref. [37].

Let us focus on the assemblage shared between Bob and Charlie in Fig. 2. From the statistics $\mathbf{p} = \{p(abc|xyzw)\}$ obtained in the activation protocol, compute the marginal $\mathbf{p}^{B'C} = \{p(bc|zw)\}$. Then, the following result holds.

Lemma 7. *Given a marginal correlation $\boldsymbol{p} = \{p(bc|zw)\}$ between Bob and Charlie, compute the following Bell functional*

$$\begin{aligned}\mathcal{I}_E = &\langle C_1B_1\rangle + \langle C_1B_2\rangle - \langle C_1B_3\rangle - \langle C_1B_4\rangle \\ &+\langle C_2B_1\rangle - \langle C_2B_2\rangle + \langle C_2B_3\rangle - \langle C_2B_4\rangle \\ &+\langle C_3B_1\rangle - \langle C_3B_2\rangle - \langle C_3B_3\rangle + \langle C_3B_4\rangle.\end{aligned} \tag{A1}$$

If $\mathcal{I}_E = 4\sqrt{3}$, then the assemblage shared between Bob and Charlie is given by $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$ with elements $\sigma^{(r)}_{c|w} = r\widetilde{\sigma}_{c|w} + (1-r)(\widetilde{\sigma}_{c|w})^T$. Here, $\widetilde{\sigma}_{c|1} = (\mathbb{I} + (-1)^c Z)/4$, $\widetilde{\sigma}_{c|2} = (\mathbb{I} + (-1)^c X)/4$ and $\widetilde{\sigma}_{c|3} = (\mathbb{I} - (-1)^c Y)/4$.

Bob's observables that saturate the functional $\mathcal{I}_E$ are given by $\{\widetilde{B_1}, \widetilde{B_2}, \widetilde{B_3}, \widetilde{B_4}\} = \{(Z+X-Y)/\sqrt{3}, (-Z+X+Y)/\sqrt{3}, (-Z-X-Y)/\sqrt{3}\}$. Therefore, in the self-testing stage of the protocol, Bob's measurement $M^{BB'}_{b|z}$ on his two qubits corresponds to the observables $\mathbb{I}^B \otimes \widetilde{B_z}^{B'}$ for $z \in \{1, 2, 3, 4\}$.

Like all self-testing statements, the self-tested assemblage has the desired form up to local isometries. For discussion about this result, including the possible isometries in the EPR scenario, we refer the reader to Ref. [37].

2. Measurement-device-independent EPR scenario: self-testing of $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{Z}}$

The same self-testing result as in the Bob-with-input scenario is used in the formulation of the protocol for the measurement-device-independent scenario. In the activation of post-quantumness of MDI assemblages, the assemblage to be self-tested in the first step of the protocol is the standard EPR assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{Z}}$ shared between Bob and Charlie in Fig. 3. Then, the self-testing result of Lemma 7 must be used for the marginal $\mathbf{p}^{B_{in}C} = \{p(bc|yz)\}$ computed from $\mathbf{p} = \{p(abc|xyz)\}$.

3. Channel EPR scenario: self-testing of $\mathbf{\Sigma}^{(r_1)}_{\mathbb{C}|\mathbb{W}}$ and $\mathbf{\Sigma}^{(r_2)}_{\mathbb{D}|\mathbb{U}}$

In the protocol for the channel EPR scenario, two assemblages are self-tested: $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}}$ shared between Bob and Charlie and $\mathbf{\Sigma}_{\mathbb{D}|\mathbb{U}}$ shared between Bob and Dani in Fig. 4. The goal of the first step of the protocol is to show that the elements of the self-tested assemblages correspond to the basis of the operators $\{F_{ax}\}$ that define the Bell functional used for post-quantumness certification. In the channel EPR scenario, the operators $\{F_{ax}\}$ can be decomposed in the basis $\{\pi_{c|w} \otimes \pi_{d|u}\}$ as $F_{ax} = \sum_{c,d,w,u} \xi^{ax}_{cdwu}(\pi_{c|w} \otimes \pi_{d|u})$ for some real numbers $\{\xi^{ax}_{cdwu}\}$.

The self-testing stage relies on the marginals $\mathbf{p}^{B_{in}C} = \{p(b_{out}c|yw)\}$ and $\mathbf{p}^{B'D} = \{p(bd|zu)\}$, both calculated from the overall statistics $\mathbf{p} = \{p(abcd|xyzwu)\}$. Here, b_{out} represents a classical outcome obtained in the first stage of the protocol, where the Hilbert space $\mathcal{H}_{B_{out}}$ encodes merely a classical outcome. If one simply uses the result of Lemma 7, the relevant assemblages will be certified to have elements $\sigma^{(r_1)}_{c|w} = r_1\widetilde{\sigma}_{c|w} + (1-r_1)(\widetilde{\sigma}_{c|w})^T$ and $\sigma^{(r_2)}_{d|u} = r_2\widetilde{\sigma}_{d|u} + (1-r_2)(\widetilde{\sigma}_{d|u})^T$, with unknown parameters r_1 and r_2. To avoid a false-positive detection in the activation protocol (which we discuss further in the Appendix B 3), this self-test need to be accompanied by a complementary procedure to certify that the two assemblages are aligned, as we outline below.

Let $\mathbf{\Sigma}^{(r)}_{\mathbb{CD}|\mathbb{WU}}$ denote the tensor product of the self-tested assemblages. The assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{CD}|\mathbb{WU}}$ has elements $\{\sigma^{(r_1)}_{c|w} \otimes \sigma^{(r_2)}_{d|u}\}$. The goal of the alignment procedure is to ensure that these elements have the form $\{r(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u}) + (1-r)(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u})^T\}$ for some parameter r. To align the assemblages, we need to introduce new measurements on the system $\mathcal{H}_{B_{out}} \otimes \mathcal{H}_{B'}$ labeled by settings $\{\lozenge, \blacklozenge\}$. The outcome set of the measurement $\lozenge$ is denoted by $\mathbb{B}^{\lozenge}$ and has 4 outcomes, while the outcome set of the measurement $\blacklozenge$ is denoted by $\mathbb{B}^{\blacklozenge}$ and has 2 outcomes. Upon these measurements, the system $\mathcal{H}_{B_{in}}$ is simply passed to the Bob's measuring device without going through the channel assemblage and measured with the system $\mathcal{H}_{B'}$. In short, this additional alignment procedure self-tests the measurements corresponding to $\lozenge$ and $\blacklozenge$ to ensure that the relevant assemblage has the desired form. This result relies on Ref. [34, Lemma 3] and is also discussed in detail in Ref. [32, Appendix B].

