Summary of Professional Accomplishments

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Contents

1 Name

Anita Magdalena Dąbrowska
Maiden name: Rezmerska

2 Diplomas, scientific or artistic degrees

- 2008, PhD degree in physics

  Nicolaus Copernicus University in Toruń, Poland (NCU)

  Title of dissertation: *Stochastic evolution of observed quantum systems*

  Supervisor: Dr habil. Przemysław Staszewski

- 2000, Master's degree in theoretical physics (NCU)

  Title of master thesis: *The role of the continuous spectrum in population transfer processes in a strong laser radiation field*

  Supervisor: Prof. Dr habil. Jarosław Zaremba

  Grade: very good

- 1998 Bachelor's degree in physics, specialising in theoretical physics (NCU)

  Title: *Lifetime of excited states of the atom due to spontaneous emission*

  Supervisor: Prof. Dr habil. Jarosław Zaremba

  Grade: very good

3 Information on employment in scientific or artistic institutes or faculties

- from 01.10.2019 assistant professor, research and teaching group, Department of Mathematical Methods of Physics, Institute of Theoretical Physics and Astrophysics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk

- 20.11.2018 – 30.09.2019 senior lecturer, Department of Mathematical Modelling in Biomedical Sciences, Department of Theoretical Foundations of Biomedical Sciences and Medical Informatics, Faculty of Pharmacy, Nicolaus Copernicus University in Toruń

- 01.02.2012 – 19.11.2018 assistant professor, Department of Mathematical Modelling in Biomedical Sciences, Department of Theoretical Foundations of Biomedical Sciences and Medical Informatics, Faculty of Pharmacy, Nicolaus Copernicus University in Toruń

- 11.2004 – 31.01.2012 assistant, Department of Mathematical Modelling in Biomedical Sciences, Department of Theoretical Foundations of Biomedical Sciences and Medical Informatics, Faculty of Pharmacy, Nicolaus Copernicus University in Toruń

- 01.10.2000 – 11.2004 assistant, Department of Mathematical Modelling in Biomedical Sciences, Department of Theoretical Foundations of Biomedical Sciences and Medical Informatics, Faculty of Pharmacy, the Medical University in Bydgoszcz

4 Description of the achievements, set out in art. 219 para 1 point 2 of the Act

4.1 Cycle of scientific articles related thematically, pursuant to art. 219 para 1. point 2b of the Act

Title of the achievement: *Filtering equations and quantum trajectories beyond the Markov regime*

Articles constituting an achievement

[H1] **Anita Dąbrowska**, Gniewomir Sarbicki, and Dariusz Chruściński. Quantum trajectories for a system interacting with environment in a single photon state: counting and diffusive processes. *Physical Review A*, 96, 053819-1-053819-11, 2017.

[H2] **Anita Dąbrowska**. From a posteriori to a priori solutions for a two-level system interacting with a single-photon wavepacket. *Journal of the Optical Society of America B*, 37(4), 1240-1248, 2020.

[H3] **Anita Dąbrowska**. Photon counting probabilities of the output field for a single-photon input. *Journal of the Optical Society of America B*, 40(5), 1299-1310, 2023.

[H4] **Anita Dąbrowska**, Dariusz Chruściński, Sagnik Chakraborty, and Gniewomir Sarbicki. Eternally non-Markovian dynamics of a qubit interacting with a single-photon wavepacket. *New Journal of Physics*, 23, 123019-1-123019-18, 2021.

[H5] **Anita Dąbrowska**, Gniewomir Sarbicki, and Dariusz Chruściński. Quantum trajectories for a system interacting with environment in N-photon state. *Journal of Physics A: Mathematical and Theoretical*, 52(10), 105303-1-105303-21, 2019.

[H6] **Anita Dąbrowska**. Quantum trajectories for environment in superposition of coherent states. *Quantum Information Processing*, 18, 224-1-224-22, 2019.

[H7] **Anita Dąbrowska**, Marcin Marciniak. Stochastic approach to evolution of a quantum system interacting with environment in squeezed number state, Quantum Information Processing, 22, 385, 2023. DOI:10.1007/s11128-023-04108-9

The papers [H1, H4, H5, H7] are collaborative papers. The articles [H2, H3, H6] are single-author papers. A description of each author's contribution is included in the statements attached to the application. My contribution is described in the attached list of scientific achievements. Papers [H1, H5, H6] were written while I was employed at Nicolaus Copernicus University in Toruń. The remaining papers were written after I began my employment at the University of Gdańsk.

4.2 Research objectives, results, and description of publications based on them

4.2.1 Introduction and research motivation

Filtering theory uses probability tools to estimate stochastic processes. Let us consider some stochastic process $\{X_t; t \in T\}$, $T = \mathbb{R}_+$, that cannot be observed directly. Instead, one can observe some other process $\{Y_t; t \in T\}$ correlated with the process $\{X_t; t \in T\}$. The filtering problem consists of estimating the process X from a measurement of Y with an assumed optimisation criterion. This means more precisely that, having the results for $\{Y_s; s \leq t\}$, we want to estimate X_t. The most common optimisation criterion is to minimise the mean-square error [1, 2]. From the physics point of view, the filtering theory is a collection of the methods to estimate the state in the dynamic system with stochastic properties. Classical filtering theory is used, for example, in engineering and financial mathematics.

The *quantum filtering theory* [3–10], formulated in the eighties of the previous century within the framework of the *quantum stochastic calculus of Itô type* [11,12], provides a description of the evolution of an open quantum system depending on the results of continuous in time observations of the environment of that system. In this model we deal with a quantum system interacting with an environment being usually a boson field (a propagating electromagnetic field). There also exists a quantum filtering theory for fermionic environments [13–15], but it has not been the subject of my research so far, and I will not be presenting this version here. It is convenient to present the filtering model using the concepts of *input and output fields*, i.e. the fields, respectively, before and after interaction with a given quantum system. One can say that the Bose field disturbs the free evolution of the quantum system; on the other hand, the measurement performed on the output field provides us with information about this system. The evolution dependent on the result of measurement is referred to as a conditional evolution. With the measurement performed continuously in time we can associate some stochastic process. Quantum filtering theory is formulated within approximations that allow to describe the measurement of the output field by a family of operators having a joint spectral measure.

The evolution of an open quantum system depending on the results of continuous in time measurements of the output field is described by an equation called the *filtering equation* or *stochastic master equation*. The form of this equation depends on the type of measurement being performed (the choice of the field observable) and the state of the input field. The solutions of this equation are called *quantum trajectories*.

In many cases, the stochastic master equation preserves the purity of the state, that is, it transforms a pure state into a pure state. In such a situation, the equation for the density operator can be replaced by the equation for the conditional vector state. The evolution of the system is then described by an equation called the *stochastic Schrödinger equation*, which is the basis of the Monte Carlo algorithm, described, for example, in the papers [18, 19]. Taking an average over all possible realisations of the considered stochastic process, we proceed to the unconditional evolution of the open system, described by a reduced density operator which, for the input field in the Gaussian state, satisfies the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [20, 21]. In this case, the evolution of the open system is called a *Markovian evolution* [22]. It should be noted that not every stochastic master equation is a filtering equation. Often, stochastic master equations are used to solve the master equation without a measurement-related context. There are stochastic equations for open systems that are not related to a measurement. It should be emphasised that stochastic methods can be used not only to solve the master equation, which does not preserve a pure state in the general case, but allow us to better understand the evolution of an open system, and provide tools to efficiently determine the properties of the output field. They are also the starting point of the quantum control model [10, 23].

Originally, the quantum version of classical filtering theory was developed for a field in a Gaussian state. In the basic version of the model, the input field is assumed to be in a vacuum state. Development of techniques for generating of wave packets in *non-classical states*, such as a superposition of coherent states [24–26], states of definite number of photons and squeezed N-photon states [27–35], and the growing area of their applications have brought the need for new stochastic tools enabling to describe the interaction of non-classical light with a quantum system. Note that we are talking about non-classical states of the continuous-mode field [36–40]. The time correlations of the input field mean that the reduced evolution of the open system is not described by the GKSL master equation. In the papers [41–47] one can find generalised master equations for open quantum systems driven by a wave packet in the single-photon or N-photon state. Note that when the system interacts with a field in the N-photon state, the evolution of the system is not described by a single master equation, but by a set of coupled differential equations. A derivation of the general form of the set of coupled master equations defining the reduced evolution of a quantum system interacting with a field in the N-photon state was given in [46].

To derive stochastic equations for non-classical fields, one cannot use directly the methods developed for the situation when an input field is in a Gaussian state. A stochastic analysis for an open quantum system interacting with the Bose field prepared in a non-classical state one can find, for instance, in [48–60] and also in [H1-H3, H5-H7, P2, P3]. Historically, the first method of determining the filtering equations for a field in a non-classical state involves extending the Hilbert space of the actual quantum system by the Hilbert space of an auxiliary system which, together with the input field in the classical state, acts as a generator of the Bose field in the chosen state [9, 53, 61–64]. The quantum system is *cascaded* to the auxiliary system [61]. The input field for this auxiliary system is prepared in the classical state. The field after interaction with the auxiliary system is the input field for the main system. It is assumed that there is no initial correlation between the main and auxiliary systems. First, the stochastic master equation is determined for the system composed of the main and auxiliary systems, then the partial trace over the auxiliary system is taken, eliminating the degrees of freedom of this system from the description. Using this method, the set of coupled stochastic equations defining the conditional evolution for a quantum system interacting with the light in the single-photon state was determined in [48]. In [52], the filtering equations for the light in the multi-photon state were obtained in a similar way. Of course, this method of determining the filtration equations has some limitations. Let us emphasise that the model of the generation of a non-classical field by an auxiliary system cannot be understood literally as a recipe for conducting an experiment. This is only a mathematical procedure allowing us to determine quantum trajectories for non-classical fields. However, depending on the complexity of the non-classical state, sometimes it can be very difficult to define a suitable auxiliary system. A different concept was used in the papers [49, 51, 55]. Here, the Hilbert space of the quantum system is extended by the Hilbert space of the auxiliary system in such a way that, for the given system, the desired non-Markovian evolution is obtained. Hence the name of the method—the non-Markovian embedding. In this method, the auxiliary system and the Bose field are initially in an entangled state. By this method, the filtering equations for the Bose field in the single-photon state, superposition of coherent states and the multi-photon state were determined in [49, 51, 55]. The authors of [57], published in 2017, determined the set of filtering equations for the field in the N-photon state using a method based on a temporal decomposition of the input state . In a similar manner, the set of stochastic master equations was obtained for a quantum system interacting with a field in a squeezed N-photon state in [60].

The evolution of open systems interacting with a wave packet in a non-classical state is quite complicated. The derivations given in [48, 49, 51, 52, 55, 57, 60] refer to the mathematical tools that are not widely used by physicists. In particular, the papers [49, 51] are very formal.

My motivation for writing papers [H1-H3, H6-H7] was the desire to develop a more fundamental method than previously proposed for determining the stochastic evolution of a quantum system interacting with a field in a non-classical state, revealing the structure of entanglement between the environment and the system. My goal was also to popularize stochastic tools that allow for the efficient determination of the evolution of quantum systems interacting with non-classical fields and the analysis of output field properties. The motivation for writing paper [H5] was the desire to present a general analysis of the non-Markovian character of the evolution of a two-level atom interacting with a single-photon field.

In the papers [H1-H3, H5-H7], which make up the scientific achievement that forms the basis of this proposal, a new way to determine the filtration equations for non-classical fields was defined. In the mentioned papers, the stochastic evolution of an open quantum system interacting with a field in non-classical states was determined based on a *model of repeated interactions and measurements* also called in the literature the *collision model* [65–87]. Descriptions of the time discretisation procedure leading to the collision model in quantum optics, its relation to the input-output formalism, and physical approximations one can find, for instance, in [71,74,80–82]. In the collision model, the environment is approximated by a certain sequence of sub-systems. When these sub-systems are initially uncorrelated, they do not interact with each other and they interact with the quantum system only once, the model of repeated interactions leads to the Markovian dynamics of an open quantum system and allows to approximate with arbitrary accuracy the evolution governed by the GKSL master equation [88]. A formulation of the discrete stochastic calculus and a description of discrete quantum trajectories for an input field in the Gaussian state can be found in [7, 66, 67, 69, 74, 84, 85, 87].

The evolution of a quantum system interacting with a field prepared in a non-classical state is non-Markovian due to the temporal correlations of the field. The influence of such correlations on the evolution of open systems has been analysed within the framework of the collision model, for instance, in [73,76–79]. The non-Markovian evolution of an open system will also be obtained if we assume that the sub-systems of the environment interact with each other or if we allow them to interact with the system more than once, and if the system is correlated with the environment at the initial time.

The following paper provides a brief introduction to quantum filtration theory in its basic version, i.e. for a field in a vacuum state, a presentation of the model approximations, and a general introduction to the collision model in quantum optics. These chapters cannot, however, be regarded as a comprehensive introduction.

4.2.2 Summary of the obtained results

In papers [H1-H4, H6-H7], which constitute the habilitation achievement, a method for determining filtering equations and quantum trajectories for non-classical fields based on the model of repeated interactions and measurements was proposed. The presented method for determining the evolution of the open system is simpler and more intuitive than those proposed by other authors. The discrete-time approach allows for a better understanding of the properties of non-classical states, the processes occurring during the interaction of the open system with the non-classical field, and how quantum trajectories and the filtering equations related to measurement are determined.

In the publications [H1-H3, H5-H7], the continuous in time evolution of a quantum system was obtained from the dynamics generated by a discrete in time sequence of (weak) interactions (collisions) of the quantum system with ancillas forming its environment, defined as a chain of qubits or harmonic oscillators prepared in an entangled state being a discrete analogue of the non-classical state of the continuous-mode field. It is assumed that after each interaction, a measurement is performed on the bath element that has just interacted with the system. The sequence of measurements carried out on the bath elements leads to a discrete in time stochastic evolution of the open system. There is no initial correlation between the system and the environment.

The essential element of each derivation is the discussion regarding the properties of the input field state. It is the entanglement of the environmental subsystems that causes the difficulty in determining the equations for the evolution of the open system. In papers [H1-H4, H6-H7], the problem of conditional evolution is defined in the Hilbert space including the states of the open system and the input field. When we follow the measurement results of the output field, the composed system consisting of the system and the input field remains in a pure state. However, unlike the case of Gaussian fields, the conditional state is not a separable state but an entangled state of the system and the input field. The papers provide general formulae for the *a posteriori* state of the system and the input field. It is shown how this state can be decomposed into the sum of components representing alternative scenarios occurring in parallel in the experiment. The approach proposed in the papers allows to simplify the problem of determining the conditional evolution of an open system. Instead of formulating it using matrix equations, one can use vector equations. The papers presents sets of coupled equations for conditional vectors related to random measurement results. If the environment is the field in the N-photon state, we obtain a set of $(N+1)^2$ coupled equations for conditional operators and only $N+1$ equations for vectors.

Starting from the discrete model, formulae were finally obtained for the evolution of the system conditioned by the results of continuous in-time observation of the output field. Two types of measurement were considered in the papers. : photon counting measurement and optical quadrature measurement. The model of repeated interactions and measurements allowed in [H1-H3, H5-H7] not only to derive differential equations describing the conditional and unconditional evolution of the quantum system, but also to enable the determination of the general structure of quantum quantum trajectories related to the measurement. In addition to the formulae for the trajectories, the papers also provide their physical interpretation. The determined analytical formulae make it possible to obtain full statistics of the photon counts of the output field and to provide analytical solutions to the generalized master equation.

In papers [H1-H4] the interaction of the quantum system with the field in the single-photon state was considered. Based on the collision model, in paper **[H1]** the sets of coupled filtering equations was determined for the Bose field in the single-photon state for two types of observations of the output field. The study considers counting observation, where the photons of the input field are directly counted, as well as quadrature observation of the field. The reservoir field is modelled by an infinite chain of qubits. The paper contains formulae for quantum trajectories related to photodetection of output field photons. Note that in the general case we have infinitely many photon counting scenarios in the output field. The paper gives the formula for the probability density of counting m photons in the output field at times $t_1, t_2, \ldots, t_m$ such that $0 < t_1 < t_2 < \ldots < t_m < t$ and no other photons in the interval from 0 to t and the expression for the probability of no count up to a given moment. The paper also contains general formulae for conditional mean values of increments of stochastic processes for counting and diffusion observations. The obtained results are general and can be applied to various quantum systems, such as atoms, ions, and resonance cavities.

The results of [H1] were utilized in paper **[H2]** to determine the conditional and unconditional evolution of a two-level atom interacting with a unidirectional field in a single-photon state. The paper derives the general form of the *a priori* state of the two-level atom for the arbitrary initial state of the atom and any profile of the single-photon field by averaging quantum trajectories for counting observations. Using the formulae for quantum trajectories, the statistics of photon counts in the output field were obtained. The general form of the positive-definite measure associated with photon counts is provided. Formulae were derived for the probability of detecting a fixed number of photons up to a given time, the mean number of counts up to a given time, the average photon counting times, and the Mandel parameter.

In paper **[H3]**, the collision model was used to describe the interaction of an open system with a bidirectional electromagnetic field. In one direction, the field is prepared in a single-photon state, and in the other direction, it is in the vacuum state. In this case, the system's environment is modeled by two chains of qubits. The supplement to the paper provides the derivation of the set of filtering equations for the bidirectional field. The conditional evolution is determined here for a two-dimensional counting process. The paper includes the derivation of general formulae for conditional vectors associated with photon counts by two detectors. The obtained results allow the determination of the output field state for any open system. The paper derives the probabilities of photon counts in the output field. From this, the probabilities of wave packet scattering, reflection, and transmission were obtained.

As an example of the application of these results, the scattering of a pulse on a two-level atom is described. The paper includes analytical formulae for any initial state of the atom and any temporal profile of the photon. It provides detailed results for a photon with an exponential profile. The influence of destructive and constructive interference of indistinguishable photons in the scattering process is particularly interesting. It is shown that the key factor here is the relationship between the lifetime of the excited state of the atom and the pulse width.

The paper **[H4]** is devoted to analysing the non-Markovian nature of the evolution of a two-level system (qubit) interacting with a bidirectional field; in one direction, the input field is prepared in a single-photon state, while in the other, it is in a vacuum state. The reduced evolution of the qubit is represented then by a hierarchy of master equations. The temporal correlations of the field are responsible for all non-Markovian memory effects of the qubit dynamics. The analytical solution to the hierarchy of equations for any initial state of the system and for an arbitrary temporal profile of the photon state is provided and used to show that the set of these equations is equivalent to a single time-local master equation. The time dependent rates governing damping (cooling), pumping, and dephasing processes being fully characterized by the wave packet profile are determined. An immediate consequence of the analysis is the observation that in general the dynamical map governing the qubit evolution is not invertible which implies the singularity of rates in the corresponding time-local master equation. It is shown that in the resonant case whenever time-local generator is regular (does not display singularities) the qubit evolution never displays information backflow. However, in general the generator might be highly singular leading to intricate non-Markovian effects.

In in article **[H5]** the results of [H1] were generalized by considering the field in the N-photon state. This field state is also called the Fock state. The set of filtering equations for a quantum system interacting with an environment prepared in a continuous-mode N-photon state was derived. Here the unidirectional field with photons of the same time profiles was considered. To determine the conditional evolution of the quantum system depending on the continuous in-time measurements of the output field, the model of repeated interactions and measurements with the environment given as an infinite chain of harmonic oscillators was used. It is assumed that the bath harmonic oscillators do not interact between themselves and they are prepared initially in an entangled state being a discrete analogue of a continuous-mode N-photon state. The continuous in time conditional evolution of the quantum system for the photon counting observation was derived starting from determination of discrete in time recurrence equations for the $N + 1$ conditional vectors. Subsequently, a set of recursive filtering equations was obtained for the system operators that depend on the observation results, ultimately leading to continuous-time stochastic evolution. The article also presents solutions to the derived set of stochastic equations. The quantum trajectories are used to find the analytical formulas defining the photon counting statistics in the output field and to provide a solution to the set of coupled master equations. The quantum trajectories in the continuous case are represented by a proposed diagrammatic technique with very transparent "Feynman rules". This technique considerably simplifies the structure of the solution and enables one to find a physical interpretation for the solution in terms of a few elementary processes. It should be emphasized that the results are general and can be applied to various open systems.