Appendix B: No false-positives

1. Bob-with-input EPR scenario

Let us start with analyzing the Bob-with-input protocol. In the second step of the protocol, we assumed that the assemblage shared between Bob and Charlie is given by $\mathbf{\Sigma}^{(r=1)}_{\mathbb{C}|\mathbb{W}} = \widetilde{\mathbf{\Sigma}}_{\mathbb{C}|\mathbb{W}}$. However, the success of first step of the protocol can only guarantee that that the assemblage $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$ that they share has elements of the form $\sigma^{(r)}_{c|w} = r\widetilde{\sigma}_{c|w} + (1-r)(\widetilde{\sigma}_{c|w})^T$, with an unknown parameter r. Here we show that if the Bob-with-input assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{XY}}$ belongs to the quantum set, it is impossible to observe a value of $I^*_{BwI}[\mathbf{p}]$ which violates the quantum bound regardless of the parameter r that defines $\mathbf{\Sigma}^{(r)}_{\mathbb{C}|\mathbb{W}}$. In other words, if the self-testing stage of the protocol is successful and a violation of the quantum bound is observed in the second step, it must be the case that the BwI assemblage shared between Alice and Bob is post-quantum.

Consider the case when $r = 0$ and $\mathbf{\Sigma}^{(r=0)}_{\mathbb{C}|\mathbb{W}} = \{(\widetilde{\sigma}_{c|w})^T\}$. Then, the observed correlations are given by

$$\begin{aligned} p(a, b = 0, c|x, y, z = \star, w)_T &= \mathrm{tr}_{\mathcal{H}_B,\mathcal{H}_{B'}} \left\{ M^{BB'}_{b=0|z=\star}(\sigma_{a|xy} \otimes (\widetilde{\sigma}_{c|w})^T) \right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{((\widetilde{\sigma}_{c|w})^T)^T \sigma_{a|xy}\right\}, \end{aligned} \tag{B1}$$

and the Bell functional can be written as

$$\begin{aligned} I^*_{BwI}[\mathbf{p}_T] &= \sum_{a,x,y,c,w} \xi^{axy}_{cw} \frac{1}{2}\mathrm{tr}\left\{((\widetilde{\sigma}_{c|w})^T)^T \sigma_{a|xy}\right\}, \\ &= \frac{1}{2}\sum_{a,x,y} \mathrm{tr}\left\{ \left(\sum_{c,w} \xi^{axy}_{cw} (\widetilde{\sigma}_{c|w})^T\right) (\sigma_{a|xy})^T \right\}, \\ &= \frac{1}{4}\mathrm{tr}\left\{ \sum_{a,x,y} F_{axy} (\sigma_{a|xy})^T \right\}. \end{aligned} \tag{B2}$$

If the assemblage $\{\sigma_{a|xy}\}$ is quantum, the assemblage $\{(\sigma_{a|xy})^T\}$ is quantum as well (due to the fact that the transpose operation on a qubit is a decomposable positive and trace-preserving map) and $I^*_{BwI}[\mathbf{p}_T] \geq 0$. It follows that if $\{\sigma_{a|xy}\}$ has a quantum realization, $I^*_{BwI}[\mathbf{p}_T] \geq 0$ and $I^*_{BwI}[\mathbf{p}] \geq 0$. Hence, any correlation generated by measurements on the convex combination $\{r\widetilde{\sigma}_{c|w} + (1 - r)(\widetilde{\sigma}_{c|w})^T\}$ must satisfy the quantum bound as well.

2. Measurement-device-independent EPR scenario

The same reasoning as in the protocol for the Bob-with-input scenario can be applied to the measurement-device-independent scenario. If Bob and Charlie share an assemblage defined by $r = 0$, i.e., $\mathbf{\Sigma}^{(r=0)}_{\mathbb{C}|\mathbb{Z}} = \{(\widetilde{\sigma}_{c|z})^T\}$, the observed correlations are given by

$$\begin{aligned} p(a, b, c|x, y = \star, z)_T &= \mathcal{N}_{ab|x}((\widetilde{\sigma}_{c|z})^T), \\ &= \mathrm{tr}\left\{ (M^C_{c|z})^T \, J(\mathcal{N}_{ab|x}) \right\}, \\ &= \mathrm{tr}\left\{ M^C_{c|z} \, J(\mathcal{N}_{ab|x})^T \right\}. \end{aligned} \tag{B3}$$

Here, we used the fact that $\widetilde{\sigma}^T_{c|z} = \mathrm{tr}_C \left\{ (\mathbb{I}^{B_{in}} \otimes (M^C_{c|z})^T) \left|\phi^+\right\rangle \left\langle\phi^+\right|_{B_{in}C} \right\}$. Then, the Bell functional reads

$$\begin{aligned} I^*_{MDI}[\mathbf{p}_T] &= \sum_{a,b,x,c,z} \xi^{abx}_{cz} \mathrm{tr}\left\{ M^C_{c|z} \, J(\mathcal{N}_{ab|x})^T \right\}, \\ &= \sum_{a,b,x} \mathrm{tr}\left\{ \left(\sum_{c,z} \xi^{abx}_{cz} \pi_{c|z}\right) J(\mathcal{N}_{ab|x})^T \right\}, \\ &= \mathrm{tr}\left\{ \sum_{a,b,x} F_{abx} \, J(\mathcal{N}_{ab|x})^T \right\}. \end{aligned} \tag{B4}$$

Using the same arguments as for the Bob-with-input case, one can see that if $J(\mathcal{N}_{ab|x})$ corresponds to a quantum assemblage, then $J(\mathcal{N}_{ab|x})^T$ has a quantum realization and any correlation generated by measurements on the convex combination $\{r\widetilde{\sigma}_{c|w} + (1 - r)(\widetilde{\sigma}_{c|w})^T\}$ must satisfy the quantum bound.

3. Channel EPR scenario

In the set-up for the channel EPR scenario, the first step of the protocol certifies that the relevant assemblages have elements $\{r(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u}) + (1-r)(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u})^T\}$ for some parameter r. In the main text, we showed that when we restrict the tensor product of these assemblages to have the form $\{\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u}\}$, the quantum bound of the Bell functional given in Eq. (28) is equal to zero. In this appendix, we show that a false positive detection of post-quantumness is not possible even if their tensor product is instead $\{(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u})^T\}$ (the result of no false-positive detection with $\{r(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u}) + (1-r)(\widetilde{\sigma}_{c|w} \otimes \widetilde{\sigma}_{d|u})^T\}$ follows from a convexity argument, similar to the BwI and MDI protocols).

Let $r=0$, hence $\mathbf{\Sigma}^{(r=0)}_{\mathbb{C}|\mathbb{W}} = \{(\widetilde{\sigma}_{c|w})^T\}$ and $\mathbf{\Sigma}^{(r=0)}_{\mathbb{D}|\mathbb{U}} = \{(\widetilde{\sigma}_{d|u})^T\}$. Then, the correlations generated in the protocol set-up can be written as

$$
\begin{aligned}
p(a, b=0, c, d|x, y=\star, z=\star, w, u)_T &= \mathrm{tr}_{\mathcal{H}_{B_{out}}, \mathcal{H}_{B'}} \left\{ M^{B_{out}B'}_{b=0|z=\star} (\mathcal{I}_{a|x}((\widetilde{\sigma}_{c|w})^T) \otimes (\widetilde{\sigma}_{d|u})^T) \right\}, \\
&= \frac{1}{2} \mathrm{tr}_{\mathcal{H}_{B_{out}}, \mathcal{H}_{B'}} \left\{ (((\widetilde{\sigma}_{d|u})^T)^T \otimes (\widetilde{M}_{c|w})^T) J(\mathcal{I}_{a|x}) \right\}, \\
&= \frac{1}{2} \mathrm{tr}_{\mathcal{H}_{B_{out}}, \mathcal{H}_{B'}} \left\{ ((\widetilde{\sigma}_{d|u})^T \otimes \widetilde{M}_{c|w})(J(\mathcal{I}_{a|x}))^T \right\},
\end{aligned}
\tag{B5}
$$

where we used the same transformations as in the Appendix B 1 for the BwI protocol and in the Appendix B 2 for the MDI protocol. The only thing left to show is that if $J(\mathcal{I}_{a|x})$ is a quantum assemblage, $J(\mathcal{I}_{a|x})^T$ has a quantum realization as well. Recall that the Choi matrix of a quantum channel assemblage has the following form

$$
J(\mathcal{I}_{a|x}) = \mathrm{tr}_A \left\{ (M_{a|x} \otimes \Gamma^{BB_{in} \to B_{out}} \otimes \mathbb{I}^C)[\rho_{AB} \otimes (|\phi^+\rangle \langle \phi^+|)_{B_{in}C}] \right\}. \tag{B6}
$$

Therefore, we can write its transpose as

$$
\begin{aligned}
J(\mathcal{I}_{a|x})^T &= \mathrm{tr}_A \left\{ (M_{a|x} \otimes \Gamma^{BB_{in} \to B_{out}} \otimes \mathbb{I}^C)[\rho_{AB} \otimes (|\phi^+\rangle \langle \phi^+|)_{B_{in}C}] \right\}^T, \\
&= \mathrm{tr}_A \left\{ (\mathbb{I}^A \otimes T^{B_{out}C})(M_{a|x} \otimes \Gamma^{BB_{in} \to B_{out}} \otimes \mathbb{I}^C)[\rho_{AB} \otimes (|\phi^+\rangle \langle \phi^+|)_{B_{in}C}] \right\}.
\end{aligned}
\tag{B7}
$$

To show that $J(\mathcal{I}_{a|x})^T$ admits a quantum realization, we will use the following lemma.

Lemma 8. *Let $\Phi(\cdot)$ and $\Psi(\cdot)$ be CPTP maps and ρ be a quantum state. The application of a CPTP map on a quantum state followed by a transpose, $(\Phi(\rho))^T$, is equivalent to first applying the transpose on the state, and then acting on it with a different CPTP map, $\Psi(\rho^T)$.*

Proof. Using Kraus decomposition of the map $\Phi(\cdot)$, we can write $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$, where $\sum_k A_k^\dagger A_k = \mathbb{I}$. The transpose of this new state can be written as

$$
\begin{aligned}
(\Phi(\rho))^T &= \left(\sum_k A_k \rho A_k^\dagger \right)^T, \\
&= \sum_k (A_k^\dagger)^T \rho^T (A_k)^T, \\
&= \sum_k B_k \rho^T B_k^\dagger,
\end{aligned}
\tag{B8}
$$

where we defined a new operator $B_k = (A_k^\dagger)^T$. This new operator defines a CPTP map $\Psi(\cdot) = \sum_k B_k(\cdot)B_k^\dagger$

as

$$\sum_k B_k^\dagger B_k = \sum_k (A_k)^T (A_k^\dagger)^T, \tag{B9}$$
$$= \sum_k (A_k^\dagger A_k)^T,$$
$$= \left(\sum_k A_k^\dagger A_k\right)^T,$$
$$= \mathbb{I}.$$

Therefore, we showed that $(\Phi(\rho))^T = \Psi(\rho^T)$. □

Due to Lemma 8, we can move the transpose from the output system of the CPTP map $\Gamma^{BB_{in}\to B_{out}}$ to its input systems. Then, we can write

$$J(\mathcal{I}_{a|x})^T = \mathrm{tr}_A\left\{(M_{a|x}\otimes\Gamma'^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)(\mathbb{I}^A\otimes T^{BB_{in}C})[\rho_{AB}\otimes(|\phi^+\rangle\langle\phi^+|)_{B_{in}C}]\right\}, \tag{B10}$$
$$= \mathrm{tr}_A\left\{(M_{a|x}\otimes\Gamma'^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}^{T_B}\otimes(|\phi^+\rangle\langle\phi^+|)^T_{B_{in}C}]\right\},$$
$$= \mathrm{tr}_A\left\{(M_{a|x}\otimes\Gamma'^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)^T_A[\rho_{AB}^{T}\otimes(|\phi^+\rangle\langle\phi^+|)^T_{B_{in}C}]\right\},$$
$$= \mathrm{tr}_A\left\{((M_{a|x})^T\otimes\Gamma'^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}^{T}\otimes(|\phi^+\rangle\langle\phi^+|)^T_{B_{in}C}]\right\}.$$

Therefore, if $J(\mathcal{I}_{a|x})$ is a valid quantum assemblage, then its transpose $J(\mathcal{I}_{a|x})^T$ admits of a quantum realization in terms of the measurements $\{(M_{a|x})^T\}$, a CPTP map $\Gamma'^{BB_{in}\to B_{out}}$ and the states ρ_{AB}^T and $(|\phi^+\rangle\langle\phi^+|)^T_{B_{in}C}$. This completes our argument.

Finally, we can revisit the additional alignment step of the self-testing stage and discuss why is it necessary. Imagine the assemblages are only self-tested to have the form $\sigma^{(r_1)}_{c|w} = r_1\widetilde{\sigma}_{c|w} + (1-r_1)(\widetilde{\sigma}_{c|w})^T$ and $\sigma^{(r_2)}_{d|u} = r_2\widetilde{\sigma}_{d|u} + (1-r_2)(\widetilde{\sigma}_{d|u})^T$, with unknown parameters r_1 and r_2. The case of $r_1 = r_2 = 1$ was discussed in the main text. The contrary situation, i.e., when $r_1 = r_2 = 0$, is discussed in this appendix. However, if the self-tested assemblages would be defined by $r_1 = 1$ and $r_2 = 0$, this could create a false-positive detection in the activation protocol. Therefore, the alignment stage of the step 1 of the protocol is necessary to eliminate the possibility of a false-positive detection.