In paper **[H6]**, the set of filtering equations describing the conditional evolution of an open quantum system interacting with the Bose field prepared in a superposition of coherent states was determined. This paper provides solutions for two measurement schemes of the output field: photon counting and homodyne detection. The collision model with the environment, represented by an infinite chain of qubits, is considered. It is assumed that the bath qubits do not interact with each other and are initially prepared in an entangled state, being a discrete analogue of a superposition of continuous-mode coherent states. Due to the temporal correlations present in the environment, the evolution of the open quantum system becomes non-Markovian. Starting from a discrete-time description of the problem, the sets of recurrence stochastic equations were obtained, and finally in the continuous-time limit differential filtering equations were determined. As an example of the application of the derived equations, the conditional evolution of a cavity mode initially prepared in a coherent state was considered. The paper presents the conditional state of the field in the cavity for both photon counting and diffusive observation.

In paper **[H7]**, the sets of filtering and master equations for a quantum system interacting with a wave packet of light in a continuous-mode squeezed number state were determined. The problem of the conditional evolution of a quantum system was formulated using the model of repeated interactions and measurements. In this approach, the quantum system undergoes a sequence of interactions with an environment defined by a chain of harmonic oscillators. It is assumed that the environment is prepared in an entangled state, which is a discrete analogue of a continuous-mode squeezed number state. The paper provides a derivation of a discrete stochastic dynamics that depends on the results of measurements performed on the field after its interaction with the system. Stochastic dynamics is considered for a photon counting measurement scheme. By taking a continuous time limit, the set of differential stochastic equations for the system was finally obtained. It should be emphasized that in this case, the stochastic evolution of the system is given by a set of infinitely many coupled equations. The paper contains a general solution of this set for an arbitrary open system. A construction of analytical formulae for quantum trajectories and exclusive probability densities that fully characterize the statistics of photons in the output field are given.

Using stochastic tools, the paper presents a solution to the problem of optimal excitation of a harmonic oscillator (field in a cavity) by a traveling field prepared in a squeezed N-photon state. It is assumed that the oscillator is initially in the vacuum state. The paper provides the condition for perfectly transferring the photons from the input field into the cavity.

4.2.3 Introduction

Markovian and non-Markovian dynamics of open quantum systems

Theory of open quantum systems theory describes of the evolution of a quantum system interacting with its environment. Interaction with the environment is a source of decoherence, energy dispersion, spontaneous emission. It is usually assumed that the evolution of a composed system consisting of a quantum system and its environment is unitary and that there is no initial correlations between the systems. The reduced density operator at time t is defined

$$\varrho(t) = \Phi_t \varrho(0) := \mathrm{Tr}_{\mathcal{E}}\{U(t)\varrho(0) \otimes \eta(0) U^{\dagger}(t)\}, \tag{1}$$

where $U(t)$ is the unitary operator describing the evolution of composed system, $\varrho(0)$ and $\eta(0)$ stand for the initial state of the system and environment, respectively. Let the initial environmental state $\eta(0)$ be fixed. For $t \geq 0$ the above equation defines a linear map

$$\Phi_t : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}), \tag{2}$$

on the open system's state space $\mathcal{T}(\mathcal{H})$ for the Hilbert space $\mathcal{H}$, thus Φ_t maps any initial state of the system to the system's state at time t

$$\varrho(0) \to \varrho(t) = \Phi_t \varrho(0). \tag{3}$$

A one-parameter family $\{\Phi_t; t \geq 0\}$ one calls a quantum dynamical map. The dynamical map preserves the Hermiticity and the trace of operators. It is a positive map, i.e., it maps positive operators to positive operators. An important feature of dynamical map is that it is not only positive but it is also completely positive. The property

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \tag{4}$$

satisfied for all $t, s \geq 0$ defines the structure of a semigroup. A dynamical map having the semigroup property has a generator $\mathcal{L}$ such that

$$\Phi_t = \exp[\mathcal{L}t]. \tag{5}$$

In this case for the reduced system density operator one obtains the equation

$$\dot{\varrho}(t) = \mathcal{L}\varrho(t). \tag{6}$$

As it was proved in [20, 21], the most general form of a semigroup dynamical map has the following form

$$\mathcal{L}\varrho = -i[H_{\mathcal{S}}, \varrho] + \sum_k \gamma_k \left(R_k \varrho R_k^{\dagger} - \frac{1}{2} R_k^{\dagger} R_k \varrho - \frac{1}{2} \varrho R_k^{\dagger} R_k \right), \tag{7}$$

where $H_{\mathcal{S}}$ is the Hamiltonian of the quantum system, R_k are system operators, often called the Lindblad operators, and $\gamma_k \geq 0, \forall k$. The equation (6) with the generator (7) is called the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation In physical models created for quantum systems interacting with the environment, obtaining the equation of the form (6) for reduced evolution requires several approximations [8]. The most important of these is the Markov approximation. In the case of a differentiable dynamical map for the reduced state $\varrho(t)$ we obtain the local time equation

$$\dot{\varrho(t)} = \mathcal{L}_t \varrho(t), \tag{8}$$

with the time dependent generator $\mathcal{L}_t$. If $\{\Phi_t, t \geq 0\}$ is invertible, then $\mathcal{L}_t = \dot{\Phi}_t \circ \Phi_t^{-1}$. If the map is non-invertible at time $t = \tau$ then Φ_t at τ displays singularities. Any time-local generator

$\mathcal{L}_t$ has the form of (7), but the effective Hamiltonian $H_{\mathcal{S}}$, operators R_k, and γ_k may be time dependent. Given generator $\mathcal{L}_t$ the formal solution for the dynamical map reads as follows

$$\Phi_t = \overleftarrow{T} \exp\left[\int_0^t \mathcal{L}_s ds\right], \tag{9}$$

where $\overleftarrow{T}$ stands for the chronological product. It should be stressed that in the case when $\mathcal{L}_t$ is time-dependent, then the transition rates γ_k are not necessarily non-negative. A quantum dynamical map $\{\Phi_t; t \geq 0\}$ is called divisible if for any $t \geq s$ one has

$$\Phi_t = V_{t,s} \circ \Phi_s, \tag{10}$$

where $V_{t,s}$ is completely positive propagator. If Φ_t is invertible, then

$$V_{t,s} = \Phi_t \circ \Phi_s^{-1}, \tag{11}$$

Thus any invertible map is divisible. One calls the map $\{\Phi_t; t \geq 0\}$ P-divisible if $V_{t,s}$ is positive and trace-preserving. If $V_{t,s}$ is completely positive and trace preserving then $\{\Phi_t; t \geq 0\}$ is called CP-divisible. Following [89], we call the evolution represented by $\{\Phi_t; t \geq 0\}$ Markovian if it is CP-divisible. An invertible map is CP-divisible if and only if all rates are non-negative. The occurrence of negative values of these coefficients is treated as an indicator of non-Markovian evolution. If the evolution is Markovian, then for any pair of the initial states ϱ_1 and ϱ_2 [90]

$$\frac{d}{dt}\|\Phi_t(\rho_1 - \rho_2)\|_1 \leq 0, \tag{12}$$

where $\|X\|_1 = \mathrm{Tr}|X|$ denotes the trace norm of X. Note, that the quantity $\|\rho_1 - \rho_2\|_1$ describes the distinguishability of ρ_1 and ρ_2. The above condition is called the BLP condition, named after the authors of the paper [90]: Breuer, Laine, and Piilo. In the literature, one can find various concepts of non-Markovian indicators of the evolution of quantum systems. For any Markovian evolution [91, 92]

$$\frac{d}{dt}|\mathrm{det}\Lambda_t| \leq 0. \tag{13}$$

The quantity $\mathrm{Vol}(t) = |\mathrm{det}\Lambda_t|\mathrm{Vol}(0)$ is the voluume of the accessible states at time t, hence (13) implies monotonic decrees of Vol(t). Any P-divisible map (and hence also CP-divisible) satisfies (13).

Quantum stochastic calculus and quantum filtering theory

In this section we recall some basic rules of quantum stochastic calculus (QSC) in the boson Fock space [11, 12]. Let us denote by $\mathcal{F}$ the symmetric Fock space over the Hilbert space $\mathcal{K} = \mathbb{C}^n \otimes L^2(\mathbb{R}_+)$ of all square integrable functions from $\mathbb{R}_+$ into $\mathbb{C}^n$. For any $f \in \mathcal{K}$ one can define a coherent vector by the formula

$$e(f) \;=\; \exp\left(-\frac{1}{2}||f||^2_{\mathcal{K}}\right)\left(1, f, (2!)^{-1/2} f \otimes f, (3!)^{-1/2} f \otimes f \otimes f, \ldots\right). \tag{14}$$

In particular, $e(0) = (1, 0, 0, \ldots) \in \mathcal{F}$ is the Fock vacuum. On the linear span of all the exponential vectors in $\mathcal{F}$ we define the annihilation $B_j(t)$, creation $B_j^\dagger(t)$, and number $\Lambda_{ij}(t)$ processes as follows [11, 12]:

$$B_j(t)e(f) \;=\; \int_0^t f_j(s)\mathrm{d}s \; e(f), \tag{15}$$

$$B_j^\dagger(t)e(f) \;=\; \frac{\partial}{\partial\epsilon_j} e\big(f + \epsilon\chi_{[0,t)}\big)\bigg|_{\epsilon=0}\,, \tag{16}$$

$$\Lambda_{ij}(t)e(f) \;=\; -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda} e\big(\exp\big(\mathrm{i}\lambda P_{ij}\chi_{[0,t)}\big)f\big)\bigg|_{\lambda=0}\,, \tag{17}$$

where $\chi_{[0,t)}$ is the indicator function of $[0,t)$, $\epsilon \equiv (\epsilon_1,\ldots,\epsilon_n) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $(P_{ij}f)_k = \delta_{ik}f_j$. The operators $B_j(t)$, $B_j^\dagger(t)$, $\Lambda_{ij}(t)$ satisfy the commutation relations of the form

$$[B_i(t), B_j(t')] = [B_i^\dagger(t), B_j^\dagger(t')] = 0, \;\; [B_i(t), B_j^\dagger(t')] = \delta_{ij}\, t \wedge t', \tag{18}$$

$$[\Lambda_{ij}(t), \Lambda_{kl}(t')] = \delta_{jk}\Lambda_{il}(t \wedge t') - \delta_{il}\Lambda_{kj}(t \wedge t'), \tag{19}$$

$$[B_j(t), \Lambda_{kl}(t')] = \delta_{jk}\, B_l(t \wedge t'), \;\; [\Lambda_{kl}(t), B_j^\dagger(t')] = \delta_{lj}\, B_k^\dagger(t \wedge t'), \tag{20}$$

where $t \wedge t' = \min(t,t')$. The operators $B_j(t)$, $B_j^\dagger(t)$, and $\Lambda_{ij}(t)$ can be written as

$$B_j(t) \;=\; \int_0^t b_j(s)\mathrm{d}s, \;\; B_j^\dagger(t) = \int_0^t b_j^\dagger(s)\mathrm{d}s, \;\; \Lambda_{ij}(t) = \int_0^t b_i^\dagger(s) b_j(s)\mathrm{d}s, \tag{21}$$

where $b_j(t)$, $b_j^\dagger(t)$ satisfy the canonical commutation relations

$$[b_j(t), b_i(t')] = [b_j^\dagger(t), b_i^\dagger(t')] = 0, \quad [b_j(t), b_i^\dagger(t')] = \delta_{ji}\delta(t-t'). \tag{22}$$

The Fock space $\mathcal{F}$ has a continuous tensor product structure, i.e.

$$\mathcal{F} = \mathcal{F}_{[0,t)} \otimes \mathcal{F}_{[t,\infty)}, \tag{23}$$

where $\mathcal{F}_{[0,t)}$ and $\mathcal{F}_{[t,\infty)}$ are the symmetric Fock spaces over $\mathbb{C}^n \otimes L^2([0,t))$ and $\mathbb{C}^n \otimes L^2([t,\infty))$, respectively. The family $\{M(t),\, t \geq 0\}$ of operators on $\mathcal{H}\otimes\mathcal{F}$ is called a quantum adapted process if $M(t)$ acts as the identity operator in $\mathcal{F}_{[t,\infty)}$ and can act non-trivially in $\mathcal{H}\otimes\mathcal{F}_{[0,t)}$. Hudson and Parthasarathy gave a rigorous meaning to the quantum stochastic differential equation (QSDE) of the type [11, 12]

$$\mathrm{d}M(t) \;=\; \sum_{j=1}^{n}\left(\sum_{i=1}^{n} F_{ji}(t)\,\mathrm{d}\Lambda_{ji}(t) + E_j(t)\,\mathrm{d}A_j(t) + D_j(t)\,\mathrm{d}A_j^\dagger(t)\right) + C(t)\mathrm{d}t\,, \tag{24}$$

where $M(t)$, $F_{ji}(t)$, $E_j(t)$, $D_j(t)$, $C(t)$ are adapted processes on $\mathcal{H}\otimes\mathcal{F}$. The increments $\mathrm{d}B_j(t) = B_j(t+dt) - B_j(t)$, $\mathrm{d}B_j^\dagger(t) = B_j^\dagger(t+dt) - B_j^\dagger(t)$, $\mathrm{d}\Lambda_{ij}(t) = \Lambda_{ij}(t+dt) - \Lambda_{ij}(t)$ commute with any adapted process $N(t)$ in $\mathcal{H}\otimes\mathcal{F}$. If $M'(t)$ is the process which satisfies an equation of the type (24), then the differential of the product $M(t)M'(t)$ is given by the formula

$$\mathrm{d}\big(M(t)M'(t)\big) = \mathrm{d}M(t)M'(t) + M(t)\mathrm{d}M'(t) + \mathrm{d}M(t)\mathrm{d}M'(t)\,, \tag{25}$$

where $\mathrm{d}M(t)\,\mathrm{d}M'(t)$ can be computed with the help of the multiplication table

$$\mathrm{d}B_i(t)\,\mathrm{d}B_j^\dagger(t) = \delta_{ij}\,\mathrm{d}t\,, \quad \mathrm{d}B_i(t)\,\mathrm{d}\Lambda_{kj}(t) = \delta_{ik}\,\mathrm{d}B_j(t)\,,$$

$$\mathrm{d}\Lambda_{kj}(t)\,\mathrm{d}B_i^\dagger(t) = \delta_{ji}\,\mathrm{d}B_k^\dagger(t)\,, \quad \mathrm{d}\Lambda_{ij}(t)\,\mathrm{d}\Lambda_{kl}(t) = \delta_{jk}\,\mathrm{d}\Lambda_{il}(t), \tag{26}$$

and all other products vanish. From here on, we will restrict our considerations to the unidirectional bosonic field, i.e. we will assume that $\mathcal{K} = \mathbb{C}\otimes L^2(\mathbb{R}_+)$.

We assume that the Bose field interacts with a quantum system (we will briefly call it a system $\mathcal{S}$). Let $\mathcal{H}_{\mathcal{S}}$ be the Hilbert space associated with $\mathcal{S}$ and $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$ be the space of linear bounded operators on $\mathcal{H}_{\mathcal{S}}$. For simplicity of notation, we will usually omit a tensor multiplication by identity operators. The evolution of the composed system consisting of $\mathcal{S}$ and the Bose field is described by the unitarity operator $U(t)$, which satisfies the quantum stochastic differential equation of the Itô type [11, 12, 17].

$$\begin{aligned} &\mathrm{d}U(t) = \left[L\mathrm{d}B^{\dagger}(t) - L^{\dagger}S\mathrm{d}B(t) + (S-I)\,d\Lambda(t) - \left(\mathrm{i}H_{\mathcal{S}} + \frac{1}{2}L^{\dagger}L\right)\mathrm{d}t\right]U(t), \\ &U(0) = I, \end{aligned} \tag{27}$$

where L, S, $H_{\mathcal{S}}$ belong to $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$, $H_{\mathcal{S}}$ stands for the Hamiltonian of $\mathcal{S}$, and S is the unitary operator describing a process of a direct scattering of light by the system $\mathcal{S}$. From a physical point of view, (27) is the equation for the evolution operator of the composed system written in the interaction picture eliminating the free evolution of the Bose field. The unitary operator $U(t)$ acts non-trivially in $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{F}_{[0,t)}$ and it commutes with the increments $dB(t)$, $dB^{\dagger}(t)$, $d\Lambda(t)$. According to the interpretation given by Gardiner and Collet [16], the operators $B(t)$, $B^{\dagger}(t)$, and $\Lambda(t)$ describe the input field, i.e. the field before the interaction with the system $\mathcal{S}$.

Any bounded operator of $\mathcal{S}$ in the Heisenberg picture

$$j_t(X) = U^{\dagger}(t)(X \otimes \mathbb{1})U(t), \tag{28}$$

is an adapted process, meaning that it acts as an identity operator in the space $\mathcal{F}_{[t,+\infty)}$. To determine the differential equation for $j_t(X)$ we apply the rules of the quantum stochastic calculus. In this way we obtain the equation of the form

$$dj_t(X) = j_t\left(\mathcal{L}^* X\right)dt + j_t\left(S^{\dagger}[X,L]\right)dB_t^{\dagger} + j_t\left([L^{\dagger},X]S\right)dB_t + j_t(S^{\dagger}XS - X)d\Lambda_t, \tag{29}$$

where

$$\mathcal{L}^* X = i[H_{\mathcal{S}}, X] + L^{\dagger}XL - \frac{1}{2}L^{\dagger}LX - \frac{1}{2}XL^{\dagger}L\,. \tag{30}$$

The initial state of the composed system is assumed to be the product state

$$\rho(0) \otimes \rho_{field}. \tag{31}$$

where $\rho(0)$ is the initial state of $\mathcal{S}$ and ρ_{field} is the state of the input field. The reduced density operator of $\mathcal{S}$ is defined by the partial trace as

$$\varrho(t) = \mathrm{Tr}_{\mathcal{F}}\left(U(t)\rho(0) \otimes \rho_{field}U^{\dagger}(t)\right). \tag{32}$$

One can check that when the input Bose field is prepared in the vacuum state, then the operator $\rho(t)$ satisfies the differential equation

$$\dot{\varrho}(t) = \mathcal{L}\varrho(t), \tag{33}$$

where

$$\mathcal{L}\varrho = -i[H_{\mathcal{S}}, \varrho] + L\varrho L^{\dagger} - \frac{1}{2}L^{\dagger}L\varrho - \frac{1}{2}\varrho L^{\dagger}L\,. \tag{34}$$

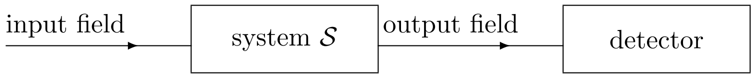

Figure 1: Measurement scheme for output field photon counting

The field after the interaction with the system $\mathcal{S}$, called the output field, is given as

$$B^{out}(t) = U^\dagger(t)B(t)U(t), \quad B^{out\dagger}(t) = U^\dagger(t)B^\dagger(t)U(t), \quad \Lambda^{out}(t) = U^\dagger(t)\Lambda(t)U(t). \tag{35}$$

Making use of the quantum stochastic calculus one can shown that

$$dB^{out}(t) = j_t(S)dB(t) + j_t(L)dt, \quad dB^{out\dagger}(t) = j_t(S^\dagger)dB^\dagger(t) + j_t(L^\dagger)dt, \tag{36}$$

$$d\Lambda^{out}(t) = d\Lambda(t) + j_t(L^\dagger S)dB(t) + j_t(S^\dagger L)dB(t)^\dagger + j_t(L^\dagger L)dt. \tag{37}$$

Note that the output field carries information about the system $\mathcal{S}$ and for this reason the field can be treated as the measurement apparatus. In the quantum filtering theory we consider a continuous in time measurement of the output field. In the basic version of the model, it is assumed that we use instantaneous detectors with 100% efficiency. The two most commonly considered measurement schemes are the measurement of $\Lambda^{out}(t)$, in which the photons of the output field are counted directly, and the measurement of optical quadrature

$$Y(t) = B^{out}(t) + dB^{out\dagger}(t). \tag{38}$$

The measurement of the number observable is illustrated in Figure 1. A schematic of heterodyne detection can be found in Figure 2. In the latter case, the output field goes to the beam-splitter and at the other input of the beam-splitter we have a strong field in the coherent state [17,93,94].