Appendix C: Optimizing the Bell functional over all measurements

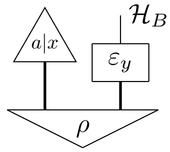
In the activation protocols for the Bob-with-input and channel EPR scenarios, we first assumed that $M_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$. Then, we have shown that under this assumption the quantum bound of the Bell functionals constructed for these scenarios is equal to zero. In this appendix, we show that the quantum bound of these functionals, when optimised over all possible measurements $M_{b=0|z=\star}$, is still equal to zero. Therefore, observing a negative value of the Bell functional in an activation set-up necessarily means that the underlying generalised EPR assemblage is post-quantum. Here we focus on the Bob-with-input scenario; the proof for the channel EPR scenario has the same structure.

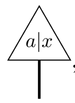
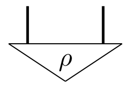
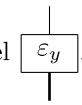
In Section II B, we have shown that

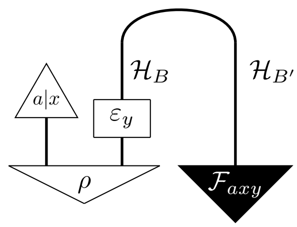
$$I^*_{BwI}[\mathbf{p}] = \frac{1}{2}\sum_{a,x,y,c,w}\xi^{axy}_{cw}\mathrm{tr}_{\mathcal{H}_B,\mathcal{H}_{B'}}\left\{M_{0|\star}(\sigma_{a|xy}\otimes\widetilde{\sigma}_{c|w})\right\}\geq 0, \tag{C1}$$

for $M_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$. The proof that this inequality holds for arbitrary $M_{b=0|z=\star}$ relies on the observations made in Ref. [32, Appendix A]. Here we present it using diagrammatic notation introduced in Ref. [39].

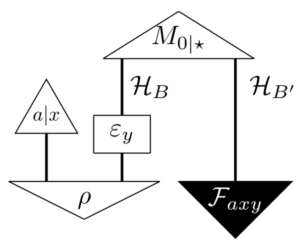
Let us represent an arbitrary quantum Bob-with-input assemblage on a system $\mathcal{H}_{B'}$ with the diagram



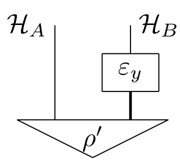
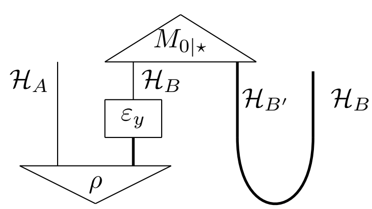
for a quantum system $\mathcal{H}_B$, an effect , a state  and a channel . Now, we can represent Eq. (C1) as

$I^*_{BwI}[\mathbf{p}] = \frac{1}{4}\sum_{a,x,y}$  $\geq 0\,.$ (C2)

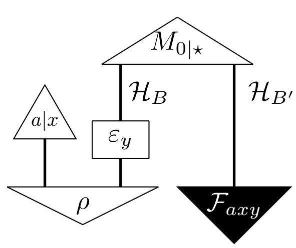
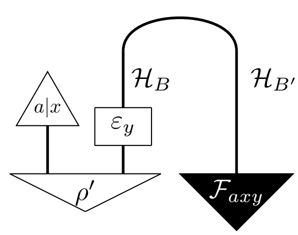
Here, the representation of the operators F_{axy} as black triangles implies that they are possibly nonphysical (post-quantum) processes. The statement of Theorem 2 can be expressed in the diagrammatic language as:

$I^*_{BwI}[\mathbf{p}] = \frac{1}{4}\sum_{a,x,y}$  $\geq 0\,,$ (C3)

where $M_{0|\star}$ now represents a general measurement. To prove this statement, first define a new quantum state ρ' as

 $=$  (C4)

This is a valid quantum state, although it can be unnormalised. Then, we can use this construction to rewrite the Bell functional as

$I^*_{BwI}[\mathbf{p}] = \frac{1}{4}\sum_{a,x,y}$  $= \frac{1}{4}\sum_{a,x,y}$  $\geq 0\,,$ (C5)

The inequality follows from Eq (C2), which is valid for all quantum states.

Appendix D: Activation in the Bob-with-input scenario - example

In this appendix, we use the construction from Theorem 2 to derive a Bell functional that certifies post-quantumness of the assemblage $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ introduced in Ref. [25, Eq. (6)], which belongs to the set $\mathbf{\Sigma}^{QC}$.

We focus on the assemblage $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ in a Bob-with-input EPR scenario where $\mathbb{X} = \{1, 2, 3\}$, $\mathbb{A} = \{0, 1\}$ and $\mathbb{Y} = \{0, 1\}$. The elements of this assemblage, $\{\sigma^{PTP}_{a|xy}\}$, can be mathematically expressed as follows:

$$\sigma^{PTP}_{a|xy} = \frac{1}{4}(\mathbb{I} + (-1)^{a+\delta_{x,2}+\delta_{y,1}}\sigma_x)\,, \tag{D1}$$

where the operators σ_1, σ_2, and σ_3 (since the choice of Alice's measurement $x \in \{1, 2, 3\}$) denote Pauli X, Y, and Z operators respectively. In Ref. [25, Appendix D], it was shown that $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ is a post-quantum assemblage and that if Bob measures his subsystem, the correlations that arise between him and Alice always admit a quantum realization.

To show that the post-quantumness of $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ can be activated, we will use the EPR functional constructed in Ref. [25, Eq. (D3)]. It is specified by the operators $F^{PTP}_{axy} = \frac{1}{2}(\mathbb{I} - (-1)^a\sigma_x)^{T^y}$, where T^y denotes that the transpose operation is applied when $y = 1$. The classical bound of this functional is $\beta^C_{PTP} = 1.2679$ and the no-signalling bound is $\beta^{NS}_{PTP} = 0$. The assemblage $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ saturates the minimum bound of this EPR functional. Although the quantum bound β^Q_{PTP} of this functional is not known exactly, it can be lower bounded by the almost-quantum bound, i.e., a bound calculated for all assemblages lying in the outer approximation of the quantum set called the almost-quantum set, which is given by $\beta^{AQ}_{PTP} = 0.4135$. On the other hand, β^Q_{PTP} is upper bounded by the classical bound. Hence, we know that $1.2679 \geq \beta^Q_{PTP} \geq 0.4135$. In order to use the activation protocol introduced in Section II B, we should normalise the operators $\{F^{PTP}_{axy}\}$ and construct an EPR functional with a quantum bound equal to zero[4]. However, since the value of β^Q_{PTP} is unknown, this is impossible. Instead, we normalise the operators $\{F^{PTP}_{axy}\}$ using the lower bound on β^Q_{PTP}, namely β^{AQ}_{PTP}. Then, for any quantum or almost-quantum assemblage $\{\sigma^{AQ}_{a|xy}\}$, the following holds

$$\mathrm{tr}\left\{\sum_{a,x,y} \widetilde{F}^{PTP}_{axy}\sigma^{AQ}_{a|xy}\right\} \geq 0, \tag{D2}$$

where the operators $\{\widetilde{F}^{PTP}\}$ are given by

$$\widetilde{F}^{PTP}_{axy} = F^{PTP}_{axy} - \frac{\beta^{AQ}_{PTP}}{|\mathbb{X}|\,|\mathbb{Y}|}\,\mathbb{I}\,. \tag{D3}$$

To certify post-qauntumness of $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$ solely from correlations $\{p(a, b, c|x, y, z, w)\}$, we need to transform the EPR functional (D2) to a Bell functional of the form (8). The coefficients $(\xi^{PTP})^{axy}_{cw}$ can be read off Eq. (D3) and are given by

$$(\xi^{PTP})^{axy}_{cw} = \delta_{c,a\oplus 1\oplus y\delta_{x,2}}\ \delta_{w,x\oplus_3 1} - \delta_{w,1}\frac{\beta^{AQ}_{PTP}}{|\mathbb{X}|\,|\mathbb{Y}|}, \tag{D4}$$

where $\oplus$ and $\oplus_3$ denote sum modulo 2 and modulo 3, respectively. Using the reasoning of Eq. (7), we know that the Bell functional specified by coefficients given in Eq. (D4), when evaluated on a post-quantum assemblage (which is not almost-quantum), must satisfy $I^{PTP}_{BwI}[\mathbf{p}] < 0$. It is easy to check that if $\mathbf{\Sigma}_{\mathbb{C}|\mathbb{W}} = \widetilde{\mathbf{\Sigma}}_{\mathbb{C}|\mathbb{W}}$, $M^{BB'}_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$ and $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{XY}} = \mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$, the functional I^{PTP}_{BwI} evaluates to $I^{PTP}_{BwI}[\mathbf{p}^{PTP}] = -\beta^{AQ}$, which shows the activation of post-quantumness of $\mathbf{\Sigma}^{PTP}_{\mathbb{A}|\mathbb{XY}}$.

---

[4] In Section II B, we assume that the quantum bound of the functional is equal to zero. This assumption is necessary for the proof that the protocol also works for measurements different than $M^{BB'}_{b=0|z=\star} = |\phi^+\rangle\langle\phi^+|$, which is given in Appendix C.

Appendix E: MDI assemblage from the set $\mathbf{N}^{QC}$

Here we show that the post-quantum measurement-device-independent assemblage $\mathbf{N}^{PTP}_{\mathbb{AB}|\mathbb{X}}$ introduced in Ref. [27, Eq. (28)] belongs to the set $\mathbf{N}^{QC}$. $\mathbf{N}^{PTP}_{\mathbb{AB}|\mathbb{X}}$ is illustrated in Fig. 5 and it can be mathematically expressed as

$$\mathbf{N}^{PTP}_{\mathbb{AB}|\mathbb{X}} = \left\{\mathcal{N}^{PTP}_{ab|x}\right\}_{a\in\mathbb{A},\, b\in\mathbb{B},\, x\in\mathbb{X}} , \tag{E1}$$

$$\text{with} \quad \begin{cases} \mathcal{N}^{PTP}_{ab|x} = \operatorname{tr}\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes N_b)(\mathbb{I}_A\otimes \mathrm{CT}^{BB_{in}\to B})[\rho_{AB}\otimes\rho_y]\right\} , \\ M_{a|1} = \frac{\mathbb{I}+(-1)^a\sigma_x}{2}, \quad M_{a|2} = \frac{\mathbb{I}+(-1)^a\sigma_y}{2}, \quad M_{a|3} = \frac{\mathbb{I}+(-1)^a\sigma_z}{2}, \end{cases}$$

where $\rho_{AB} = |\phi^+\rangle\langle\phi^+|$ and σ_x, σ_y, and σ_z are the Pauli operators. The processing $\mathrm{CT}^{BB_{in}\to B}$ can be viewed as a controlled-transpose operation on BB_{in}, where B_{in} acts as the control qubit and B is the system that is being transposed. Then, the system B_{in} is traced-out and the system B is measured by Bob to generate a classical outcome b. The measurement elements are $N_0 = \frac{1}{3}\mathbb{I} + \frac{1}{3}\sigma_y$ and $N_1 = \frac{2}{3}\mathbb{I} - \frac{1}{3}\sigma_y$. In Ref. [27], it was shown that $\mathbf{N}^{PTP}$ is a post-quantum assemblage.

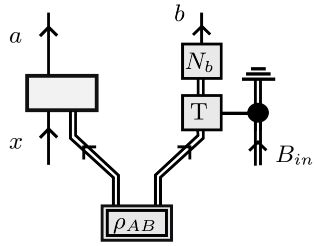


FIG. 5. Mathematical depiction of the MDI assemblage $\mathbf{N}^{PTP}$: Alice and Bob share a quantum state ρ_{AB}; Alice performs measurements on her system, while Bob performs a controlled-transpose operation $\mathrm{CT}^{BB_{in}\to B}$ followed by a measurement N_b.

Let us examine a situation when Bob's input state defined on $\mathcal{H}_{B_{in}}$ is fixed and given by ρ_y. The correlations in such set-up are given by $p(ab|xy) = \mathcal{N}^{PTP}_{ab|x}(\rho_y)$ and can be written as

$$p(ab|xy) = \operatorname{tr}\left\{(M_{a|x}\otimes \mathbb{I}_{B_{out}})(\mathbb{I}_A\otimes N_b)(\mathbb{I}_A\otimes \mathrm{CT}^{BB_{in}\to B})[\rho_{AB}\otimes\rho_y]\right\} . \tag{E2}$$

Here, ρ_y acts only as a control qubit. Hence, we can represent the controlled-transpose operation as labeled by the classical label y: $(\mathbb{I}_A\otimes \mathrm{CT}^{BB_{in}\to B})[\rho_{AB}\otimes\rho_y] \equiv (\mathbb{I}_A\otimes \mathrm{CT}^B_y)\rho_{AB}$. Transpose is a positive and trace-preserving map, hence it has a dual map which is positive and unital. For the transpose map, its dual is also a transpose map. Therefore, instead of applying the transpose map on the system B, one can imagine the controlled transpose is applied on the measurement N_b instead; hence, we can rewrite Eq. (E2) as

$$p(ab|xy) = \operatorname{tr}\left\{(M_{a|x}\otimes N_{b|y})\rho_{AB}\right\} . \tag{E3}$$

The new general measurement $N_{b|y}$ depends on the control qubit and can be given by the original measurement N_b, its transpose, or superposition thereof. It follows that the correlations defined as in Eq. (E3) admit of a quantum realization.