The process $\{\Lambda^{\mathrm{out}}(t), t \geq 0\}$ is called self-nondemolition, it means that the operators of this family commute with each other [3, 4]:

$$[\Lambda^{out}(t), \Lambda^{out}(t')] = 0, \quad \forall t, \, t' \geq 0 \tag{39}$$

and therefore there exists a joint spectral measure for them. The process $\{Y(t), t \geq 0\}$ also satisfies the self-nondemolition condition. The measurements of these observables are nondemolition for the reason that [3, 4]

$$[\Lambda^{out}(t'), U^\dagger(t)(X \otimes \mathbb{1})U(t)] = 0, \quad [Y(t'), U^\dagger(t)(X \otimes \mathbb{1})U(t)] = 0, \qquad 0 \leq t' \leq t \tag{40}$$

for any operator X of the system $\mathcal{S}$. According to (40), the measurement of a given output process disturbs neither the present nor the future state of the system $\mathcal{S}$.

Quantum filtering theory describes the estimation of the state of $\mathcal{S}$ based on the results of observations of the output processes. Here, the filtering equations for the two types of observations are presented. They correspond to the situation when $S = \mathbb{1}$ and the input boson field is

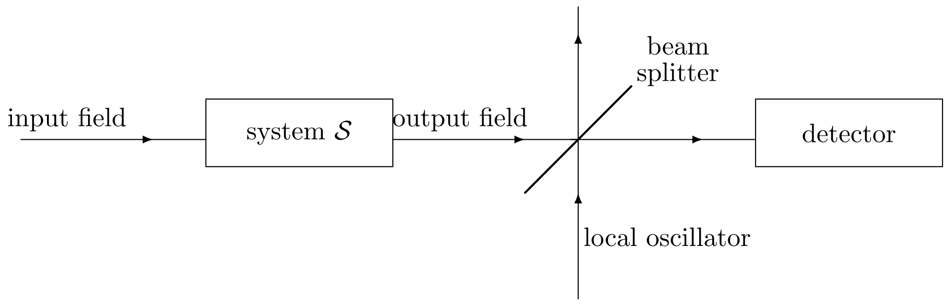

Figure 2: Optical field quadrature detection scheme

in the vacuum state. For the observation of the process $\Lambda^{out}(t)$, we obtain the filtering equation of the form

$$d\rho(t) = \mathcal{L}\rho(t) + \left(\frac{L\rho(t)L^\dagger}{\mathrm{Tr}\left\{L^\dagger L\rho(t)\right\}} - \rho(t) \right) dN(t), \tag{41}$$

where $\mathcal{L}$ is the superoperator defined by (34), $N(t)$ is the counting process such that

$$dN(t) = d\Lambda^{out}(t) - \mathrm{Tr}\{L^\dagger L\rho(t)\}dt. \tag{42}$$

The conditional expectation of the increment $d\Lambda^{out}(t)$ given all results of the measurement up to t is

$$\mathbb{E}[d\Lambda^{out}(t)|\rho(t)] = \mathrm{Tr}\{L^\dagger L\rho(t)\}dt. \tag{43}$$

For the observation of the optical quadrature, we get the stochastic differential equation

$$d\rho(t) = \mathcal{L}\rho(t) + \Big(L\rho(t) + \rho(t)L^\dagger - \rho(t)\mathrm{Tr}\{(L+L^\dagger)\rho(t)\} \Big) dW(t), \tag{44}$$

where $W(t)$ is the Wiener process related to $Y(t)$ by the formula

$$dW(t) = dY(t) - \mathrm{Tr}\{(L+L^\dagger)\rho(t)\}dt. \tag{45}$$

The conditional mean value of $dY(t)$ reads as

$$\mathbb{E}[dY(t)|\rho(t)] = \mathrm{Tr}\{(L+L^\dagger)\rho(t)\}dt. \tag{46}$$

The conditional state $\rho(t)$ depends on the measurement results of a given process up to time t and it is called the *a posteriori* state. Solutions to the filtering equations are also known as quantum trajectories. Averaging the state of $\rho(t)$ over all possible realizations of the considered stochastic process up to time t yields the *a priori* state of $\mathcal{S}$ that satisfies the master equation (33).

Approximations

Let us consider a quantum system $\mathcal{S}$, which interacts with a propagating electromagnetic field. We describe here the interaction with the one-dimensional boson field running in one direction. In this review $\hbar = 1$. The system $\mathcal{S}$ can be an atom, an ion, a field in a resonant cavity, we will not specify this here. The Hamiltonian of a composed system consisting of the system $\mathcal{S}$ and its environment $\mathcal{E}$ can be written as

$$H = H_{\mathcal{S}} + H_{\mathcal{E}} + H_{int}, \tag{47}$$

where $H_{\mathcal{S}}$ is the Hamiltonian of $\mathcal{S}$, $H_{\mathcal{E}}$ is the Hamiltonian of the bosonic field, and H_{int} represents the Hamiltonian of the interaction between systems. The Hamiltonian generating the free field evolution has the form

$$H_{\mathcal{E}} = \int_0^{+\infty} \omega b^\dagger(\omega) b(\omega) d\omega, \tag{48}$$

where $b(\omega)$ and $b^\dagger(\omega)$ are respectively the annihilation and creation operators given in the frequency domain. These operators satisfy the canonical commutation relations

$$[b(\omega), b(\omega')] = [b^\dagger(\omega), b^\dagger(\omega')] = 0, \quad [b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega'). \tag{49}$$

We assume that the Hamiltonian of the interaction has the linear form

$$H_{int} = i \int_0^{+\infty} \kappa(\omega) \left[b^\dagger(\omega) + b(\omega)\right] \left[R - R^\dagger\right] d\omega, \tag{50}$$

where R is the system operator and $\kappa(\omega)$ is a real function describing the coupling between the systems. Making the rotating wave approximation (RWA) and transforming the Hamiltonian into the interaction picture with respect to free dynamics the field we obtain from (50)

$$\tilde{H}_{int}(t) = i \int_0^{+\infty} \kappa(\omega) \left[R b^\dagger(\omega) e^{i\omega t} - R^\dagger b(\omega) e^{-i\omega t}\right] d\omega. \tag{51}$$

We assume that we deal with a narrowband field, i.e. the frequency spread $\Delta\omega$ (bandwidth) is much smaller compared to the central frequency ω_c. Such field is called the *quasi-monochromatic*. The central frequency is located near the characteristic frequency ω_0 of the system $\mathcal{S}$. We make the flat spectrum approximation thus we replace $\kappa(\omega)$ by a constant value. The last approximation consists of extend the lower limit of integration to infinity. In this way we obtain the interaction Hamiltonian

$$\tilde{H}_{int}(t) = i\sqrt{\Gamma}(R b^\dagger(t) - R^\dagger b(t)), \tag{52}$$

where

$$b(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} b(\omega) e^{-i\omega t} d\omega. \tag{53}$$

The operators $b(t)$ and $b^\dagger(\omega)$ are called white noise operators due to the fact that they satisfy the singular commutation relation

$$[b(t), b^\dagger(t')] = \delta(t - t'). \tag{54}$$

Hence the evolution operator has the form

$$U(t) = \overleftarrow{T} \exp \left[\int_0^t \left(-iH_{\mathcal{S}} + \sqrt{\Gamma} \left(Rb^\dagger(s) - R^\dagger b(s) \right) \right) ds \right], \tag{55}$$

where $\overleftarrow{T}$ is the time-ordering operator.

Collision models in quantum optics

The evolution of an open system can be studied in a convenient and intuitive way using collision models [82,88]. Here we consider a system $\mathcal{S}$ interacting with an environment $\mathcal{E}$ modelled by a sequence of identical quantum systems. We assume that initially there is no correlation between the system $\mathcal{S}$ and its environment. Thus we write the initial state of the composed system $\mathcal{S}$ plus $\mathcal{E}$ as

$$\rho(0) \otimes \eta(0), \tag{56}$$

where $\rho(0)$ and $\eta(0)$ are respectively the states of $\mathcal{S}$ and its environment at time zero. In the basic version of the collision model, we assume that the quantum systems constituting the environment do not interact with each other. We envisage the dynamics of a composed system as a series of interactions (collisions) of $\mathcal{S}$ with subsequent elements of the environment. It is assumed that each of them it interacts with $\mathcal{S}$ only once. Each collision (each interaction) has the same duration and it will be denoted by τ. The interaction in the time-interval from $k\tau$ to $(k+1)\tau$ is described by the unitarity operator V_k. The state of the composed system after n interactions is given as

$$\sigma_n = V_{n-1} \dots V_0(\rho(0) \otimes \eta)(V_{n-1} \dots V_0)^\dagger. \tag{57}$$

Eliminating the degrees of freedom associated with the environment, i.e. taking the partial trace over $\mathcal{E}$, we obtain the state of the system $\mathcal{S}$,

$$\varrho_n = \mathrm{Tr}_{\mathcal{E}}(\sigma_n). \tag{58}$$

If $\eta(0)$ is a factorized state and all sub-systems are prepared in the same state η, i.e. $\eta(0) = \eta^{\otimes n}$, then

$$\varrho_n = \Phi_n(\rho(0)) = \mathrm{Tr}_{\mathcal{E}_n}[V_{n-1} \left(\varrho_{n-1} \otimes \eta \right) V_{n-1}^\dagger], \tag{59}$$

where by $\mathcal{E}_n$ we denoted the n-th element of the environment. In this case, systems that will interact with $\mathcal{S}$ in the future are not correlated with it. Clearly, the system $\mathcal{S}$ is correlated with all systems which interacted with $\mathcal{S}$ in the past. It is easy to check that the (discrete) quantum map Φ_n, defined by (59), fulfils the semigroup property

$$\Phi_n = \Phi_{n-m} \circ \Phi_m \tag{60}$$

for any integer $0 \leq m \leq n$.

To obtain the collision model from the model described in the previous section, let us divide the time of interaction T into equal intervals, each of length τ, such that $T = N\tau$. The evolution operator of the composed system in the interval from $k\tau$ to $(k+1)\tau$, defined as

$$\overleftarrow{T} \exp \left[\int_{k\tau}^{(k+1)\tau} \left(-iH_{\mathcal{S}} + \sqrt{\Gamma} \left(Rb^\dagger(s) - R^\dagger b(s) \right) \right) ds \right], \tag{61}$$

will be approximated by

$$\exp\left[\int_{k\tau}^{(k+1)\tau}\left(-iH_{\mathcal{S}}+\sqrt{\Gamma}\left(Rb^{\dagger}(s)-R^{\dagger}b(s)\right)\right)ds\right]. \tag{62}$$

One can show that the difference between (61) and (62) is a quantity of the order $O(\tau^{3/2})$. When $\tau \to 0$ this difference goes to zero and we move from a discrete to a continuous-time description. The integral over the interaction operator can be written in the form

$$\int_{k\tau}^{(k+1)\tau}\sqrt{\Gamma}\left(Rb^{\dagger}(s)-R^{\dagger}b(s)\right)ds=\sqrt{\Gamma}\left(Rb_k^{\dagger}-R^{\dagger}b_k\right)\sqrt{\tau}, \tag{63}$$

where we introduced the field operators

$$b_k=\frac{1}{\sqrt{\tau}}\int_{k\tau}^{(k+1)\tau}b(s)ds. \tag{64}$$

Due to (54), the discrete operators satisfy the commutation rules

$$[b_n,b_m]=[b_n^{\dagger},b_m^{\dagger}]=0,\quad [b_n,b_m^{\dagger}]=\delta_{nm}. \tag{65}$$

The interaction from $k\tau$ to $(k+1)\tau$ is described by the operators

$$\sqrt{\frac{\Gamma}{\tau}}\left(Rb_k^{\dagger}-R^{\dagger}b_k\right). \tag{66}$$

4.2.4 Filtering equations and quantum trajectories for an open system interacting with a field in a single-photon state

I start the review of the publications [H1-H7] by presenting the results for the Bose field in the single-photon state. In the paper **[H1]** an unidirectional boson field is modelled by an infinite chain of non-interacting with themselves qubits. A formulation of the discrete quantum stochastic calculus for a field modelled by a sequence of two-level systems can be found in the [65,66,86,87]. Using the discrete approximation of the Fock space for the field modelled by a sequence of qubits, it is not only possible to determine the stochastic evolution for the field in the vacuum state [67,69,84–87], but also in the coherent, squeezed and thermal states [74]. In [H1], the qubit chain is prepared in an entangled state that is a discrete analogue of the continuous-mode single-photon state [37,39]. It is assumed that environmental qubits do not interact with each other, but they interact successively with the quantum system $\mathcal{S}$. Each of the bath qubits interacts with $\mathcal{S}$ only once. The initial state of the total system is a product state, i.e. initially there is no correlation between the quantum system and the field. The stochastic evolution of the system $\mathcal{S}$ was determined assuming that the measurement is performed on the environment qubits just after their interaction with the system $\mathcal{S}$. The entanglement of qubits indicates the presence of nonclassical temporal correlations of the input field and it is the reason of the non-Markovianity of evolution of the open system. Starting from the discrete description, the model of the continuous in time measurement and evolution of $\mathcal{S}$ was obtained in the limit, when the time of the interaction of individual qubits with $\mathcal{S}$ tends to zero. The filtering equations were obtained for two types of the measurement of the output field: photodetection and heterodyne measurement. In the paper the formulae for the quantum trajectories for the counting process and the photon counting probabilities which define the whole statistics of photons in the output field were determined. Moreover, an intuitive and rigorous interpretation to the quantum trajectories was given. The stochastic equations derived in this paper are consistent with the results obtained [43, 48, 57] and [P2].

The paper [H1] considers a quantum system $\mathcal{S}$ of the Hilbert space $\mathcal{H}_{\mathcal{S}}$ interacting with the environment consisting of a sequence of two-level systems which interact in turn one by one with the system $\mathcal{S}$ each during the time interval of the length τ. The Hilbert space of the environment is

$$\mathcal{H}_{\mathcal{E}} = \bigotimes_{k=0}^{+\infty} \mathcal{H}_{\mathcal{E},k}, \tag{67}$$

where $\mathcal{H}_{\mathcal{E},k} = \mathbb{C}^2$ is the Hilbert space of the qubit interacting with $\mathcal{S}$ in the time interval $[k\tau, (k+1)\tau)$. Let us notice that the Hilbert space $\mathcal{H}_{\mathcal{E}}$ can be split as a tensor product

$$\mathcal{H}_{\mathcal{E}} = \mathcal{H}_{\mathcal{E}}^{j-1]} \otimes \mathcal{H}_{\mathcal{E}}^{[j}, \quad \mathcal{H}_{\mathcal{E}}^{j-1]} = \bigotimes_{k=0}^{j-1} \mathcal{H}_{\mathcal{E},k}, \quad \mathcal{H}_{\mathcal{E}}^{[j} = \bigotimes_{k=j}^{+\infty} \mathcal{H}_{\mathcal{E},k}. \tag{68}$$

If $j\tau$ is the current moment then $\mathcal{H}_{\mathcal{E}}^{j-1]}$ can be interpreted as the part of the space of the environment which refers to two-level systems which have already interacted with $\mathcal{S}$ and $\mathcal{H}_{\mathcal{E}}^{[j}$ as the space referring to two-level systems which have not interacted with $\mathcal{S}$ yet. The ground and excited states of the qubit of number k we will denote respectively by $|0\rangle_k$ and $|1\rangle_k$. It was assumed that the environment is prepared in the state

$$|1_\xi\rangle = \sum_{k=0}^{+\infty} \sqrt{\tau}\xi_k \sigma_k^+ |vac\rangle, \tag{69}$$

where $|vac\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \ldots$ is the vacuum vector defined in $\mathcal{H}_{\mathcal{E}}$, the operators $\sigma_k^- = |0\rangle_k\langle 1|$, $\sigma_k^+ = |1\rangle_k\langle 0|$ act non-trivially only in the space $\mathcal{H}_{\mathcal{E},k}$, and $\sum_{k=0}^{+\infty}|\xi_k|^2\tau = 1$. In order to simplify the notation we omit a multiplication by identity operators. Note that $|1_\xi\rangle$ has the additive decomposition property

$$|1_\xi\rangle = \sqrt{\tau}\sum_{k=0}^{j} \xi_k \sigma_k^+ |vac\rangle + \sqrt{\tau}\sum_{k=j+1}^{+\infty} \xi_k \sigma_k^+ |vac\rangle \tag{70}$$

and one can check the identities

$$\sigma_k^- |1_\xi\rangle = \sqrt{\tau}\xi_k |vac\rangle, \quad \sigma_k^+ \sigma_k^- |1_\xi\rangle = \sqrt{\tau}\xi_k |1_k\rangle. \tag{71}$$

The vector $|1_\xi\rangle$ can be written in the form

$$|1_\xi\rangle = \sum_{k=0}^{+\infty} \sqrt{\tau}\xi_k |1_k\rangle, \tag{72}$$

where $|1_k\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \ldots |0\rangle_{k-1} \otimes |1\rangle_k \otimes |0\rangle_{k+1} \otimes |0\rangle_{k+2} \ldots$. Thus $|\xi_k|^2\tau$ is the probability that qubit of number k is prepared in the upper state and all the others qubits are in their ground states. Let us stress that the environmental state $|1_\xi\rangle$ is highly entangled (depending on the profile ξ_k). The state $|1_\xi\rangle$ is a discrete analogue of a single-photon state in the model of continuous modes [37, 39, 40]

$$|1_\xi\rangle = \int_0^{+\infty} \xi(t) dB_t^\dagger |vac\rangle \tag{73}$$

with $\xi \in \mathbb{C}$ and the normalization $\langle 1_\xi | 1_\xi \rangle = \int_0^\infty |\xi(t)|^2 dt = 1$. In the model of repeated interactions the unitary operator defining the evolution of the total system up to the time $j\tau$ is given by

$$U_{j\tau} = \mathbb{V}_{j-1}\mathbb{V}_{j-2}\dots\mathbb{V}_0, \quad U_0 = \mathbb{1}, \tag{74}$$

where $\mathbb{V}_k$ acts non-trivially only in the space $\mathcal{H}_{\mathcal{E},k} \otimes \mathcal{H}_\mathcal{S}$ and $\mathbb{V}_k = \exp(-i\tau H_k)$, where H_k is a bipartite Hamiltonian acting on $\mathcal{H}_{\mathcal{E},k} \otimes \mathcal{H}_S$ such that

$$H_k = H_\mathcal{S} + \frac{i}{\sqrt{\tau}} \left(\sigma_k^+ \otimes L - \sigma_k^- \otimes L^\dagger \right). \tag{75}$$

We set $\hbar = 1$ throughout the text. The Hamiltonian H_k is written in the interaction picture eliminating the free evolution of the field. Here $H_\mathcal{S}$ stands for the Hamiltonian of $\mathcal{S}$, σ_k^+ and σ_k^- denote respectively the raising and lowering operators acting in $\mathcal{H}_{\mathcal{E},k}$. From the mathematical point of view, L is an arbitrary bounded operator on $\mathcal{H}_S$. One can consider $\mathcal{S}$ being a two-level atom and define L as $\sqrt{\Gamma}\sigma_-$, where Γ is a positive coupling constant and σ_- is the atom lowering operator. If $\mathcal{S}$ is a one-sided cavity, then $L = \sqrt{\Gamma}a$, where a is the annihilation operator of a cavity mode. Since $\mathcal{H}_{\mathcal{E},k} = \mathbb{C}^2$ one has the following representation

$$\mathbb{V}_k = \begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix}, \tag{76}$$

with V_{ij} being the system operators and one easily finds

$$V_{00} = \mathbb{1}_\mathcal{S} - i\tau H_\mathcal{S} - \tau\frac{1}{2}L^\dagger L + O(\tau^2), \quad V_{10} = \sqrt{\tau}L + O(\tau^{3/2}), \tag{77}$$

$$V_{01} = -\sqrt{\tau}L^\dagger + O(\tau^{3/2}), \quad V_{11} = \mathbb{1}_\mathcal{S} + O(\tau). \tag{78}$$

The initial state of the composed $\mathcal{E} + \mathcal{S}$ system is the pure product state of the form

$$|1_\xi\rangle \otimes |\psi\rangle. \tag{79}$$

The state of the composed system at time $j\tau$ (after j interaction) is thus given as $U_{j-1}|1_\xi\rangle \otimes |\psi\rangle$. Taking the partial trace over the environment, we obtain the reduced state of $\mathcal{S}$ at time $j\tau$:

$$\varrho_j = \mathrm{Tr}_\mathcal{E} \left[U_{j-1} |1_\xi\rangle\langle 1_\xi| \otimes |\psi\rangle\langle\psi| U_{j-1}^\dagger \right]. \tag{80}$$

It is assumed that that after each interaction a measurement is performed on the last qubit which has just interacted with $\mathcal{S}$. In the paper an evolution of $\mathcal{S}$ conditioned on the results of the measurements performed subsequently on the environment qubits at the time instances $\tau, 2\tau, 3\tau, \dots$ was determined. First the measurement of the observable

$$\sigma_k^+\sigma_k^- = |1\rangle_k\langle 1|, \quad k = 0, 1, 2, \dots \tag{81}$$

was considered. The following theorem was proved.