Appendix F: Activation protocol beyond qubit assemblages

In this appendix, we show how to adapt the activation protocols such that post-quantumness of higher-dimensional assemblages can be certified. First, we use methods introduced in Ref. [32, Appendix B] to generalise the self-testing stage of the protocol. Then, we adapt the Bell inequalities used in the protocols to the higher-dimensional case for each scenario.

In all three scenarios, we consider assemblages with the dimension of Bob's output system being $d > 2$. We embed these systems in a higher-dimensional Hilbert space, where each qudit lives on a Hilbert space arising from a parallel composition of n qubits. The basis for this space is given by tensor products of Pauli operators. Following the intuition of the qubit-assemblages protocols, the first step of the protocol need to self-test the basis elements of the relevant Hilbert space. This step of the protocol is the same for the Bob-with-input, measurement-device-independent and channel EPR scenarios.

The fact that one can only self-test a mixture of an assemblage and its transpose will again pose a problem here. This is analogous to the problem that occurs in the protocol for the channel EPR scenario, where if the self-tested assemblages would form a tensor product of the form $\widetilde{\sigma}_{c|w} \otimes (\widetilde{\sigma}_{d|u})^T$, a false-positive detection of post-quantumness could be detected (we discuss this problem in the Appendix B 3). Therefore, instead of just self-testing tensor product of Pauli operators, the self-testing stage of the protocol must guarantee that the assemblage is of the form

$$\sigma^{(r)}_{\mathbf{c}|\mathbf{w}} = r \bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i} + (1-r) \bigotimes_{i=1}^{n} (\widetilde{\sigma}_{c_i|w_i})^T. \tag{F1}$$

Here, $\mathbf{c} = (c_1, ..., c_n)$ with $c_i \in \{0, 1\}$ and $\mathbf{w} = (w_1, ..., w_n)$ with $w_i \in \{1, 2, 3\}$. The technique to self-test $\{\sigma^{(r)}_{\mathbf{c}|\mathbf{w}}\}$ of the form (F1) was first introduced in Ref. [34] and later used in the activation protocol introduced in Ref. [32]. We will not recall the exact self-testing statement here; all the technical details can be found in Refs. [32, 34].

If the first step of the activation protocol is successful, the probabilities generated in the protocol set-up can be used for post-quantumness certification using tailored Bell functionals. Below, we adapt the Bell functional for each scenario to the higher-dimensional case.

1. Bob-with-input EPR scenario

Consider a Bob-with-input assemblage with the dimension of the system $\mathcal{H}_B$ being greater than 2. Analogous to the case of a qubit assemblage, define the Bell functional as

$$I^*_{BwI}[\mathbf{p}] \equiv \sum_{a,x,y,c,w} \xi^{axy}_{cw} p(a, b = 0, c|x, y, z = \star, w). \tag{F2}$$

To compute the quantum bound of the functional, recall that

$$p(a, b = 0, c|x, y, z = \star, w) = \operatorname{tr}\left\{ M^{BB'}_{b=0|z=\star} (\sigma_{a|xy} \otimes \sigma^{(r)}_{\mathbf{c}|\mathbf{w}}) \right\}, \tag{F3}$$

where $\sigma^{(r)}_{\mathbf{c}|\mathbf{w}}$ is given by Eq. (F1). For now, assume that $\sigma^{(r=1)}_{\mathbf{c}|\mathbf{w}} = \bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i}$ and $M^{BB'}_{b=0|z=\star} = |\phi^n\rangle\langle\phi^n|$, with $|\phi^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |kk\rangle_{BB'}$. Then, the Bell functional can be written as

$$\begin{aligned} I^*_{BwI}[\mathbf{p}] &= \sum_{a,x,y,c,w} \xi^{axy}_{cw} \frac{1}{2^n} \operatorname{tr}\left\{ (\bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i})^T \sigma_{a|xy} \right\}, \\ &= \frac{1}{2^{n+1}} \sum_{a,x,y} \operatorname{tr}\left\{ F_{axy} \sigma_{a|xy} \right\}. \end{aligned} \tag{F4}$$

It follows that $I^*_{BwI}[\mathbf{p}] \geq 0$ for any $\mathbf{p}$ arising from a quantum assemblage $\mathbf{\Sigma}_{\mathbb{A}|\mathbb{XY}}$ with elements $\{\sigma_{a|xy}\}$.

To have a universal protocol for activation of post-quantumness in high-dimensional Bob-with-input assemblages, there are two things left to show. Firstly, it is necessary to show that the quantum bound of the functional I^*_{BwI} does not change when it is optimised over measurements different than $M^{BB'}_{b=0|z=\star} = |\phi^n\rangle\langle\phi^n|$.

The proof of this claim is analogous to the one presented in Appendix C. Secondly, the protocol cannot show false-positive detection when the self-tested assemblage is of the form given in Eq. (F1). Let us first focus on the case when $\sigma_{\mathbf{c}|\mathbf{w}}^{(r=0)} = (\bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i})^T$. Then, the Bell functional is given by

$$I_{BwI}^{*}[\mathbf{p}] = \frac{1}{2^{n+1}} \sum_{a,x,y} \mathrm{tr}\left\{F_{axy}(\sigma_{a|xy})^T\right\}. \tag{F5}$$

The transpose map applied on a quantum Bob-with-input assemblage transforms its elements as follows

$$\begin{aligned}(\sigma_{a|xy})^T &= \left(\mathrm{tr}_A\left\{(M_{a|x}\otimes\mathcal{E}_y)\rho_{AB}\right\}\right)^T, \\ &= \mathrm{tr}_A\left\{(\mathbb{I}\otimes T)(M_{a|x}\otimes\mathcal{E}_y)\rho_{AB}\right\}.\end{aligned} \tag{F6}$$

To show that $(\sigma_{a|xy})^T$ is a quantum assemblage if $(\sigma_{a|xy})$ has a quantum realization, we will use Lemma 8. Let $\mathcal{E}'_y$ be a CPTP map that is indexed by y. Due to Lemma 8, we can shift the transpose in Eq. (F6) to the quantum state as

$$\begin{aligned}(\sigma_{a|xy})^T &= \mathrm{tr}_A\left\{(\mathbb{I}\otimes T)(M_{a|x}\otimes\mathcal{E}_y)\rho_{AB}\right\}, \\ &= \mathrm{tr}_A\left\{(M_{a|x}\otimes\mathcal{E}'_y)\rho_{AB}^{T_B}\right\}, \\ &= \mathrm{tr}_A\left\{(M_{a|x}\otimes\mathcal{E}'_y)^{T_A}\rho_{AB}^{T}\right\}, \\ &= \mathrm{tr}_A\left\{((M_{a|x})^T\otimes\mathcal{E}'_y)\rho'_{AB}\right\},\end{aligned} \tag{F7}$$

where $\{(M_{a|x})^T\}$ are valid POVMs and ρ' is a valid quantum state. Therefore, we can see that if $\{\sigma_{a|xy}\}$ form a valid quantum assemblage, the elements $\{(\sigma_{a|xy})^T\}$ also have a quantum realization in terms of the measurements $\{(M_{a|x})^T\}$, CPTP maps $\{\mathcal{E}'_y\}$ and the state ρ'_{AB}. It follows that $I_{BwI}^{*}[\mathbf{p}] \geq 0$ for any $\mathbf{p}$ arising from a quantum assemblage, even when $\sigma_{\mathbf{c}|\mathbf{w}}^{(r=0)} = (\bigotimes_{i=1}^{n} \widetilde{\sigma}_{c_i|w_i})^T$. Hence, by convexity, any correlation generated when the self-tested assemblage has the form (F1) cannot create a false-positive detection of post-quantumness.