Theorem 1 *The conditional state of $\mathcal{S}$ and the part of the environment which has not interacted with $\mathcal{S}$ up to $j\tau$ for the measurement of (81) at the moment $j\tau$ is given by*

$$|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle = \frac{|\Psi_{j|\boldsymbol{\eta}_j}\rangle}{\sqrt{\langle\Psi_{j|\boldsymbol{\eta}_j}|\Psi_{j|\boldsymbol{\eta}_j}\rangle}}, \tag{82}$$

where $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ is the unnormalized conditional vector of the form

$$|\Psi_{j|\boldsymbol{\eta}_j}\rangle \quad = \quad \sqrt{\tau}\sum_{k=j}^{+\infty}\xi_k\sigma_k^+|vac\rangle_{[j,+\infty)}\otimes|\alpha_{j|\boldsymbol{\eta}_j}\rangle + |vac\rangle_{[j,+\infty)}\otimes|\beta_{j|\boldsymbol{\eta}_j}\rangle \tag{83}$$

where $\boldsymbol{\eta}_j$ is a binary j-vector $\boldsymbol{\eta}_j = (\eta_j, \eta_{j-1}, \ldots, \eta_1)$ with $\eta_k \in \{0,1\}$, which represents results of all measurements of (81) up to the time $j\tau$. We use the notation with $|vac\rangle_{[j,+\infty)} = |0\rangle_j \otimes |0\rangle_{j+1}\otimes\ldots$ The vectors $|\alpha_{j|\boldsymbol{\eta}_j}\rangle$, $|\beta_{j|\boldsymbol{\eta}_j}\rangle$ from the Hilbert space $\mathcal{H}_{\mathcal{S}}$ satisfy the recurrence stochastic equations

$$|\alpha_{j+1|\boldsymbol{\eta}_{j+1}}\rangle \quad = \quad V_{\eta_{j+1}0}|\alpha_{j|\boldsymbol{\eta}_j}\rangle, \tag{84}$$

$$|\beta_{j+1|\boldsymbol{\eta}_{j+1}}\rangle \quad = \quad V_{\eta_{j+1}0}|\beta_{j|\boldsymbol{\eta}_j}\rangle + \sqrt{\tau}\xi_j V_{\eta_{j+1}1}|\alpha_{j|\boldsymbol{\eta}_j}\rangle \tag{85}$$

and the initial condition $|\alpha_0\rangle = |\psi\rangle$, $|\beta_0\rangle = 0$.

The conditional vectors $|\alpha_{j|\boldsymbol{\eta}_j}\rangle$, $|\beta_{j|\boldsymbol{\eta}_j}\rangle$ depend on all results of the measurements of (81) up to the time $j\tau$. The vector $|\widetilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle$ indicates that the system $\mathcal{S}$ becomes entangled with this part of the environment which has not interacted with $\mathcal{S}$ yet. The physical interpretation of $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ is very intuitive. Namely, the first term in (83) represents the following scenario: all qubits of the environment up to time $j\tau$ were prepared in the ground state and the qubit prepared in the excited state appears only in the future. The second term represents the situation in which $\mathcal{S}$ has already interacted with the qubit prepared in the excited state and in the future it will interact with the environment being in the vacuum. Clearly, sooner or later, it depends on the profile ξ_k, the system $\mathcal{S}$ meets the qubit prepared in the excited state, so finally only the second term gives non-zero contribution to (83), and $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ becomes separable. The paper [H1] provides a general solution to the set of equations (84) and (85).

In order to obtain the state of $\mathcal{S}$ conditioned on all results of the measurements up to the time $j\tau$, one has to perform a partial trace of $|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle\langle\tilde{\Psi}_{j|\boldsymbol{\eta}_j}|$ with respect to the environment degrees of freedom (the future space of the environment). Thus the *a posteriori* state of $\mathcal{S}$ at the time $j\tau$ is

$$\tilde{\rho}_{j|\boldsymbol{\eta}_j} = \frac{\rho_{j|\boldsymbol{\eta}_j}}{\mathrm{Tr}\rho_{j|\boldsymbol{\eta}_j}}, \tag{86}$$

where

$$\rho_{j|\boldsymbol{\eta}_j} = |\alpha_{j|\boldsymbol{\eta}_j}\rangle\langle\alpha_{j|\boldsymbol{\eta}_j}|\sum_{k=j}^{+\infty}\tau|\xi_k|^2 + |\beta_{j|\boldsymbol{\eta}_j}\rangle\langle\beta_{j|\boldsymbol{\eta}_j}|. \tag{87}$$

The probability of a particular trajectory $\boldsymbol{\eta}_j$ registered from time 0 to $j\tau$ is therefore given by $\mathrm{Tr}\rho_{j|\boldsymbol{\eta}_j}$. η_k $(k = 1, 2, \ldots)$ are random variables with values $\{0, 1\}$ and hence one deals with the discrete stochastic process $(\eta_j, \eta_{j-1}, \ldots, \eta_1)$. A single realization of this process consists of zeros and ones. Let us simplify our notation by skipping the condition $\boldsymbol{\eta}_j$. The conditional expectations for η_{j+1} were determined:

$$\mathbb{E}[\eta_{j+1}|\tilde{\rho}_j] = k_j\tau + O(\tau^2), \quad \mathbb{E}[(\eta_{j+1})^2|\tilde{\rho}_j] = k_j\tau + O(\tau^2), \tag{88}$$

where

$$k_j = \mathrm{Tr}\left(L^\dagger L\tilde{\rho}_j + \frac{\xi_j^*}{\mathrm{Tr}\rho_j}\, L|\beta_j\rangle\langle\alpha_j| + \frac{\xi_j}{\mathrm{Tr}\rho_j}\, |\alpha_j\rangle\langle\beta_j|L^\dagger + \frac{|\xi_j|^2}{\mathrm{Tr}\rho_j}\, |\alpha_j\rangle\langle\alpha_j|\right). \tag{89}$$

They define expectation values that depend on all the results of measurements from previous moments. The paper defines a discrete stochastic process

$$n_j = \sum_{k=1}^{j} \eta_k \tag{90}$$

with the increment $\eta_{j+1} = n_{j+1} - n_j =: \Delta n_j$ having two possible values: 0 and 1. It was shown that the stochastic evolution of $\mathcal{S}$ corresponding to the process (90) is described by the set of coupled discrete filtering equations:

$$\begin{aligned}
\tilde{\rho}_{j+1} &= \tilde{\rho}_j - i[H_S, \tilde{\rho}_j]\tau - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_j\right\}\tau + L\rho_j L^\dagger\tau + [|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|, L^\dagger]\xi_j\tau + [L, |\tilde{\beta}_j\rangle\langle\tilde{\alpha}_j|]\xi_j^*\tau \quad (91)\\
&+ \left\{\frac{1}{k_j}\left(L\tilde{\rho}_j L^\dagger + L|\tilde{\beta}_j\rangle\langle\tilde{\alpha}_j|\xi_j^* + |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|L^\dagger\xi_j + |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j||\xi_j|^2\right) - \tilde{\rho}_j\right\}(\Delta n_{j+1} - k_j\tau),
\end{aligned}$$

$$\begin{aligned}
|\tilde{\alpha}_{j+1}\rangle\langle\tilde{\beta}_{j+1}| &= |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j| - i\left[H_S, |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\right]\tau - \frac{1}{2}\left\{L^\dagger L, |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\right\}\tau\\
&+ L|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|L^\dagger\tau + [L, |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|]\,\xi_j^*\tau\\
&+ \left\{\frac{1}{k_j}\left(L|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|L^\dagger + L|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|\xi_j^*\right) - |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\right\}(\Delta n_{j+1} - k_j\tau), \qquad (92)
\end{aligned}$$

$$\begin{aligned}
|\tilde{\alpha}_{j+1}\rangle\langle\tilde{\alpha}_{j+1}| &= |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j| - i\,[H_S, |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|]\,\tau - \frac{1}{2}\left\{L^\dagger L, |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|\right\}\tau + L|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|L^\dagger\tau\\
&+ \left\{\frac{1}{k_j}L|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|L^\dagger - |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|\right\}(\Delta n_{j+1} - k_j\tau) \qquad (93)
\end{aligned}$$

with the initial condition $\tilde{\rho}_0 = |\psi\rangle\langle\psi|$, $|\tilde{\alpha}_0\rangle\langle\tilde{\beta}_0| = 0$, and $|\tilde{\alpha}_0\rangle\langle\tilde{\alpha}_0| = |\psi\rangle\langle\psi|$. Here we have introduced $|\tilde{\alpha}_j\rangle = |\alpha_j\rangle\sqrt{\mathrm{Tr}\rho_j}$, $|\tilde{\beta}_j\rangle = |\beta_j\rangle\sqrt{\mathrm{Tr}\rho_j}$. These discrete stochastic equations specify the stochastic evolution of the system $\mathcal{S}$ for the measurement of (81). The operator $\tilde{\rho}_j$ is the normalised conditional state of the system $\mathcal{S}$. The paper simplifies the notation of vectors and conditional operators by dropping the subscript $\boldsymbol{\eta}_j$. In the continuous-time limit, $\tau \to 0$, one obtains from (91)-(93) the set of differential filtering equations

$$\begin{aligned}
d\tilde{\rho}_t &= -i[H_S, \tilde{\rho}_t]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t\right\}dt + L\tilde{\rho}_t L^\dagger dt + [\tilde{\rho}_t^{01}, L^\dagger]\xi_t dt + [L, \tilde{\rho}_t^{10}]\xi_t^* dt\\
&+ \left\{\frac{1}{k_t}\left(L\tilde{\rho}_t L^\dagger + L\tilde{\rho}_t^{10}\xi_t^* + \tilde{\rho}_t^{01}L^\dagger\xi_t + \tilde{\rho}_t^{00}|\xi_t|^2\right) - \tilde{\rho}_t\right\}(dn(t) - k_t dt), \qquad (94)
\end{aligned}$$

$$\begin{aligned}
d\tilde{\rho}_t^{01} &= -i[H_S, \tilde{\rho}_t^{01}]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{01}\right\}dt + L\tilde{\rho}_t^{01}L^\dagger dt + \left[L, \tilde{\rho}_t^{00}\right]\xi_t^* dt\\
&+ \left\{\frac{1}{k_t}\left(L\tilde{\rho}_t^{01}L^\dagger + L\tilde{\rho}_t^{00}\xi_t^*\right) - \tilde{\rho}_t^{01}\right\}(dn(t) - k_t dt), \qquad (95)
\end{aligned}$$

$$\begin{aligned} d\tilde{\rho}_t^{00} &= -i[H_S, \tilde{\rho}_t^{00}]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{00}\right\}dt + L\tilde{\rho}_t^{00}L^\dagger dt \\ &+ \left(\frac{1}{k_t}L\tilde{\rho}_t^{00}L^\dagger - \tilde{\rho}_t^{00}\right)(dn(t) - k_t dt)\,, \end{aligned} \tag{96}$$

where $\tilde{\rho}_t^{10} = (\tilde{\rho}_t^{01})^\dagger$ and the initial condition $\tilde{\rho}_0 = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{01} = 0$, and $\tilde{\rho}_0^{00} = |\psi\rangle\langle\psi|$. Here $n(t)$ is the continuous-time stochastic counting process describing the counting of the photons in the output field. For the increment $dn(t) = n(t+dt) - n(t)$, one gets the Itô rule $(dn(t))^2 = dn(t)$. The mean conditional value of $dn(t)$ was obtained

$$\mathbb{E}[dn(t)|\tilde{\rho}_t] = k_t dt, \tag{97}$$

where

$$k_t = \mathrm{Tr}\left(L^\dagger L\tilde{\rho}_t + L\tilde{\rho}_t^{10}\xi_t^* + \tilde{\rho}_t^{01}L^\dagger\xi_t + \tilde{\rho}_t^{00}|\xi_t|^2\right). \tag{98}$$

This is the conditional mean value of the number of photons in the interval from t to $t+dt$ dependent on the measurement results up to time t. In the time interval of the length dt, one can measure at most one photon.

For the non-selective measurement, we obtain from (94)-(96) *a priori* evolution of $\mathcal{S}$ given by

$$\dot{\tilde{\rho}}_t = -i[H_S, \tilde{\rho}_t] - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t\right\} + L\tilde{\rho}_t L^\dagger + [\tilde{\rho}_t^{01}, L^\dagger]\xi_t + [L, \tilde{\rho}_t^{10}]\xi_t^*, \tag{99}$$

$$\dot{\tilde{\rho}}_t^{01} = -i[H_S, \tilde{\rho}_t^{01}] - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{01}\right\} + L\tilde{\rho}_t^{01}L^\dagger + \left[L, \tilde{\rho}_t^{00}\right]\xi_t^*, \tag{100}$$

$$\dot{\tilde{\rho}}_t^{00} = -i[H_S, \tilde{\rho}_t^{00}] - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{00}\right\} + L\tilde{\rho}_t^{00}L^\dagger. \tag{101}$$

The paper [H1] also describes the statistics of photons in the output field. Let us note that all realization of the counting process $n(t)$ may be divided into disjoint sectors that contain realizations with exactly m counts at some moments $t_m > \ldots > t_2 > t_1 > 0$ and no other photons from 0 to t. Denote by $p_0^t(t_m, t_{m-1}, \ldots, t_2, t_1)$ the probability density of observing a particular trajectory corresponding to m counts at $t > t_m > \ldots > t_2 > t_1 > 0$ and no other photons from zero to t (called also an exclusive probability density). It was shown that

$$p_0^t(t_m, t_{m-1}, \ldots, t_2, t_1)dt_m dt_{m-1}\ldots dt_1 = ||\,|\alpha_{t|t_m,\ldots,t_1}\rangle\,||^2 \int_t^{+\infty} dt'|\xi_{t'}|^2 + ||\,|\beta_{t|t_m,\ldots,t_1}\rangle\,||^2.$$

Thus having in disposal the recipes for the conditional vectors, one can describe the whole statistics of photons in the output field. The probability of no counts up to time t is

$$P_0^t(0) = ||\,|\alpha_{t|\mathbf{0}_t}\rangle\,||^2 \int_t^{+\infty} dt'|\xi_{t'}|^2 + ||\,|\beta_{t|\mathbf{0}_t}\rangle\,||^2. \tag{102}$$

The probability of having exactly m counts up to time t reads

$$P_0^t(m) = \int_0^t dt_m \int_0^{t_m} dt_{m-1}\ldots\int_0^{t_2} dt_1 p_0^t(t_m, t_{m-1}, \ldots, t_2, t_1). \tag{103}$$

Introducing a non-Hermitian Hamiltonian $G = H_{\mathcal{S}} - \frac{i}{2}L^\dagger L$ and the corresponding (non-unitary) propagator $\mathbf{T}_t = e^{-iGt}$, one finds

$$|\alpha_{t|\mathbf{0}_t}\rangle = \mathbf{T}_t|\psi\rangle, \quad |\beta_{t|\mathbf{0}_t}\rangle = -\int_0^t dt'\mathbf{T}_{t-t'}\xi_{t'}L^\dagger\mathbf{T}_{t'}|\psi\rangle. \tag{104}$$

For a count at the time t' and no other counts in the interval from zero to t we have the conditional vectors

$$|\alpha_{t|t'}\rangle = \sqrt{dt'}\mathbf{T}_{t-t'}L\mathbf{T}_{t'}|\psi\rangle, \tag{105}$$

$$\begin{aligned} |\beta_{j|t'}\rangle \quad = \quad & \sqrt{dt'}\Big[\mathbf{T}_t\xi_{t'} - \mathbf{T}_{t-t'}L\left(\int_0^{t'} ds\, \mathbf{T}_{t'-s}\xi_s L^\dagger \mathbf{T}_s\right) \\ & - \left(\int_{t'}^t ds\mathbf{T}_{t-s}\xi_s L^\dagger\mathbf{T}_{s-t'}\right) L\mathbf{T}_{t'}\Big]|\psi\rangle. \end{aligned} \tag{106}$$

For two counts at t' and t'', where $0 < t' < t''$, and no other counts in the interval from zero to t, one gets

$$|\alpha_{t|t'',t'}\rangle = \sqrt{dt''dt'}\mathbf{T}_{t-t''}L\mathbf{T}_{t''-t'}L\mathbf{T}_{t'}|\psi\rangle, \tag{107}$$

$$\begin{aligned} |\beta_{t|t'',t'}\rangle \quad = \quad & \sqrt{dt''dt'}\Big[\mathbf{T}_{t-t''}L\mathbf{T}_{t''}\xi_{t'} + \mathbf{T}_{t-t'}\xi_{t''}L\mathbf{T}_{t'} \\ & -\mathbf{T}_{t-t''}L\mathbf{T}_{t''-t'}L\Big(\int_0^{t'} ds\mathbf{T}_{t'-s}\xi_s L^\dagger\mathbf{T}_s\Big) \\ & -\mathbf{T}_{t-t''}L\Big(\int_{t'}^{t''} ds\mathbf{T}_{t''-s}\xi_s L^\dagger\mathbf{T}_{s-t'}\Big)L\mathbf{T}_{t'} \\ & -\Big(\int_{t''}^{t} ds\mathbf{T}_{t-s}\xi_s L^\dagger\mathbf{T}_{s-t''}\Big)L\mathbf{T}_{t''-t'}L\mathbf{T}_{t'}\Big]|\psi\rangle. \end{aligned} \tag{108}$$

At first sight the above formulae seem to be complicated but the physical interpretation of individual terms are very intuitive. The term defined by the conditional vector $|\alpha_t\rangle$ gives the contribution to the probability of particular trajectory conditioned on the assumption that the two level system of the environment prepared in the upper state will appear after the time t, so all photons measured by us up to t were emitted by the system $\mathcal{S}$. The conditional vector $|\beta_t\rangle$ gives a contribution to the probability based on the assumption that the system $\mathcal{S}$ has already interacted with the two level system of the environment prepared in the upper state.

In the paper [H1] the measurement of the observable

$$\sigma_k^x = \sigma_k^+ + \sigma_k^- = |+\rangle_k\langle +| - |-\rangle_k\langle -|, \tag{109}$$

with $k = 0, 1, 2, \ldots$, and

$$|+\rangle_k = \frac{1}{\sqrt{2}}\left(|0\rangle_k + |1\rangle_k\right), \quad |-\rangle_k = \frac{1}{\sqrt{2}}\left(|0\rangle_k - |1\rangle_k\right), \tag{110}$$

being vectors from the Hilbert space $\mathcal{H}_{\mathcal{E},k}$, was also considered.

Theorem 2 *The conditional state of $\mathcal{S}$ and this part of the environment which has not interacted with $\mathcal{S}$ up to the time $j\tau$ at the moment $j\tau$ for the measurement of (109) can be written in the form of*

$$|\tilde{\Psi}_j\rangle = \frac{|\Psi_j\rangle}{\sqrt{\langle\Psi_j|\Psi_j\rangle}}, \tag{111}$$

where $|\Psi_j\rangle$ is the unnormalized conditional vector of the form

$$|\Psi_j\rangle = \sqrt{\tau}\sum_{k=j}^{+\infty}\xi_k\sigma_k^+|vac\rangle_{[j,+\infty)}\otimes|\alpha_j\rangle + |vac\rangle_{[j,+\infty)}\otimes|\beta_j\rangle \tag{112}$$

with the conditional vectors $|\alpha_j\rangle$, $|\beta_j\rangle$ which satisfy the following recurrence equations

$$|\alpha_{j+1}\rangle = \frac{1}{\sqrt{2}}\left(V_{00}+q_{j+1}V_{10}\right)|\alpha_j\rangle, \tag{113}$$

$$|\beta_{j+1}\rangle = \frac{1}{\sqrt{2}}\left[\left(V_{00}+q_{j+1}V_{10}\right)|\beta_j\rangle + \sqrt{\tau}\xi_j\left(V_{01}+q_{j+1}V_{11}\right)|\alpha_j\rangle\right] \tag{114}$$

with the initial condition $|\alpha_0\rangle = |\psi\rangle$, $|\beta_0\rangle = 0$. By $q_{j+1} = 1, -1$ we indicated the result of the measurement performed at $(j+1)\tau$ on the j-th qubits in the basis (110).

In the paper we defined the stochastic process

$$w_j = \sqrt{\tau}\sum_{k=1}^{j}\left(q_k - r_{k-1}\sqrt{\tau}\right), \tag{115}$$

where

$$r_j = \mathrm{Tr}\left(L\tilde{\rho}_j + \tilde{\rho}_jL^\dagger + |\tilde{\beta}_j\rangle\langle\tilde{\alpha}_j|\xi_j^* + |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\xi_j\right), \tag{116}$$

where $\tilde{\rho}_j$ is the *a posteriori* state of $\mathcal{S}$ at $j\tau$, $|\tilde{\alpha}_j\rangle = |\alpha_j\rangle/\sqrt{\mathrm{Tr}\rho_j}$, and $|\tilde{\beta}_j\rangle = |\beta_j\rangle/\sqrt{\mathrm{Tr}\rho_j}$. The mean conditional values $\mathbb{E}[q_k|\tilde{\rho}_{k-1}] \simeq r_{k-1}\sqrt{\tau}$, $\mathbb{E}[(q_k)^2|\tilde{\rho}_{k-1}] \simeq 1$. It was shown in the paper that the set of difference stochastic equations of $\mathcal{S}$ for the process (115) has the form

$$\begin{aligned}\tilde{\rho}_{j+1}-\tilde{\rho}_j &= -i[H_S,\tilde{\rho}_j]\tau - \frac{1}{2}\{L^\dagger L,\tilde{\rho}_j\}\tau + L\tilde{\rho}_jL^\dagger\tau + [|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|, L^\dagger]\xi_j\tau + [L,|\tilde{\beta}_j\rangle\langle\tilde{\alpha}_j|]\xi_j^*\tau \\ &+ \left(L\tilde{\rho}_j + \tilde{\rho}_jL^\dagger + |\tilde{\beta}_j\rangle\langle\tilde{\alpha}_j|\xi_j^* + |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\xi_j - \tilde{\rho}_jr_j\right)\Delta w_{j+1},\end{aligned} \tag{117}$$

$$\begin{aligned}|\tilde{\alpha}_{j+1}\rangle\langle\tilde{\beta}_{j+1}| &= |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j| - i[H_S,|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|]\tau - \frac{1}{2}\{L^\dagger L,|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|\}\tau \\ &+ L|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|L^\dagger\tau + [L,|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|]\xi_j^*\tau \\ &+ \left(L|\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j| + |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|L^\dagger + |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|\xi_j^* - |\tilde{\alpha}_j\rangle\langle\tilde{\beta}_j|r_j\right)\Delta w_{j+1},\end{aligned} \tag{118}$$

$$\begin{aligned}|\tilde{\alpha}_{j+1}\rangle\langle\tilde{\alpha}_{j+1}| &= |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j| - i[H_S,|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|]\tau - \frac{1}{2}\{L^\dagger L,|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|\}\tau + L|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|L^\dagger\tau \\ &+ \left(L|\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j| + |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|L^\dagger - |\tilde{\alpha}_j\rangle\langle\tilde{\alpha}_j|r_j\right)\Delta w_{j+1},\end{aligned} \tag{119}$$

where $\Delta w_{j+1} = w_{j+1} - w_j = q_{j+1}\sqrt{\tau} - r_j\tau$. The process w_j in the limit $\tau \to 0$ converges to the Wiener process. In the limit $\tau \to 0$ we obtain the following stochastic differential equations of the form

$$\begin{aligned} d\tilde{\rho}_t &= -i[H_S, \tilde{\rho}_t]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t\right\}dt + L\tilde{\rho}_t L^\dagger dt + [\tilde{\rho}_t^{01}, L^\dagger]\xi_t dt + [L, \tilde{\rho}_t^{10}]\xi_t^* dt \\ &+ \left(L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger + \tilde{\rho}_t^{01}\xi_t + \tilde{\rho}_t^{10}\xi_t^* - \tilde{\rho}_t r_t\right) dw(t), \end{aligned} \tag{120}$$

$$\begin{aligned} d\tilde{\rho}_t^{01} &= -i[H_S, \tilde{\rho}_t^{01}]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{01}\right\}dt + L\tilde{\rho}_t^{01} L^\dagger dt + \left[L, \tilde{\rho}_t^{00}\right]\xi_t^* dt \\ &+ \left(L\tilde{\rho}_t^{01} + \tilde{\rho}_t^{01} L^\dagger + \tilde{\rho}_t^{00}\xi_t^* - \tilde{\rho}_t^{01} r_t\right) dw(t), \end{aligned} \tag{121}$$

$$\begin{aligned} d\tilde{\rho}_t^{00} &= -i[H_S, \tilde{\rho}_t^{00}]dt - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{00}\right\}dt + L\tilde{\rho}_t^{00} L^\dagger dt \\ &+ \left(L\tilde{\rho}_t^{00} + \tilde{\rho}_t^{00} L^\dagger - \tilde{\rho}_t^{00} r_t\right) dw(t), \end{aligned} \tag{122}$$

where $r_t = \mathrm{Tr}\left(L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger + \tilde{\rho}_t^{10}\xi_t^* + \tilde{\rho}_t^{01}\xi_t\right)$, $\tilde{\rho}_t^{10} = (\tilde{\rho}_t^{01})^\dagger$, and initially we have $\tilde{\rho}_0 = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{01} = 0$, and $\tilde{\rho}_0^{00} = |\psi\rangle\langle\psi|$.

The results published in [H1] can be applied to various quantum systems interacting with the wave packet in the single-photon state. In the paper **[H2]**, as the system $\mathcal{S}$, I considered a two-level atom. Using the general formulae from [H1], I determined the conditional evolution of the two-level atom for the counting observation. The aim of the study was to show the connection between conditional and unconditional evolution and use the quantum trajectories to determine the statistics of photons in the output field. An important part of the paper was also a presentation of the interpretation of quantum trajectories for the case when the Bose field interacts with a two-level atom.

The *a priori* state of a quantum system interacting with the field in the single-photon state in the representation of the counting stochastic process $n(t)$ has the form

$$\tilde{\varrho}_t = \rho_{t|0} + \sum_{m=1}^{+\infty} \int_0^t dt_m \ldots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \rho_{t|t_m,\ldots,t_2,t_1}, \tag{123}$$

where $\rho_{t|t_m,\ldots,t_2,t_1}$ is a conditional operator defined by the

$$\rho_{t|cond} = |\alpha_{t|cond}\rangle\langle\alpha_{t|cond}| \int_t^{+\infty} ds|\xi_s|^2 + |\beta_{t|cond}\rangle\langle\beta_{t|cond}|, \tag{124}$$

where $|\alpha_{t|cond}\rangle$, $|\beta_{t|cond}\rangle$ are conditional vectors depending on the results of photon counts in the output field up to time t. The sum is taken over all possible pathways of the photon detection for the number of photons ranging from $m = 0$ to $m = \infty$ from the time 0 to t. The quantity $\mathrm{Tr}\rho_{t|0}$ is the probability of lack of any detections from 0 to t, while $\mathrm{Tr}\rho_{t|t_m,\ldots,t_2,t_1}$ for all $m \geq 1$ is the probability density of detecting photons at times $t_1, t_2, \ldots, t_m$ such that $0 < t_1 < t_2 < \ldots < t_m$ and no other photons from 0 to t. Clearly, for the two-level system, the sum has at most two non-zero terms. In the case where initially the atom is in an excited state, two photons will be eventually counted in the output field. The formula given in this paper is a generalisation of the well-known formula for solving the master equation for the field in the vacuum state, which can be found, for example, in [6].

I assumed in the paper [H2] that the system $\mathcal{S}$ is a two-level atom with the eigenstates $|g\rangle$ and $|e\rangle$, and

$$H_{\mathcal{S}} = -\Delta_0\sigma_z, \quad L = \sqrt{\Gamma}\sigma_-, \tag{125}$$

where $\Gamma \in \mathbb{R}_+$, $\sigma_- = |g\rangle\langle e|$, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, and $\Delta_0 = \omega_c - \omega_0$, where ω_0 is the frequency transition between atomic states, ω_c is the carrier frequency of the input wave packet. Using the formulae determined in [H1], I obtained the conditional vectors relating the absence of photons in the interval from 0 up to t of the form

$$|\alpha_{t|0}\rangle = \left(e^{-i\Delta_0 t}|g\rangle\langle g| + e^{\left(i\Delta_0-\frac{\Gamma}{2}\right)t}|e\rangle\langle e|\right)|\psi_0\rangle, \tag{126}$$

$$|\beta_{t|0}\rangle = -\sqrt{\Gamma}e^{\left(i\Delta_0-\frac{\Gamma}{2}\right)t}\int_0^t ds\xi_s e^{\left(-2i\Delta_0+\frac{\Gamma}{2}\right)s}|e\rangle\langle g|\psi_0\rangle, \tag{127}$$

where $|\psi_0\rangle$ is the initial state of the system. It is seen from the formulae that when the two-level atom was initially in the ground state and we have not observed any photons up to the time t, it means that the atom has not met the external photon yet and it is still in the ground state or it has already met the photon, it absorbed this photon and after the absorption the system stayed in the excited state up to t. If the atom was initially in the excited state and we have not observed any photons up to t, it implies that only one scenario has made real—the system has not met the external photon yet and it is still in the excited state (only the vector $|\alpha_{t|0}\rangle$ gives non-zero contribution to the conditional state of the system). Knowledge of the vectors (126), (127) allowed me to obtain the unnormalized conditional state of the two-level atom for not observing any photon up to the time t,

$$\begin{aligned}\rho_{t|0} &= \left(e^{-i\Delta_0 t}|g\rangle\langle g| + e^{\left(i\Delta_0-\frac{\Gamma}{2}\right)t}|e\rangle\langle e|\right)|\psi_0\rangle\langle\psi_0| \\ &\times \left(e^{i\Delta_0 t}|g\rangle\langle g| + e^{\left(-i\Delta_0-\frac{\Gamma}{2}\right)t}|e\rangle\langle e|\right)\int_t^{+\infty} ds|\xi_s|^2 \\ &+ \Gamma e^{-\Gamma t}\left|\int_0^t ds\xi_s e^{\left(-2i\Delta_0+\frac{\Gamma}{2}\right)s}\right|^2 |\langle\psi_0|g\rangle|^2|e\rangle\langle e|. \end{aligned}\tag{128}$$

By taking the trace of (128), I obtained the probability of not detecting any photons up to t,

$$\begin{aligned}P_t(0) &= \left(|\langle\psi_0|g\rangle|^2 + e^{-\Gamma t}|\langle\psi_0|e\rangle|^2\right)\int_t^{+\infty} ds|\xi_s|^2 \\ &+ \Gamma e^{-\Gamma t}|\langle\psi_0|g\rangle|^2\left|\int_0^t ds\xi_s e^{\left(-2i\Delta_0+\frac{\Gamma}{2}\right)s}\right|^2. \end{aligned}\tag{129}$$

The first expression refers to the possibility that the system will meet the photon after the time t, the second term is the probability of absorption of the photon before t and staying after this absorption in the excited state up to t. Of course, one can easily generalize this expression to the case of an arbitrary initial state of the system.

For a detection of one photon in the interval $[t_1, t_1 + dt)$ and no other photons from 0 to t, I obtained

$$|\alpha_{t|t_1}\rangle = \sqrt{\Gamma}e^{-i\Delta_0 t}e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)t_1}|g\rangle\langle e|\psi_0\rangle, \tag{130}$$

$$|\beta_{t|t_1}\rangle = e^{-i\Delta_0 t}\left[\left(\xi_{t_1} - \Gamma\int_0^{t_1} ds\xi_s e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)(t_1-s)}\right)|g\rangle\langle g|\right.$$
$$\left. +e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)t}\left(\xi_{t_1} - \Gamma\int_{t_1}^{t} ds\xi_s e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)(t_1-s)}\right)|e\rangle\langle e|\right]|\psi_0\rangle. \tag{131}$$

Thus when the two-level atom was initially prepared in the ground state, then we have $|\alpha_{t|t_1}\rangle = 0$ and the only two scenarios of events are possible. Namely, we detected the photon coming from the external field or the atom absorbed the photon before time t_1, then emitted it at interval $[t_1, t_1+dt)$, and stayed in the ground state up to t. These two scenarios are described respectively by the first and second terms of the formula for $|\beta_{t|t_1}\rangle$. If the atom was initially in the excited state, it might not meet the photon before t, and we observed the photon emitted by the system or the atom has met the external photon before t, then we detected it directly from the field or the atom emitted the photon at t_1, then absorbed the photon from the field, and after this stayed in the excited state up to t.

For detection of two photons at the intervals $[t_1, t_1 + dt)$ and $[t_2, t_2 + dt)$ such that $0 < t_1 < t_2 < t$ and no other photons from 0 to t, we have

$$|\alpha_{t|t_2,t_1}\rangle = 0, \tag{132}$$

$$\begin{aligned}|\beta_{t|t_2,t_1}\rangle &= \sqrt{\Gamma}e^{-i\Delta_0 t}e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)(t_1+t_2)}\left(\xi_{t_1}e^{-\left(2i\Delta_0-\frac{\Gamma}{2}\right)t_1} + \xi_{t_2}e^{-\left(2i\Delta_0-\frac{\Gamma}{2}\right)t_2}\right.\\ &- \left.\Gamma\int_{t_1}^{t_2} ds\xi_s e^{\left(-2i\Delta_0+\frac{\Gamma}{2}\right)s}\right)|g\rangle\langle e|\psi_0\rangle.\end{aligned} \tag{133}$$

Thus if we observed two photons we are certain that the system has already met the photon ($|\alpha_{t|t_2,t_1}\rangle = 0$). The terms in (133) correspond respectively to the following scenarios:

- the first photon came directly from the field and the second one was emitted by the atom,
- the first photon was emitted by the atom and the second one came from the field,
- the first photon was emitted by the atom, then the atom absorbed the photon from the field, and it emitted the photon at t_2.

All these possibilities we have to consider when the initial value of the probability of being in the excited state is non-zero. Of course, all the others conditional vectors vanish according to the fact that in our scheme we can not detect more than two photons.

Taking an average over all quantum trajectories, I have determined a general formula for the *a priori* state of an atom of the form

$$\begin{aligned}
\tilde{\varrho}_t &= \left[1-\Gamma e^{-\Gamma t}\left|\int_0^t ds\xi_s e^{\gamma s}\right|^2\right.\\
&\left.-\langle e|\rho_0|e\rangle e^{-\Gamma t}\left(1-4\Gamma\mathrm{Re}\int_0^t dt_1\xi_{t_1}^* e^{\gamma^* t_1}\int_0^{t_1} ds\xi_s e^{-\gamma^* s}\right)\right]|g\rangle\langle g|\\
&+\langle e|\rho_0|g\rangle e^{-\gamma t}\left(1-2\Gamma\int_0^t dt_1\xi_{t_1} e^{-\gamma^* t_1}\int_0^{t_1} ds\xi_s^* e^{\gamma^* s}\right)|e\rangle\langle g|\\
&+\langle g|\rho_0|e\rangle e^{-\gamma^* t}\left(1-2\Gamma\int_0^t dt_1\xi_{t_1}^* e^{-\gamma t_1}\int_0^{t_1} ds\xi_s e^{\gamma s}\right)|g\rangle\langle e|\\
&+\left[\Gamma e^{-\Gamma t}\left|\int_0^t ds\xi_s e^{\gamma s}\right|^2\right.\\
&\left.+\langle e|\rho_0|e\rangle e^{-\Gamma t}\left(1-4\Gamma\mathrm{Re}\int_0^t dt_1\xi_{t_1}^* e^{\gamma^* t_1}\int_0^{t_1} ds\xi_s e^{-\gamma^* s}\right)\right]|e\rangle\langle e|, \qquad (134)
\end{aligned}$$

where $\gamma=-2i\Delta_0+\frac{\Gamma}{2}$ and ρ_0 is the initial state of the atom. This is the general formula for the state of the two-level atom interacting with the single-photon wavepacket. It is seen that it depends on the initial state of the system and on the shape of the wavepackage. For the atom being initially in the ground state, we can get from (134) the known in the literature [45, 47] the probability of the excitation of the system

$$P(t)=\Gamma e^{-\Gamma t}\left|\int_0^t ds\xi_s e^{\gamma s}\right|^2. \qquad (135)$$

The decomposition of the reduced state into quantum trajectories benefits from formulae for the probability densities of photon counts in the output field. These formulae allow to fully characterize the photon statistics of the output field. In the paper [H2], I have also shown how conditional vectors can be used to derive formulae for the probability distributions of waiting times for photons in the output field. They allow, of course, to obtain the mean times of photon counts. The mean time of the first count is given by the formula

$$\bar{t}_1=\int_0^{+\infty} dt_1 t_1 p(t_1), \qquad (136)$$

where

$$\begin{aligned}
p(t_1) &= e^{-\Gamma t_1}\left(\Gamma\int_{t_1}^{+\infty} ds|\xi_s|^2+|\xi_{t_1}|^2\right)\langle e|\rho_0|e\rangle\\
&+\left|\xi_{t_1}-\Gamma\int_0^{t_1} ds\xi_s e^{\left(2i\Delta_0-\frac{\Gamma}{2}\right)(t_1-s)}\right|^2\langle g|\rho_0|g\rangle. \qquad (137)
\end{aligned}$$

For the mean time of the second count we have the formula

$$\bar{t}_2=\int_0^{+\infty} dt_2 t_2\int_0^{t_2} dt_1 p(t_2,t_1) \qquad (138)$$

with

$$p(t_2,t_1)=\Gamma e^{-\Gamma(t_1+t_2)}\left|\xi_{t_1} e^{-\left(2i\Delta_0-\frac{\Gamma}{2}\right)t_1}+\xi_{t_2} e^{-\left(2i\Delta_0-\frac{\Gamma}{2}\right)t_2}-\Gamma\int_{t_1}^{t_2} ds\xi_s e^{\left(-2i\Delta_0+\frac{\Gamma}{2}\right)s}\right|^2\langle e|\rho_0|e\rangle (139)$$

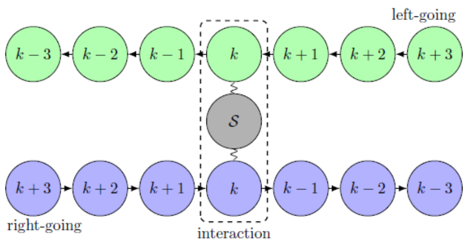

Figure 3: The system $\mathcal{S}$ interacts with a bidirectional field: the right-going single-photon pulse $|1_\xi\rangle$ and the left-going vacuum.