2. Measurement-device-independent EPR scenario

Recall that the Bell functional for the MDI scenario reads

$$I_{MDI}^{*}[\mathbf{p}] \equiv \sum_{a,b,x,c,z} \xi_{cz}^{abx} p(a,b,c|x,y=\star,z). \tag{F8}$$

If the first step of the protocol was successful, the relevant probabilities can be written as

$$p(a,b,c|x,y=\star,z) = \mathcal{N}_{ab|x}(\sigma_{\mathbf{c}|\mathbf{z}}^{(r)}), \tag{F9}$$

where

$$\sigma_{\mathbf{c}|\mathbf{z}}^{(r)} = r\bigotimes_{i=1}^{n}\widetilde{\sigma}_{c_i|z_i} + (1-r)\bigotimes_{i=1}^{n}(\widetilde{\sigma}_{c_i|z_i})^T. \tag{F10}$$

For now, assume that $\sigma_{\mathbf{c}|\mathbf{z}}^{(r=1)} = \bigotimes_{i=1}^{n}\widetilde{\sigma}_{c_i|z_i}$ Then, the Bell functional can be written as

$$I_{MDI}^{*}[\mathbf{p}] \equiv \sum_{a,b,x,c,z} \xi_{cz}^{abx}\mathcal{N}_{ab|x}(\bigotimes_{i=1}^{n}\widetilde{\sigma}_{c_i|z_i}). \tag{F11}$$

Notice that a Choi matrix of a d-dimensional qudit assemblage $\{\mathcal{N}_{ab|x}(\cdot)\}$ is expressed in terms of a maximally entangled state $|\phi^n\rangle\langle\phi^n|$, with $|\phi^n\rangle = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}|kk\rangle_{B'C}$ and $n = d$. In what follows, we encode this state in a space that arises from a parallel composition of many qubits. Then, similar to the qubit case, we can express the probabilities $p(a,b,c|x,y=\star,z)$ as

$$p(a,b,c|x,y=\star,z) = \mathrm{tr}\left\{(\bigotimes_{i=1}^{n}\widetilde{M}^{C_i}_{c_i|z_i})\, J(\mathcal{N}_{ab|x})\right\}, \tag{F12}$$

and the functional is given by

$$\begin{aligned} I^*_{MDI}[\mathbf{p}] &= \sum_{a,b,x,c,z}\xi^{abx}_{cz}\mathrm{tr}\left\{(\bigotimes_{i=1}^{n}\widetilde{M}^{C_i}_{c_i|z_i})\, J(\mathcal{N}_{ab|x})\right\}, \\ &= \mathrm{tr}\left\{\sum_{a,b,x}F_{abx}\, J(\mathcal{N}_{ab|x})\right\}. \end{aligned} \tag{F13}$$

Therefore, $I^*_{MDI}[\mathbf{p}] \geq 0$ for any $\mathbf{p}$ arising from a quantum assemblage $\mathbf{N}_{\mathbb{A}\mathbb{B}|\mathbb{X}}$ with elements $\{\mathcal{N}_{ab|x}\}$.

Now, we will show that a false-positive detection is not possible even when the self-tested assemblage is of the form given in Eq. (F10). Let $\sigma^{(r=0)}_{\mathbf{c}|\mathbf{z}} = (\bigotimes_{i=1}^{n}\widetilde{\sigma}_{c_i|z_i})^T$. Then, the Bell functional can be written as

$$I^*_{MDI}[\mathbf{p}] = \mathrm{tr}\left\{\sum_{a,b,x}F_{abx}\,\left(J(\mathcal{N}_{ab|x})\right)^T\right\}. \tag{F14}$$

To show that a transpose operation on the Choi matrix of an MDI assemblage cannot generate post-quantumness, we will again use Lemma 8. Recall that any quantum MDI assemblage is defined by a set of measurement channels $\Theta_b^{BB'\to B_{out}}$. For each $b\in\mathbb{B}$, we can decompose $\Theta_b^{BB_{in}\to B_{out}}$ in terms of its Kraus operators to see that a transpose on the output space of this measurement channel B_{out} can be passed onto the input of the channel defined on BB_{in}. It follows that

$$\begin{aligned} \left(J(\mathcal{N}_{ab|x})\right)^T &= \mathrm{tr}_A\left\{(M_{a|x}\otimes\Theta_b^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}\otimes|\phi^n\rangle\langle\phi^n|_{B_{in}C}]\right\}^T, \\ &= \mathrm{tr}_A\left\{(\mathbb{I}^A\otimes T^{B_{out}C})(M_{a|x}\otimes\Theta_b^{BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}\otimes|\phi^n\rangle\langle\phi^n|_{B_{in}C}]\right\}, \\ &= \mathrm{tr}_A\left\{(M_{a|x}\otimes\Theta_b^{'BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}^{T_B}\otimes(|\phi^n\rangle\langle\phi^n|_{B_{in}C})^T]\right\}, \\ &= \mathrm{tr}_A\left\{(M_{a|x}\otimes\Theta_b^{'BB_{in}\to B_{out}}\otimes\mathbb{I}^C)^{T_A}[\rho_{AB}^{T}\otimes(|\phi^n\rangle\langle\phi^n|_{B_{in}C})^T]\right\}, \\ &= \mathrm{tr}_A\left\{((M_{a|x})^T\otimes\Theta_b^{'BB_{in}\to B_{out}}\otimes\mathbb{I}^C)[\rho_{AB}^{T}\otimes(|\phi^n\rangle\langle\phi^n|_{B_{in}C})^T]\right\}. \end{aligned} \tag{F15}$$

Therefore, if $J(\mathcal{N}_{ab|x})$ corresponds to a valid quantum assemblage, its transpose $\left(J(\mathcal{N}_{ab|x})\right)^T$ also have a quantum realization in terms of the measurements $\{(M_{a|x})^T\}$, measurement channels $\{\Theta_b^{'BB_{in}\to B_{out}}\}$ and the states ρ^T_{AB} and $(|\phi^n\rangle\langle\phi^n|_{B_{in}C})^T$. Since this assemblage has a quantum realization, it cannot create a false-positive detection in the activation protocol.