In the paper [H2], I also showed that the stochastic process, $n(t)$, describing photon counts in the output field, can be associated for a fixed time t with a positive-operator valued measure $\{M_{t|m}\}$ labelled by the number of photons recorded $m = 0, 1, 2$ up to time t. This allowed me to obtain a general formula for the mean number of photons recorded between moments 0 and t and the Mandel parameter, Q_t, for the output field. In the final section, the results are illustrated graphically for two pulses: rectangular and exponential with decreasing values.

The paper **[H3]** is devoted to the scattering of a wave packet in the single-photon state on a quantum system. The scattering of light on a quantum system is one of core issues in quantum optics. Many efforts have been put recently to describe the scattering of light prepared in the N-photon state on quantum systems. The aim of the paper [H3] was to show that the collision model and quantum trajectories provide convenient tools for analysing the scattering of light in non-classical states on quantum systems. I formulated the problem of interaction of quantum system $\mathcal{S}$ with a bidirectional boson field defined as two collections of "ancillas" which are two-level systems (qubits). One of these collections is prepared in an entangled state being a discrete analogue of the continuous-mode single photon state, and the second collection is prepared in the vacuum, which means that all its qubits are in the ground state. One can imagine that one chain describes the field going to the right and the second one refers to the field going to the left. I assumed that the qubits do not interact with each other but qubits of each chain interact subsequently with the system $\mathcal{S}$. At a given moment $\mathcal{S}$ interacts with only two qubits: one coming from the left and the other one coming from the right. Any interaction ("collision") has the same duration and each of the bath qubits interacts with the system only once. After the interaction with the system $\mathcal{S}$, a double measurement is performed: one on the field going to the right and the other on the field going to the left. The schematic of the collision dynamics is shown in Fig. 3.

Using the collision model and stochastic tools, I determined the photon counting statistics for a bidirectional output field. First, I determined analytical formulae for discrete quantum trajectories associated with a two-dimensional counting process to then obtained formulae for a continuous in time observation of the output field. These expressions have general form and can be used to characterise the scattering of light on different quantum systems.

As an example of application of the formulae obtained, I considered the light scattering on a two-level atom. In addition to a full characterisation of the number of photons counted for the light in both directions, the paper [H3] also contains expressions for the probability densities of times of photon detections. These formulae were determined for any photon profile and any initial state of the atom. Using the formulae for the conditional vectors, the probabilities of no photon counts up to a given moment, one photon count on the right and zero photons on the left, $P_R(t)$, and vice versa, one photon count on the left and zero photons on the right, $P_L(t)$, were derived. Moreover, the probability of two counts on the right, $P_{RR}(t)$, and the probability of two counts on the left, $P_{LL}(t)$, up to t were determined. And probabilities of detection from the left and then from the right, $P_{RL}(t)$, and respectively first from the right and then from the left, $P_{LR}(t)$, up to time t. The corresponding probability densities depend on the initial state of the atom, the profile of the photon, and the value of the detuning. On this basis, the mean number of counts on the left, $\langle N_L(t)\rangle$, and the mean number of counts on the right, $\langle N_R(t)\rangle$, up to a given moment were found. Example results for an exponentially decaying pulse are shown in Figure 4. Of course, these results also allow us to find asymptotic probabilities of photon transition and reflection, which are in agreement with the results obtained in the papers [95–98] by the other methods. In the paper [H3], I provided general formulae for the asymptotic values of the probabilities characterising the scattering for an arbitrary initial state of the atom and for a photon with a decaying exponential profile.

The effect of the single-photon field on the evolution of the two-level atom, in addition to the detuning magnitude, depends on the shape of the pulse and how long the pulse lasts, or more precisely on the relationship between the time of interaction of the pulse with the two-level system and the lifetime of the excited state of the atom due to spontaneous emission. The results of the paper [H3] show how by changing the parameters of the pulse one can influence the process of scattering. In order for a photon to be scattered in another direction, it must interact long enough with an atom to be absorbed beforehand. In [H3] I also determined expressions for the mean time of photon counts for the given profile.

The results of the paper [H3] can be directly used to determine statistics for the light scattering on the other quantum systems, for instance, on a cavity mode. The results also allow us to obtain a formula for the state of the output field. A comprehensive supplement to the paper [H3] contains the derivation of a set of filtering equations for a quantum system interacting with a bidirectional field.

4.2.5 Non-Markovian dynamics of a qubit interacting with a single-photon wavepacket

Any real quantum system is never perfectly isolated and hence has to be treated as an open system [8]. Therefore, its evolution is no longer unitary and gives rise to well-known processes of dissipation, decay, and decoherence induced by the nontrivial system–environment interaction. When the interaction between the system and the environment is sufficiently weak and the experimental conditions allow the use of the Markov approximation, the evolution of the open system is given by the master equation $\dot{\rho} = \mathcal{L}\rho$, with $\mathcal{L}$ being the GKSL generator

$$\mathcal{L}(\rho) = -i[H_S, \rho] + \sum_k \gamma_k \Big(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\Big), \tag{140}$$

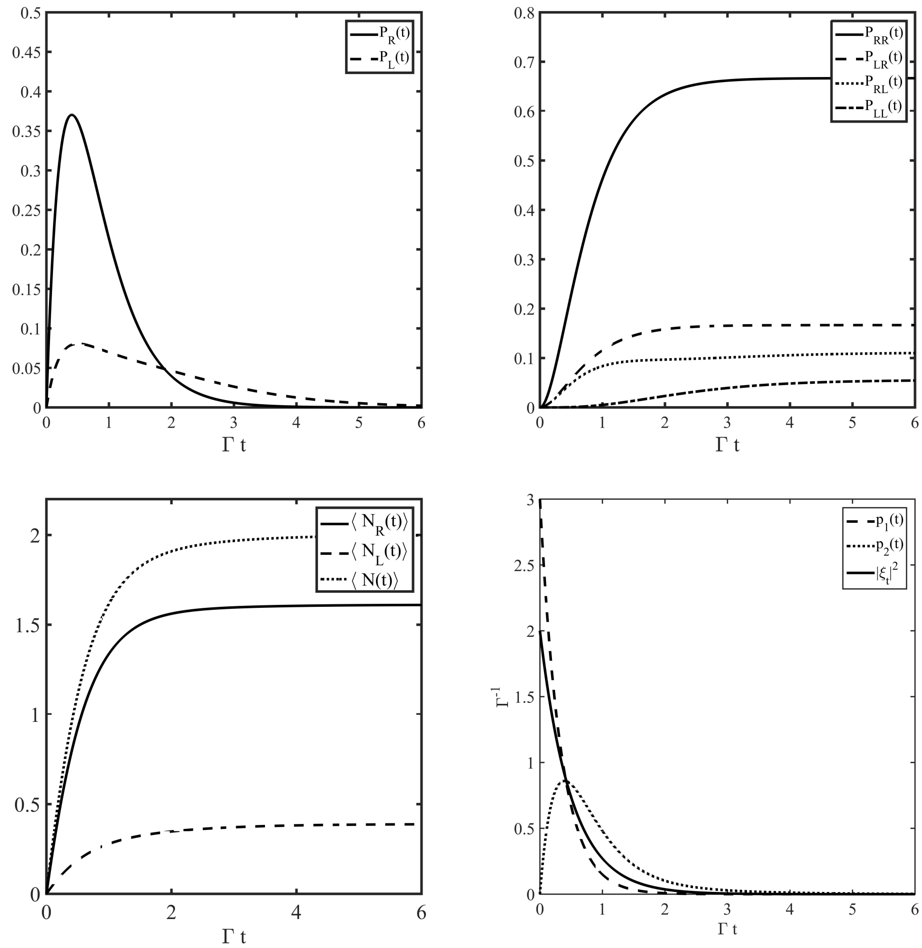

Figure 4: Photon counting characteristics of the output field for the decaying exponential pulse for the resonance case and chosen parameters together with probability densities of photon count times for the excited atom at the initial moment.

where H_S stands for the effective system's Hamiltonian (including Lamb shift correction), L_k are Lindblad operators, and $\gamma_k \geq 0$. In non-Markovian regime due to correlations between the system and environment the reduced evolution of the system is no longer governed by (140). One observes characteristic memory effects such as information bacflow or recoherence.

In the paper [**H4**], the non-Markovian dynamics of the two-level system (qubit) interacting with a single-photon wave packet was studied. The single-photon field is characterized by a central frequency and a temporal profile $\xi(t)$, where $t \in \mathbb{R}$. It should be pointed out that $\xi(t)$ is the slowly varying envelope of the single-photon pulse. The temporal correlations of the field are responsible for all non-Markovian memory effects of the qubit dynamics. The reduced evolution of the qubit, obtained within the input-output formalism, is represented then by a hierarchy of coupled equations. The analytical solution to this hierarchy of equations for any initial state of the system and an arbitrary profile $\xi(t)$ was presented in the paper and used to show that the set of these equations is equivalent to a single time-local master equation. The price one pays for this reduction is a highly nontrivial structure of time-dependent rates in the time-local master equation. The coefficients $\gamma_\pm(t)$, $\gamma_z(t)$, governing the damping (cooling), heating, and decoherence processes, are fully characterized by the wave packet profile and the detuning value. Knowledge of the formulae defining these coefficients allowed us to determine the non-Markovian indices of the qubit dynamics. It was established that in the general case, the dynamical map governing the evolution of the qubit is not reversible, which implies singularities of the coefficients $\gamma_\pm(t)$, $\gamma_z(t)$, i.e. at the moments when the map is irreversible, the coefficients are not defined. Interestingly, although the coefficients are not defined at some moments, the master equation has a regular solution at any moment.

In this paper, a general condition for the invertibility of the dynamical map for a qubit is found. It is shown that in the case of resonance, the evolution of a qubit can satisfy the BLP condition, but at the same time it is never CP-divisible. It is also checked that the evolution is *eternally non-Markovian* [99,100], i.e. at any time at least one of the coefficients of $\gamma_\pm(t)$, $\gamma_z(t)$ is negative.

The situation of a single-photon field with a decaying exponential profile is described in detail. Here, for resonance, a general condition for the reversibility of the dynamical map is given. The asymptotic behavior of the coefficients defining the local equation in time is investigated analytically. It is verified that always, regardless of the photon parameters, at least one of the coefficients at a given moment has a negative value. The paper also gives the results of numerical analyses for the case of resonance and off-resonance. The off-resonance scenario is much more complicated, because the number of singular points for the time-local generator depends to a large extent on the detuning parameter.

We consider a two-level system interacting by two channels with a continuous-mode electromagnetic field (bidirectional field). The system interacts with the single-photon field incoming from one side and the vacuum field incoming from the other side. The interaction Hamiltonian in the interaction-picture has the form

$$H_{int}(t) = i\sqrt{\Gamma_1}\left(\sigma_- a_1^\dagger(t) - \sigma_+ a_1(t)\right) + i\sqrt{\Gamma_2}\left(\sigma_- a_2^\dagger(t) - \sigma_+ a_2(t)\right), \tag{141}$$

where $a_i(t)$ and $a_i^\dagger(t)$ are quantum white-noise operators [9] obeying the communication relation

$$[a_i(t), a_j^\dagger(t')] = \delta_{ij}\delta(t-t'), \tag{142}$$

and Γ_1, Γ_2 are non-negative coupling constants, $\sigma_- = |g\rangle\langle e|$, $\sigma_+ = |e\rangle\langle g|$, where $|e\rangle$ and $|g\rangle$ denote the excited and ground states of the qubit, respectively. In this case we obtain the following hierarchy of coupled equations for the family of qubit operators ϱ^{kl} with $k, l = 0, 1$

$$\dot{\varrho}^{11}(t) = \mathcal{L}\varrho^{11}(t) + \sqrt{\Gamma_1}\,\xi^*(t)[\sigma_-, \varrho^{10}(t)] - \sqrt{\Gamma_1}\,\xi(t)[\sigma_+, \varrho^{01}(t)], \tag{143}$$

$$\dot{\varrho}^{10}(t) = \mathcal{L}\varrho^{10}(t) - \sqrt{\Gamma_1}\,\xi(t)\left[\sigma_+, \varrho^{00}(t)\right], \tag{144}$$

$$\dot{\varrho}^{01}(t) = \mathcal{L}\varrho^{01}(t) + \sqrt{\Gamma_1}\,\xi^*(t)\left[\sigma_-, \varrho^{00}(t)\right], \tag{145}$$

$$\dot{\varrho}^{00}(t) = \mathcal{L}\varrho^{00}(t) \tag{146}$$

with the super-operator

$$\mathcal{L}(\varrho) = -\frac{i\Delta_0}{2}[\varrho, \sigma_z] - \frac{\Gamma}{2}\{\sigma_+\sigma_-, \varrho\} + \Gamma\sigma_-\varrho\sigma_+, \tag{147}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $\Delta_0 = \omega_c - \omega_0$, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. Here ω_c is the central frequency of the field and ω_0 is the transition frequency of the two-level system. One defines the density operator of the system $\rho(t) := \varrho^{11}(t)$, and initially

$$\varrho^{11}(0) = \varrho^{00}(0) = \rho(0)\ , \quad \varrho^{01}(0) = \varrho^{10}(0) = 0. \tag{148}$$

Note, that $\varrho^{10}(t) = \left(\varrho^{01}(t)\right)^{\dagger}$. The qubit is initially in an arbitrary state, $\rho(0)$. Let us emphasis that ξ_t is a slowly-varying envelope of the pulse.

In this paper the solution to the hierarchy of equations (143)-(146) is given. One can check that the dynamical map Λ_t representing qubit evolution has the form

$$\rho(0) \ \rightarrow \ \rho(t) = \Lambda_t \rho(0) = \begin{pmatrix} 1 - P_e(t) & C(t)\rho_{ge}(0) \\ C^*(t)\rho_{eg}(0) & P_e(t) \end{pmatrix}, \tag{149}$$

where the population of the excited state $P_e(t) = \rho_{ee}(t)$ reads as follows

$$P_e(t) = A(t) + B(t)P_e(0), \tag{150}$$

together with

$$A(t) = \kappa\Gamma e^{-\Gamma t}\left|\int_0^t ds \xi(s) e^{\left(-i\Delta_0 + \frac{\Gamma}{2}\right)s}\right|^2, \tag{151}$$

$$B(t) = e^{-\Gamma t}\left(1 - 4\kappa\Gamma \mathrm{Re}\int_0^t ds \xi^*(s) e^{\left(i\Delta_0 + \frac{\Gamma}{2}\right)s}\int_0^s d\tau \xi(\tau) e^{\left(-i\Delta_0 - \frac{\Gamma}{2}\right)\tau}\right), \tag{152}$$

and

$$C(t) = e^{\left(-i\Delta_0 - \frac{\Gamma}{2}\right)t}\left(1 - 2\kappa\Gamma\int_0^t ds \xi^*(s) e^{\left(i\Delta_0 - \frac{\Gamma}{2}\right)s}\int_0^s d\tau \xi(\tau) e^{\left(-i\Delta_0 + \frac{\Gamma}{2}\right)\tau}\right). \tag{153}$$

Here $\kappa = \Gamma_1/\Gamma$, hence $\kappa \in [0, 1]$. Thus, if $\kappa = 0$, one deals with the system interacting only with the vacuum part, and if $\kappa = 1$, we observe only an interaction with the single-photon field. The above formulae provide a complete description of the qubit evolution for an *arbitrary* photon profile $\xi(t)$. The asymptotic state of the system is universal: the atom eventually relaxes to the ground state $|g\rangle$ irrespective of $\xi(t)$. It should be stressed, however, that $|g\rangle\langle g|$ is not an invariant state of the evolution. This is essential difference between Markovian semigroup and non-Markovian evolution. For a semigroup an asymptotic state always defines an invariant state.

There are two essentially different scenario for the qubit evolution:

1. In this case when the quantities $B(t) > 0$ and $|C(t)| > 0$ for all $t \geq 0$, the dynamical map Λ_t is invertible (eigenvalues $\{B(t), C(t), C^*(t)\}$ do not vanish), that is, knowing a qubit state $\rho(t)$ at time t one may reconstruct the initial state $\rho(0)$.

2. The dynamical map Λ_t is no longer invertible, i.e. there exists $t < \infty$ such that either $B(t) = 0$ or $C(t) = 0$.

To discuss non-Markovianity of qubit evolution it is convenient to introduce the corresponding time-local master equation for the density matrix. The formula (149) for time evolution of the qubit density operator $\rho(t)$ defines a dynamical map $\rho(0) \rightarrow \Lambda_t\rho(0)$. By differentiating (149) with respect to time, one easily finds that Λ_t satisfies time-local master equation $\dot{\Lambda}_t = \mathcal{L}_t \Lambda_t$, with the following time-local generator

$$\begin{aligned} \mathcal{L}_t(\rho) &= -i\frac{\omega(t)}{2}[\sigma_z,\rho] + \frac{\gamma_+(t)}{2}\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+,\rho\}\right) \\ &+ \frac{\gamma_-(t)}{2}\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho\}\right) + \frac{\gamma_z(t)}{2}(\sigma_z\rho\sigma_z - \rho), \end{aligned} \tag{154}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, and σ_x, σ_y, σ_z are the Pauli operators. The real valued time-dependent rates $\gamma_+(t)$, $\gamma_-(t)$, and $\gamma_z(t)$, describing, respectively, pumping, damping and pure dephasing, are defined as follows

$$\gamma_+(t) = 2\frac{\dot{A}(t)B(t) - A(t)\dot{B}(t)}{B(t)}\ , \quad \gamma_-(t) = -2\frac{\dot{B}(t)}{B(t)} - \gamma_+(t), \tag{155}$$

$$\gamma_z(t) = \frac{1}{2}\frac{\dot{B}(t)}{B(t)} - \frac{\frac{d}{dt}|C(t)|}{|C(t)|} = \frac{1}{2}\frac{\dot{B}(t)}{B(t)} - \mathrm{Re}\,\frac{\dot{C}(t)}{C(t)}, \tag{156}$$

and

$$\omega(t) = \frac{\dot{C}(t)|C(t)| - C(t)\frac{d}{dt}|C(t)|}{iC(t)|C(t)|} = \mathrm{Im}\,\frac{\dot{C}(t)}{C(t)}. \tag{157}$$

For the time-local generator (154) one can provide the following characterization [100, 101].

Theorem 3 *The qubit evolution generated by (154)*

- *is CP-divisible iff* $\gamma_\pm(t) \geq 0$ *and* $\gamma_z(t) \geq 0$,
- *is P-divisible iff*

$$\gamma_\pm(t) \geq 0, \quad \sqrt{\gamma_+(t)\gamma_-(t)} + 2\gamma_z(t) \geq 0, \tag{158}$$

- *satisfies BLP condition iff*

$$\gamma_+(t) + \gamma_-(t) \geq 0, \quad \gamma_+(t) + \gamma_-(t) + 4\gamma_z(t) \geq 0, \tag{159}$$

- *satisfies a geometric criterion (13) iff*

$$\gamma_+(t) + \gamma_-(t) + 2\gamma_z(t) \geq 0, \tag{160}$$

for all $t \geq 0$.

The following theorem was proved in the article [H4].