3. Channel EPR scenario

In the protocol for the channel EPR scenario, we used the self-test of a tensor product of Pauli operators (accompanied by an alignment procedure) even when dim($\mathcal{H}_{B_{out}}$)= 2. Moreover, the technique to show that there are no false-positive detections does not rely on any of the subsystems having a particular dimension. Therefore, the generalization of this protocol to the qudit case is straight-forward.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Physical Review **47**, 777 (1935).
[2] E. Schrödinger, Discussion of probability relations between separated systems, Mathematical Proceedings of the Cambridge Philosophical Society **31**, 555 (1935).
[3] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Physical Review A **80**, 032112 (2009).
[4] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics Physique Fizika **1**, 195 (1964).
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Physical review letters **23**, 880 (1969).
[6] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Reviews of Modern Physics **86**, 419 (2014).
[7] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[8] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox, Physical review letters **98**, 140402 (2007).
[9] R. Uola, A. C. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Reviews of Modern Physics **92**, 015001 (2020).
[10] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Physical Review A **85**, 010301 (2012).
[11] Y. Xiang, I. Kogias, G. Adesso, and Q. He, Multipartite gaussian steering: Monogamy constraints and quantum cryptography applications, Phys. Rev. A **95**, 010101 (2017).
[12] D. Cavalcanti, P. Skrzypczyk, G. Aguilar, R. Nery, P. S. Ribeiro, and S. Walborn, Detection of entanglement in asymmetric quantum networks and multipartite quantum steering, Nature communications **6**, 1 (2015).
[13] A. Máttar, P. Skrzypczyk, G. Aguilar, R. Nery, P. S. Ribeiro, S. Walborn, and D. Cavalcanti, Experimental multipartite entanglement and randomness certification of the w state in the quantum steering scenario, Quantum Science and Technology **2**, 015011 (2017).
[14] E. Passaro, D. Cavalcanti, P. Skrzypczyk, and A. Acín, Optimal randomness certification in the quantum steering and prepare-and-measure scenarios, New Journal of Physics **17**, 113010 (2015).
[15] Y. Z. Law, J.-D. Bancal, V. Scarani, *et al.*, Quantum randomness extraction for various levels of characterization of the devices, Journal of Physics A: Mathematical and Theoretical **47**, 424028 (2014).
[16] M. Reid, P. Drummond, W. Bowen, E. G. Cavalcanti, P. K. Lam, H. Bachor, U. L. Andersen, and G. Leuchs, Colloquium: the Einstein-Podolsky-Rosen paradox: from concepts to applications, Reviews of Modern Physics **81**, 1727 (2009).
[17] A. B. Sainz, L. Aolita, M. Piani, M. J. Hoban, and P. Skrzypczyk, A formalism for steering with local quantum measurements, New Journal of Physics **20**, 083040 (2018).
[18] P. J. Cavalcanti, J. H. Selby, J. Sikora, T. D. Galley, and A. B. Sainz, Post-quantum steering is a stronger-than-quantum resource for information processing, npj Quantum Information **8**, 1 (2022).
[19] B. Zjawin, D. Schmid, M. J. Hoban, and A. B. Sainz, Quantifying EPR: the resource theory of nonclassicality of common-cause assemblages, Quantum **7**, 926 (2023).
[20] V. P. Rossi, M. J. Hoban, and A. B. Sainz, On characterising assemblages in einstein–podolsky–rosen scenarios, Journal of Physics A: Mathematical and Theoretical **55**, 264002 (2022).
[21] M. F. Pusey, Negativity and steering: A stronger Peres conjecture, Physical Review A **88**, 032313 (2013).
[22] N. Gisin, Stochastic quantum dynamics and relativity, Helvetica Physica Acta **62**, 363 (1989).
[23] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, Physics Letters A **183**, 14 (1993).
[24] A. B. Sainz, N. Brunner, D. Cavalcanti, P. Skrzypczyk, and T. Vértesi, Postquantum steering, Physical review letters **115**, 190403 (2015).
[25] A. B. Sainz, M. J. Hoban, P. Skrzypczyk, and L. Aolita, Bipartite postquantum steering in generalized scenarios, Physical Review Letters **125**, 050404 (2020).
[26] E. G. Cavalcanti, M. J. Hall, and H. M. Wiseman, Entanglement verification and steering when alice and bob cannot be trusted, Physical Review A **87**, 032306 (2013).

[27] B. Zjawin, D. Schmid, M. J. Hoban, and A. B. Sainz, The resource theory of nonclassicality of channel assemblages, Quantum **7**, 1134 (2023).
[28] M. Piani, Channel steering, JOSA B **32**, A1 (2015).
[29] D. Schmid, D. Rosset, and F. Buscemi, The type-independent resource theory of local operations and shared randomness, Quantum **4**, 262 (2020).
[30] D. Rosset, D. Schmid, and F. Buscemi, Type-independent characterization of spacelike separated resources, Physical Review Letters **125**, 210402 (2020).
[31] M. J. Hoban, T. Drescher, and A. B. Sainz, A hierarchy of semidefinite programs for generalised einstein-podolsky-rosen scenarios, arXiv preprint arXiv:2208.09236 https://doi.org/10.48550/arXiv.2208.09236 (2022).
[32] M. J. Hoban, P. Skrzypczyk, , and A. B. Sainz, Activation of post-quantum steering, in preparation.
[33] J. Bowles, I. Šupić, D. Cavalcanti, and A. Acín, Device-independent entanglement certification of all entangled states, Physical review letters **121**, 180503 (2018).
[34] J. Bowles, I. Šupić, D. Cavalcanti, and A. Acín, Self-testing of pauli observables for device-independent entanglement certification, Physical Review A **98**, 042336 (2018).
[35] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2011).
[36] D. Schmid, K. Ried, and R. W. Spekkens, Why initial system-environment correlations do not imply the failure of complete positivity: A causal perspective, Phys. Rev. A **100**, 022112 (2019).
[37] S.-L. Chen, H.-Y. Ku, W. Zhou, J. Tura, and Y.-N. Chen, Robust self-testing of steerable quantum assemblages and its applications on device-independent quantum certification, Quantum **5**, 552 (2021).
[38] D. Schmid, H. Du, M. Mudassar, G. Coulter-de Wit, D. Rosset, and M. J. Hoban, Postquantum common-cause channels: the resource theory of local operations and shared entanglement, Quantum **5**, 419 (2021).
[39] B. Coecke and A. Kissinger, Picturing quantum processes: A first course on quantum theory and diagrammatic reasoning, in *Diagrammatic Representation and Inference: 10th International Conference, Diagrams 2018, Edinburgh, UK, June 18-22, 2018, Proceedings 10* (Springer, 2018) pp. 28–31.