Theorem 4 *Let* $\Delta_0 = 0$ *and* $\xi(t) \in \mathbb{R}_+$. *If* $B(t) > 0$ *for all* $t > 0$, *then there is no information backflow (BLP condition holds). One has* $C(t) > 0$ *for all* $t > 0$ *and the coherence,* $C(t)|\rho_{ge}(0)|$, *decreases monotonically in the course of time. Moreover,*

$$\gamma_+(t) \geq 0\ , \quad \gamma_-(t) \geq 2\Gamma\ , \quad \gamma_z(t) \leq 0, \tag{161}$$

and hence the evolution is eternally non-Markovian.

The problem of non-Markovianity of the qubit evolution was illustrated by the case of the exponential profile

$$\xi(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sqrt{\Gamma_p} e^{-\Gamma_p t/2} & \text{for } t \geq 0 \end{cases} \tag{162}$$

with $\Gamma_p = \alpha\Gamma$ and $\alpha > 0$. In this case, analytical formulas for the functions $A(t)$, $B(t)$ and $C(t)$ were determined and the coefficients $\gamma_\pm(t)$, $\gamma_z(t)$ were explicitly written for the resonance case. It is worth noting that quite complicated formulas were obtained even in this simple example. The following theorems were proved in this work.

Theorem 5 *The maximal excitation probability for an exponential profile reads*

$$P_e^{\max} = \frac{4\kappa}{e^2}. \tag{163}$$

and is realized for $\alpha = 1$ and $\Delta_0 = 0$ at $t = 2/\Gamma$.

Theorem 6 *In the resonance the evolution is invertible iff $\alpha \geq 8\kappa + 1$.*

Corollary 1 *In the regular case, i.e. $\alpha \geq 8\kappa + 1$, the evolution satisfies BLP condition, i.e. there is no information backflow.*

Theorem 7 *One has the following asymptotic behaviour for $\gamma_\pm(t)$ and $\gamma_z(t)$:*

- *for $\alpha \geq 1$*

$$\lim_{t\to\infty} \gamma_+(t) = \lim_{t\to\infty} \gamma_z(t) = 0 \ , \quad \lim_{t\to\infty} \gamma_-(t) = 2\Gamma, \tag{164}$$

- *for $\alpha \in (0, 1)$*

$$\lim_{t\to\infty} \gamma_+(t) = 0 \ , \quad \lim_{t\to\infty} \gamma_z(t) = \frac{1}{4}\Gamma(1-\alpha) \ , \quad \lim_{t\to\infty} \gamma_-(t) = \Gamma(1+\alpha). \tag{165}$$

It implies that asymptotically one has

- for $\alpha \geq 1$

$$\mathcal{L}_t(\rho) \ \rightarrow \ \Gamma\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right), \tag{166}$$

- for $\alpha \in (0, 1)$

$$\mathcal{L}_t(\rho) \ \rightarrow \ \frac{\Gamma(1+\alpha)}{2}\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) + \frac{\Gamma(1-\alpha)}{8}(\sigma_z\rho\sigma_z - \rho). \tag{167}$$

The paper also contains the results of numerical analyses. Here, sample plots for the resonance case for two situations: reversible and irreversible dynamics are given, see Fig. 5 and Fig.

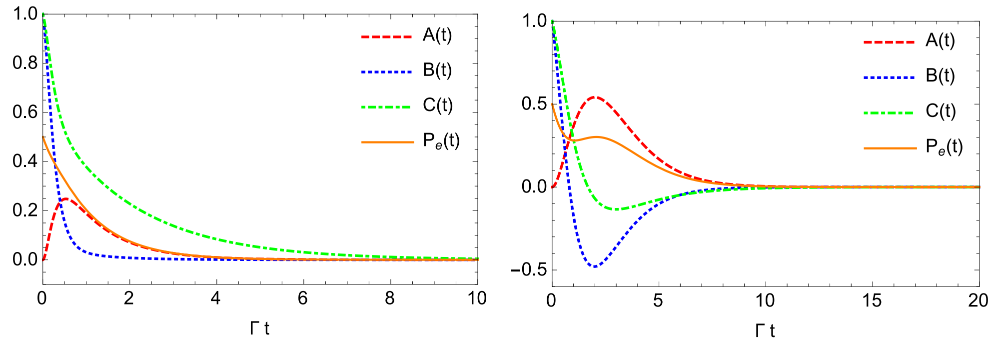

Figure 5: The plots of $A(t)$, $B(t)$, $C(t)$, and $P_e(t)$ for $\kappa = 1$, $\Delta_0 = 0$ and $P_e(0) = 0.5$. Left: regular case $\alpha = 9.5$. Right: singular case $\alpha = 1$.

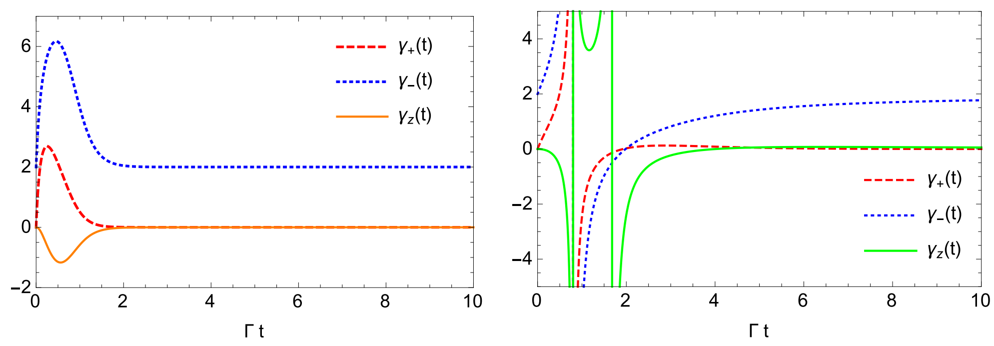

Figure 6: The plots of $\gamma_+(t)$, $\gamma_-(t)$, and $\gamma_z(t)$ for $\Gamma = 1$, $\kappa = 1$, $\Delta_0 = 0$. Left: regular case $\alpha = 9.5$. Right: singular case $\alpha = 1$.

6. Numerical analysis also shows that in the resonant case, P-divisibility and BLP condition coincide. Moreover,

$$\forall t \geq 0 \quad \gamma_+(t)\gamma_z(t) \leq 0. \tag{168}$$

In general, one sees that at least one of the three rates is negative at a given time.

The paper describes in detail the off-resonance results. The behavior of different non-Markovian evolution indicators was compared. The reversibility of the dynamics does not guarantee the BLP condition here. It is checked that in this case P-divisibility and the BLP condition do not coincide.

4.2.6 Filtering equations and quantum trajectories for an open system interacting with a field in the Fock state

The paper [**H5**] describes the stochastic evolution of a quantum system interacting with the Bose field prepared in a Fock state. The paper considers the case where the field photons have the same time profiles. To determine the conditional evolution of the quantum system, a model of repeated interactions and measurements was used. The environment is given there as an chain of harmonic oscillators prepared initially in an entangled state being a discrete analogue of a continuous-mode N-photon state. The initial correlations between the environment elements lead to non-Markovianity of evolution of the quantum system. The conditional evolution, described by the set of $(N+1)^2$ coupled filtering equations, was determined for the counting observation of the output field. The set of differential filtering equations was obtained by first determining the set of difference equations for discrete evolution. In this paper, formulae for the quantum trajectories associated with the stochastic counting process were obtained and the photon statistics of the output field were described. The solution of the set of master equations for both discrete and continuous evolution is also presented. For the conditional vectors that define quantum trajectories, a diagrammatic technique is presented to facilitate their determination and give their physical interpretation. It should be emphasised that the results obtained are of a general nature and can be applied to various quantum open systems.

The Hilbert space of the bath is given as

$$\mathcal{H}_{\mathcal{E}} = \bigotimes_{k=0}^{+\infty} \mathcal{H}_{\mathcal{E},k}, \tag{169}$$

where $\mathcal{H}_{\mathcal{E},k}$ stands for the Hilbert space of the k-th harmonic oscillator which interacts with $\mathcal{S}$ in the interval $[k\tau,(k+1)\tau)$.

The annihilation and creation operators associated with the k-th bath harmonic oscillator are denoted respectively by b_k and $b_k^\dagger$, thus we have

$$b_k|N\rangle_k = \sqrt{N}|N-1\rangle_k, \quad b_k^\dagger|N\rangle_k = \sqrt{(N+1)}|N+1\rangle_k, \tag{170}$$

where $|N\rangle_k$ is the number state in the Hilbert space $\mathcal{H}_{\mathcal{E},k}$. These operators satisfy the standard canonical commutation relations:

$$[b_l, b_k] = 0, \quad [b_l^\dagger, b_k^\dagger] = 0, \quad [b_l, b_k^\dagger] = \delta_{lk}. \tag{171}$$

The unitary evolution of the composed $\mathcal{E}+\mathcal{S}$ system describing the repeated interactions up to the time $j\tau$ for $j \geq 1$ is given by

$$U_{j\tau} = \mathbb{V}_{j-1}\mathbb{V}_{j-2}\ldots\mathbb{V}_0, \qquad U_0 = \mathbb{1}, \tag{172}$$

where the unitary operator $\mathbb{V}_k$ acts non-trivially only on $\mathcal{H}_{\mathcal{E},k} \otimes \mathcal{H}_{\mathcal{S}}$, that is,

$$\mathbb{V}_k = \mathbb{1}_{\mathcal{E}}^{k-1]} \otimes \mathbb{V}_{[k}, \tag{173}$$

and $\mathbb{V}_{[k} = \exp(-i\tau H_k)$ with

$$H_k = \mathbb{1}_{\mathcal{E}}^{[k} \otimes H_{\mathcal{S}} + \frac{i}{\sqrt{\tau}}\left(b_k^\dagger \otimes \mathbb{1}_{\mathcal{E}}^{[k+1} \otimes L - b_k \otimes \mathbb{1}_{\mathcal{E}}^{[k+1} \otimes L^\dagger\right). \tag{174}$$

Here $H_{\mathcal{S}}$ is the Hamiltonian of $\mathcal{S}$ and $L \in \mathcal{B}(\mathcal{H}_S)$, where $\mathcal{B}(\mathcal{H}_S)$ is a linear space of bounded operators acting on $\mathcal{H}_{\mathcal{S}}$. For simplicity, the Planck constant $\hbar = 1$. Using the Fock representation it was written down

$$\exp(-i\tau H_k) = \sum_{MM'} |M\rangle_k\langle M'|_k \otimes \mathbb{1}_{\mathcal{E}}^{[k+1} \otimes V_{MM'}, \quad M, M' = 0,1,2,\ldots, \tag{175}$$

where $V_{MM'} \in \mathcal{B}(\mathcal{H}_S)$.

It was assumed that the initial state of the composed $\mathcal{E}+\mathcal{S}$ system is the vector of the form

$$|N_\xi\rangle \otimes |\psi\rangle, \tag{176}$$

where $|\psi\rangle$ is the initial state of $\mathcal{S}$ and $|N_\xi\rangle$ is the N-photon state of the environment defined as

$$|N_\xi\rangle = \frac{1}{\sqrt{N!}}\left(\mathbf{b}_\xi^\dagger\right)^N |vac\rangle, \tag{177}$$

and $|vac\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \ldots$ is the vacuum vector in $\mathcal{H}_\mathcal{E}$,

$$\mathbf{b}_\xi^\dagger := \sum_{k=0}^{+\infty} \sqrt{\tau}\xi_k \mathbf{b}_k^\dagger, \quad \mathbf{b}_k^\dagger = \mathbb{1}_\mathcal{E}^{k-1]} \otimes b_k^\dagger \otimes \mathbb{1}_\mathcal{E}^{[k+1}. \tag{178}$$

It was assumed that $\mathbf{b}_0^\dagger = b_0^\dagger \otimes \mathbb{1}_\mathcal{E}^{[1}$, $\xi_k \in \mathbb{C}$ and $\sum_{k=0}^{+\infty} \tau|\xi_k|^2 = 1$. One can check that

$$\mathbf{b}_k|N_\xi\rangle = \sqrt{N\tau}\xi_k|(N-1)_\xi\rangle. \tag{179}$$

The vector (177) is a discrete version of the continuous-mode Fock state of the form [36–38, 40]

$$|N_\xi\rangle = \frac{1}{\sqrt{N!}}\left(\int_0^{+\infty} \xi_t dB_t^\dagger\right)^N |vac\rangle. \tag{180}$$

After each interaction the measurement is performed on the element of the bath chain just after its interaction with $\mathcal{S}$. In [H5] the measurement of the bath observable

$$N_k = b_k^\dagger b_k, \quad k = 0, 1, \ldots \tag{181}$$

was considered. To represent results of all measurements of (181) obtained up to time $j\tau$, the stochastic vector $\boldsymbol{\eta}_j = (\eta_j, \eta_{j-1}, \ldots, \eta_1)$ was used. The basic result for the discrete model can be written in the form of the following theorem.

Theorem 8 *The conditional state vector of $\mathcal{S}$ and the part of the environment which has not interacted with $\mathcal{S}$ up to $j\tau$ for the initial state (176) and the measurement of (181) at the moment $j\tau$ is given by*

$$|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle = \frac{|\Psi_{j|\boldsymbol{\eta}_j}\rangle}{\sqrt{\langle\Psi_{j|\boldsymbol{\eta}_j}|\Psi_{j|\boldsymbol{\eta}_j}\rangle}}, \tag{182}$$

where the unnormalized conditional state vector $|\Psi_{j|\boldsymbol{\eta}_j}\rangle \in \mathcal{H}_\mathcal{E}^{[j} \otimes \mathcal{H}_\mathcal{S}$ has the following structure

$$|\Psi_{j|\boldsymbol{\eta}_j}\rangle = \sum_{M=0}^{N} |M_\xi\rangle_{[j,+\infty)} \otimes |\psi^M_{j|\boldsymbol{\eta}_j}\rangle. \tag{183}$$

Moreover, $|M_\xi\rangle_{[j,+\infty)}$ is the unnormalized vector from $\mathcal{H}_\mathcal{E}^{[j}$ given by

$$|M_\xi\rangle_{[j,+\infty)} = \frac{1}{\sqrt{M!}}\left(\sqrt{\tau}\xi_j b_j^\dagger \otimes \mathbb{1}_\mathcal{E}^{[j+1} + \sum_{k=j+1}^{+\infty} \mathbb{1}_\mathcal{E}^{[j,k-1]} \otimes \sqrt{\tau}\xi_k b_k^\dagger \otimes \mathbb{1}_\mathcal{E}^{[k+1}\right)^M |vac\rangle_{[j,+\infty)}, \tag{184}$$

where $|vac\rangle_{[j,+\infty)} = |0\rangle_j \otimes |0\rangle_{j+1} \otimes \ldots$, *and the conditional vectors* $|\psi^0_{j|\boldsymbol{\eta}_j}\rangle$, $|\psi^1_{j|\boldsymbol{\eta}_j}\rangle, \ldots, |\psi^N_{j|\boldsymbol{\eta}_j}\rangle$ *from* $\mathcal{H}_\mathcal{S}$ *satisfy the set of coupled recurrence equations*

$$|\psi^M_{j+1|\boldsymbol{\eta}_{j+1}}\rangle = \sum_{M'=0}^{N-M} \sqrt{\binom{M+M'}{M'}} \left(\sqrt{\tau}\xi_j\right)^{M'} V_{\eta_{j+1}M'} |\psi^{M+M'}_{j|\boldsymbol{\eta}_j}\rangle, \tag{185}$$

where the operators $V_{\eta_{j+1}M'} \in \mathcal{B}(\mathcal{H}_S)$ *are defined in (175), and initially* $|\psi^N_{j=0}\rangle = |\psi\rangle$, *and* $|\psi^M_{j=0}\rangle = 0$ *for* $0 \leq M \leq N-1$.

The following notation is used here $|0_\xi\rangle_{[j,+\infty)} = |vac\rangle_{[j,+\infty)}$. The vector $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ is the entangled state vector of $\mathcal{S}$ and the part of the environment which has not interacted with $\mathcal{S}$ yet. Note, that '$N+1$' vectors $|M_\xi\rangle_{[j,+\infty)}$ are mutually orthogonal for different 'M'. The state vector $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ has the following physical interpretation: it represents a superposition of $N+1$ possible scenarios: the future part of the environment can be in the vacuum state $|vac\rangle_{[j,+\infty)}$ or in one of the states $|M_\xi\rangle_{[j,+\infty)}$ for $1 \leq M \leq N$.

The conditional probability of detecting M photons at moment $(j+1)\tau$ when the conditional state of $\mathcal{S}$ and the future part of the environment at $j\tau$ was $|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle$ is defined as

$$p_{j+1}\left(M \middle| |\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle\right) = \frac{\langle\Psi_{j|\boldsymbol{\eta}_j}|\mathbb{V}^\dagger_{[j}\left(|M\rangle_j\langle M|_j \otimes \mathbb{1}^{[j+1}_\mathcal{E} \otimes \mathbb{1}_\mathcal{S}\right)\mathbb{V}_{[j}|\Psi_{j|\boldsymbol{\eta}_j}\rangle}{\langle\Psi_{j|\boldsymbol{\eta}_j}|\Psi_{j|\boldsymbol{\eta}_j}\rangle}. \tag{186}$$

Expanding (175) in the Taylor series one check that

$$p_{j+1}\left(0 \middle| |\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle\right) = 1 + O(\tau), \tag{187}$$

and for all $M > 0$

$$p_{j+1}\left(M \middle| |\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle\right) = O(\tau^M). \tag{188}$$

Thus the probability of detecting more than one photon behaves like $O(\tau^2)$. In the continuous time limit, when $\tau \to dt$, the probability of detecting more than one photon in the time interval of the length dt vanishes. By neglecting all terms of order higher than one in τ and the processes of detecting more than one photon, we get from (185) the set of $N+1$ difference equations of the form

$$|\psi^N_{j+1|\boldsymbol{\eta}_{j+1}}\rangle = V_{\eta_{j+1}0}|\psi^N_{j|\boldsymbol{\eta}_j}\rangle, \tag{189}$$

and for $0 \leq M < N-1$

$$|\psi^M_{j+1|\boldsymbol{\eta}_{j+1}}\rangle \quad = \quad V_{\eta_{j+1}0}|\psi^M_{j|\boldsymbol{\eta}_j}\rangle + \sqrt{(M+1)\tau}\xi_j V_{\eta_{j+1}1}|\psi^{M+1}_{j|\boldsymbol{\eta}_j}\rangle, \tag{190}$$

where $\eta_{j+1} = \{0, 1\}$ and

$$V_{00} = \mathbb{1}_\mathcal{S} - i\tau H_\mathcal{S} - \tau\frac{1}{2}L^\dagger L + O(\tau^2), \quad V_{10} = \sqrt{\tau}L + O(\tau^{3/2}),$$
$$V_{01} = -\sqrt{\tau}L^\dagger + O(\tau^{3/2}), \quad V_{11} = \mathbb{1}_\mathcal{S} + O(\tau). \tag{191}$$

The paper [H5] provides a general solution to the above set of equations.

The *a posteriori* state of the system $\mathcal{S}$ at $j\tau$ has then the form

$$\tilde{\rho}_{j|\boldsymbol{\eta}_j} = \frac{\rho_{j|\boldsymbol{\eta}_j}}{\mathrm{Tr}_{\mathcal{S}}\rho_{j|\boldsymbol{\eta}_j}}, \tag{192}$$

where

$$\rho_{j|\boldsymbol{\eta}_j} = \sum_{M=0}^{N} p_j^M \, |\psi_{j|\boldsymbol{\eta}_j}^M\rangle\langle\psi_{j|\boldsymbol{\eta}_j}^M|. \tag{193}$$

The operator $\tilde{\rho}_{j|\boldsymbol{\eta}_j}$ is the conditional state of $\mathcal{S}$ depending on all results of measurements preformed on the environment elements after their interaction with $\mathcal{S}$ up to $j\tau$. The quantity

$$\mathrm{Tr}_{\mathcal{S}}\rho_{j|\boldsymbol{\eta}_j} = \sum_{M=0}^{N} p_j^M \, \langle\psi_{j|\boldsymbol{\eta}_j}^M|\psi_{j|\boldsymbol{\eta}_j}^M\rangle \tag{194}$$

is the probability of a given trajectory.

This paper presents the derivation of the set of filtering equations for the evolution of an open system conditioned by measurement results. The derivation of the stochastic equations omits, for simplicity, the index $\boldsymbol{\eta}_j$. The operators

$$\rho_j^{M,M'} = \frac{\mathrm{Tr}_{\mathcal{E}^{[j}}\left[\left(b_j^{N-M}\otimes \mathbb{1}_{\mathcal{E}}^{[j+1}\otimes\mathbb{1}_{\mathcal{S}}\right)|\Psi_j\rangle\langle\Psi_j|\left(\left(b_j^{\dagger}\right)^{N-M'}\otimes \mathbb{1}_{\mathcal{E}}^{[j+1}\otimes\mathbb{1}_{\mathcal{S}}\right)\right]}{\tau^{(2N-M-M')/2}\xi_j^{N-M}(\xi_j^*)^{N-M'}}, \tag{195}$$

were introduced, where $0 \leq M \leq N$ and $0 \leq M' \leq N$. One can check that $\rho_j^{N,N} = \rho_j$ and $\rho_j^{M,M'} = \left(\rho_j^{M',M}\right)^{\dagger}$. In the next step, operators $\tilde{\rho}_j^{M,M'} = \frac{\rho_j^{M,M'}}{\mathrm{Tr}_{\mathcal{S}}\rho_j}$ were defined, such that $\tilde{\rho}_j^{N,N} = \tilde{\rho}_j$.

The discrete stochastic process

$$n_j = \sum_{k=1}^{j} \eta_k, \tag{196}$$

with the increment $\Delta n_j = n_{j+1} - n_j = \eta_{j+1}$ was introduced. Let us emphasis that Δn_j has only two possible values: 0 or 1. One obtains the conditional mean value

$$\mathbb{E}[\Delta n_j|\tilde{\rho}_j] = k_j\tau + O(\tau^2), \tag{197}$$

where $k_j = \mathrm{Tr}_{\mathcal{S}}\left(L^{\dagger}L\tilde{\rho}_j + \xi_j^* L\tilde{\rho}_j^{N,N-1} + \xi_j\tilde{\rho}_j^{N-1,N}L^{\dagger} + |\xi_j|^2\tilde{\rho}_j^{N-1,N-1}\right)$.

For *a posteriori* state of $\mathcal{S}$ the difference stochastic equation

$$\begin{aligned}\tilde{\rho}_{j+1} &= \tilde{\rho}_j + \tau\Big(-i[H_{\mathcal{S}},\tilde{\rho}_j] - \frac{1}{2}\left\{L^{\dagger}L,\tilde{\rho}_j\right\} + L\rho_jL^{\dagger} + [\tilde{\rho}_j^{N-1,N},L^{\dagger}]\xi_j + [L,\tilde{\rho}_j^{N,N-1}]\xi_j^*\Big) \\ &\quad + \left\{\frac{1}{k_j}\left(L\tilde{\rho}_jL^{\dagger} + L\tilde{\rho}_j^{N,N-1}\xi_j^* + \tilde{\rho}_j^{N-1,N}L^{\dagger}\xi_j + \tilde{\rho}_j^{N-1,N-1}|\xi_j|^2\right) - \tilde{\rho}_j\right\}(\Delta n_j - k_j\tau)\end{aligned} \tag{198}$$

with the initial conditions: $\tilde{\rho}_{j=0} = |\psi\rangle\langle\psi|$, $\tilde{\rho}_{j=0}^{N,N-1} = \tilde{\rho}_{j=0}^{N-1,N} = 0$, and $\tilde{\rho}_{j=0}^{N-1,N-1} = N|\psi\rangle\langle\psi|$ was derived. In order to determine the *a posteriori* state of $\mathcal{S}$ at any time $j\tau$ where $j > 0$ one needs the set of $(N+1)^2$ coupled equations depending on the stochastic trajectory up to time $j\tau$:

$$\begin{aligned}\tilde{\rho}_{j+1}^{M,M'} &= \tilde{\rho}_j^{M,M'} + \tau\Big(-i[H_{\mathcal{S}}, \tilde{\rho}_j^{M,M'}] - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_j^{M,M'}\right\} + L\rho_j^{M,M'}L^\dagger \\ &\quad + [\tilde{\rho}_j^{M-1,M'}, L^\dagger]\xi_j + [L, \tilde{\rho}_j^{M,M'-1}]\xi_j^*\Big) \\ &\quad + \Bigg\{ \frac{1}{k_j}\Big(L\tilde{\rho}_j^{M,M'}L^\dagger + L\tilde{\rho}_j^{M,M'-1}\xi_j^* + \tilde{\rho}_j^{M-1,M'}L^\dagger\xi_j \\ &\quad + \tilde{\rho}_j^{M-1,M'-1}|\xi_j|^2\Big) - \tilde{\rho}_j^{M,M'}\Bigg\}(\Delta n_j - k_j\tau) \end{aligned} \tag{199}$$

with the initial condition $\tilde{\rho}_{j=0}^{M,M'} = \frac{N!}{\sqrt{M!M'!}}\delta_{MM'}\,|\psi\rangle\langle\psi|$ and $0 \le M \le N$, $0 \le M' \le N$. From $\tilde{\rho}_j^{M',M} = \left(\tilde{\rho}_j^{M,M'}\right)^\dagger$ it follows that there are at most $(N+1)(N+2)/2$ independent equations. Let us stress that the determined equations for $N=1$ are consistent with the equations derived for the environment in single-photon state.

Taking an average of $\tilde{\rho}_j$ over all realisations of the stochastic process, one obtains the *a priori* state

$$\sigma_j = \langle\tilde{\rho}_j\rangle_{st}, \tag{200}$$

For the *a priori* state, σ_j, one gets the difference equation

$$\sigma_{j+1} = \sigma_j + \tau\Big(-i[H_{\mathcal{S}}, \sigma_j] - \frac{1}{2}\left\{L^\dagger L, \sigma_j\right\} + L\sigma_j L^\dagger + [\sigma_j^{N-1,N}, L^\dagger]\xi_j + [L, \sigma_j^{N,N-1}]\xi_j^*\Big). \tag{201}$$

For the operators $\sigma_j^{M,M'} = \langle\tilde{\rho}_j^{M,M'}\rangle_{st}$, one obtains the set of the difference equations

$$\begin{aligned}\sigma_{j+1}^{M,M'} &= \sigma_j^{M,M'} + \tau\Big(-i[H_{\mathcal{S}}, \sigma_j^{M,M'}] - \frac{1}{2}\left\{L^\dagger L, \sigma_j^{M,M'}\right\} + L\sigma_j^{M,M'}L^\dagger \\ &\quad + [\sigma_j^{M-1,M'}, L^\dagger]\xi_j + [L, \sigma_j^{M,M'-1}]\xi_j^*\Big) \end{aligned} \tag{202}$$

with the initial condition: $\sigma_{j=0}^{M,M'} = N!/\sqrt{M!M'!}\delta_{MM'}|\psi\rangle\langle\psi|$, where $0 \le M \le N$ and $0 \le M' \le N$. One can easily check that $\sigma_j^{N,N} = \sigma_j$.

In the continuous time limit from (199) one obtains the set of stochastic differential equations

$$\begin{aligned} d\tilde{\rho}_t^{M,M'} &= dt\Big(-i[H_{\mathcal{S}}, \tilde{\rho}_t^{M,M'}] - \frac{1}{2}\left\{L^\dagger L, \tilde{\rho}_t^{M,M'}\right\} + L\rho_t^{M,M'}L^\dagger \\ &\quad + [\tilde{\rho}_t^{M-1,M'}, L^\dagger]\xi_t + [L, \tilde{\rho}_t^{M,M'-1}]\xi_t^*\Big) \\ &\quad + \Bigg\{ \frac{1}{k_t}\Big(L\tilde{\rho}_t^{M,M'}L^\dagger + L\tilde{\rho}_t^{M,M'-1}\xi_t^* + \tilde{\rho}_t^{M-1,M'}L^\dagger\xi_t \\ &\quad + \tilde{\rho}_t^{M-1,M'-1}|\xi_t|^2\Big) - \tilde{\rho}_t^{M,M'}\Bigg\}(dn_t - k_t dt)\,, \end{aligned} \tag{203}$$

where $k_t = \mathrm{Tr}_{\mathcal{S}}\left(L^\dagger L\tilde{\rho}_t + \xi_t^* L\tilde{\rho}_t^{N,N-1} + \xi_t\tilde{\rho}_t^{N-1,N}L^\dagger + |\xi_t|^2\tilde{\rho}_t^{N-1,N-1}\right)$, $\xi_t \in \mathbb{C}$, $\int_0^\infty |\xi_t|^2 dt = 1$, and initially $\tilde{\rho}_{t=0}^{M,M'} = \frac{N!}{\sqrt{M!M'!}}\delta_{M,M'}\,|\psi\rangle\langle\psi|$. Here n_t is the couting process that describes the photon counting in the output field from 0 till t. For the increment $dn_t = n_{t+dt} - n_t$ one obtains the conditional mean value

$$\mathbb{E}[dn_t|\tilde{\rho}_t] = k_t dt. \tag{204}$$

Note that $(dn_t)^2 = dn_t$. This means that in the interval $[t, t+dt)$ one can count at most one photon. Clearly, $\tilde{\rho}_t = \tilde{\rho}_t^{N,N}$, therefore one of the equations is the equation for the conditional state. The derived equations are consistent with those determined using the quantum stochastic calculus of the Itô type in the paper [57].

For the non-selective measurement one get from (202) the set of the differential equations:

$$\begin{aligned}\frac{d\sigma_t^{M,M'}}{dt} &= -i[H_{\mathcal{S}}, \sigma_t^{M,M'}] - \frac{1}{2}\left\{L^\dagger L, \sigma_t^{M,M'}\right\} + L\sigma_t^{M,M'}L^\dagger \\ &\quad +[\sigma_t^{M-1,M'}, L^\dagger]\xi_t + [L, \sigma_t^{M,M'-1}]\xi_t^*,\end{aligned} \tag{205}$$

where $\sigma_t^{M,M'} = \langle\tilde{\rho}_t^{M,M'}\rangle_{st}$ and initially $\sigma_{t=0}^{M,M'} = N!/\sqrt{M!M'!}\delta_{MM'}|\psi\rangle\langle\psi|$, where $0 \leq M \leq N$ and $0 \leq M' \leq N$.

The paper also determines the conditional vectors that define the quantum trajectories for a continuous counting observation of the output field. Note that all realization of the counting process n_t may be divided into disjoint sectors containing realizations with exactly s counts in the interval from time 0 to time t, one in each of the nonoverlapping intervals $[t_1, t_1 + dt_1)$, $[t_2, t_2 + dt_2)$, …, $[t_s, t_s + dt_s)$, where $t_1 < t_2 < \ldots < t_s$. The general formula for $|\psi_{t|t_s \ldots t_1}^{N-M}\rangle$ is rather involved. Instead of providing this general formula, it can be shown that the rules for conditional vectors consist of some simple formulae - "bricks" - which correspond to some basic processes. The formula for any conditional vector is built from these elementary bricks. A graphical representation of these formulae is given in the paper. In the formulae for conditional vectors, one is dealing with the following processes:

1. free propagation described by the non-unitary operator

$$\mathbf{T}_t = e^{-iGt}, \tag{206}$$

where $G = H_{\mathcal{S}} - \frac{i}{2}L^\dagger L$ is a non-Hermitian Hamiltonian,

2. absorption of a photon by the system $\mathcal{S}$ from the environment at time t,

$$W_t = -\mathbf{T}_{-t}\xi_t L^\dagger \mathbf{T}_t \quad \longleftrightarrow \quad - - \bullet - -, \tag{207}$$

3. emission of a photon by the systems $\mathcal{S}$ to the detector at time t,

$$\widetilde{L}_t = \mathbf{T}_{-t} L \mathbf{T}_t \quad \longleftrightarrow \quad - - \circ - -, \tag{208}$$

4. absorption of a photon by a detector from the environment at time t,

$$\xi_t \quad \longleftrightarrow \quad *. \tag{209}$$

Here are some examples of conditional vectors:

1. for zero counts from time 0 to time t one obtains dla zera

$$|\psi_{t|0}^N\rangle = \mathbf{T}_t|\psi\rangle \tag{210}$$

and the corresponding trivial diagram: $-----$, and

$$|\psi^{N-M}_{t|0}\rangle = \sqrt{\frac{N\,!}{(N-M)\,!}}\,\mathbf{T}_t \int_0^t dt_M \dots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 W_{t_M} \dots W_{t_2} W_{t_1} |\psi\rangle, \tag{211}$$

with the diagram: $--\bullet--\bullet--\dots--\bullet--$ with M processes (absorptions) $--\bullet--$,

2. for a count at the time t' and no other counts in the interval from zero to t:

$$|\psi^{N}_{t|t'}\rangle = \sqrt{dt'}\mathbf{T}_t\,\widetilde{L}_{t'}|\psi\rangle, \tag{212}$$

with the diagram: $--\circ--$,

$$|\psi^{N-1}_{t|t'}\rangle \;=\; \sqrt{Ndt'}\,\mathbf{T}_t\Big[\xi_{t'} + \widetilde{L}_{t'}\int_0^{t'} dsW_s + \int_{t'}^{t} dsW_s\widetilde{L}_{t'}\Big]|\psi\rangle$$

with the diagram: $-*-\;+\;-\circ-\bullet-\;+\;-\bullet-\circ-$,

$$\begin{aligned} |\psi^{N-2}_{t|t'}\rangle \;=\; & \sqrt{N(N-1)dt'}\,\mathbf{T}_t\,\Big[\widetilde{L}_{t'}\int_0^{t'} dt_2 \int_0^{t_2} dt_1 W_{t_2}W_{t_1} + \int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1 W_{t_2}W_{t_1}\,\widetilde{L}_{t'} \\ & + \int_{t'}^{t} dt_2 W_{t_2}\,\widetilde{L}_{t'}\int_0^{t'} dt_1 W_{t_1} + \xi_{t'}\int_0^{t'} dt_1 W_{t_1} + \int_{t'}^{t} dt_1 W_{t_1}\,\xi_{t'}\Big]|\psi\rangle \end{aligned} \tag{213}$$

with the diagram: $-\circ-\bullet-\bullet-\;+\;-\bullet-\bullet-\circ-\;+\;-\bullet-\circ-\bullet-\;+\;-*-\bullet-\;+\;-\bullet-*-$.

The paper gives a physical interpretation of conditional vectors and the rules defining their structure.

In the counting representation, the *a priori* state has the form

$$\sigma_t = \rho_{t|0} + \sum_{s=1}^{+\infty}\int_0^t dt_s \int_0^{t_s} dt_{s-1}\dots\int_0^{t_2} dt_1 \rho_{t|t_s,t_{s-1},\dots,t_2,t_1}, \tag{214}$$

where

$$\rho_{t|0} = \sum_{M=0}^{N} p_t^M\,|\psi^M_{t|0}\rangle\langle\psi^M_{t|0}|, \tag{215}$$

with $p_t := \int_t^{+\infty} dt'|\xi_{t'}|^2$, and

$$dt_s dt_{s-1}\dots dt_1 \rho_{t|t_s,t_{s-1},\dots,t_2,t_1} = \sum_{M=0}^{N} p_t^M\,|\psi^M_{t|t_s,t_{s-1},\dots,t_2,t_1}\rangle\langle\psi^M_{t|t_s,t_{s-1},\dots,t_2,t_1}| \tag{216}$$

with $|\psi^M_{j=0}\rangle = \delta_{NM}|\psi\rangle$ for $0 \leq M \leq N$. The operator $\rho_{t|t_s,t_{s-1},\dots,t_2,t_1}$ is unnormalized conditional state of the system $\mathcal{S}$. The integrals are taken over all possible realization of the stochastic process n_t. From this, the full statistics of photon counts in the output field can be determined. In particular, the probability of no counts (no detections) until t is given as

$$P_0^t(0) = \sum_{M=0}^{N} p_t^M\,\langle\psi^M_{t|0}|\psi^M_{t|0}\rangle. \tag{217}$$

The probability density $p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1)$ of observing a particular trajectory corresponding to s counts in the interval from 0 to t, one in each of the nonoverlapping intervals $[t_1, t_1 + dt_1)$, $[t_2, t_2 + dt_2)$, ..., $[t_s, t_s + dt_s)$, where $t_1 < t_2 < \ldots < t_s$ is defined by

$$p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1)dt_s dt_{s-1} \ldots dt_1 = \sum_{M=0}^{N} p_t^M \langle \psi^M_{t|t_s,t_{s-1},\ldots,t_2,t_1} | \psi^M_{t|t_s,t_{s-1},\ldots,t_2,t_1} \rangle. \tag{218}$$

The probability of having exactly s counts up to time t reads as

$$P_0^t(s) = \int_0^t dt_s \int_0^{t_s} dt_{s-1} \ldots \int_0^{t_2} dt_1 p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1). \tag{219}$$

4.2.7 Filtering equations and quantum trajectories for an open system interacting with a field in a superopositions of coherent states

In the paper **[H6]**, I derived the sets of filtering equations for an open quantum system interacting with the unidirectional Bose field prepared in a superposition of two coherent states. In order to determine a conditional evolution of the quantum system, I used a collision model with an environment given as an infinite chain of non-interacting between themselves qubits prepared in an entangled state being an analogue of a superposition of continuous-mode coherent states of the boson field. The elements of the environment chain interact with the quantum system in turn one by one. Because of the temporal correlations in the input field, the evolution of open quantum system is non-Markovian in this case. I assumed that the initial state of the composed system is a product state. I derived the conditional evolution assuming that after each interaction (collision), the measurement is performed on the last qubit interacted with the system. I described in this paper the conditional evolution for the two stochastic processes: counting and diffusion. Starting from the discrete in time description of the problem, I obtained in the continuous-time limit differential filtering equations which are consistent with the results published in [48, 51]. I showed that the model of repeated interactions and measurements allows in the continuous time limit to reproduce all results for the superposition of coherent state received within quantum stochastic calculus of Itô type. I would like to stress that the presented method is more straight and intuitive than the methods described in [48, 51]. It not only allows us to derive the equations describing the conditional evolution of the system but also enables us to find the general structure of quantum trajectories.

In the paper [H6], I considered a quantum system $\mathcal{S}$ interacting with an environment $\mathcal{E}$ modelled by a sequence of qubits, in the same way as in the article [H1]. The coherent state of the field is defined as [71, 74]

$$|\alpha\rangle = \bigotimes_{k=0}^{+\infty} |\alpha_k\rangle_k, \tag{220}$$

where

$$|\alpha_k\rangle_k = e^{\sqrt{\tau}\left(\alpha_k \sigma_k^+ - \alpha_k^* \sigma_k^-\right)} |0\rangle_k \tag{221}$$

is the vector from the Hilbert space, $\mathcal{H}_{\mathcal{E},k}$, associated with the qubit of the k-th number. It is assumed that $\alpha_k \in \mathbb{C}$ and $\sum_{k=0}^{+\infty} |\alpha_k|^2 \tau < \infty$. One can check that the vector $|\alpha_k\rangle_k$ has the following properties

$$|\alpha_k\rangle_k = \left(1 - \frac{|\alpha_k|^2}{2}\tau\right)|0\rangle_k + \alpha_k\sqrt{\tau}|1\rangle_k + O(\tau^{3/2}), \tag{222}$$

$$\langle\alpha_k|\sigma_k^-|\alpha_k\rangle = \sqrt{\tau}\alpha_k + O(\tau^{3/2}), \;\; \langle\alpha_k|\sigma_k^+\sigma_k^-|\alpha_k\rangle = \tau|\alpha_k|^2 + O(\tau^2). \tag{223}$$

The vector $|\alpha\rangle$ is a discrete analogue of the coherent state [36, 37]

$$|\{\alpha\}\rangle = \exp\left[\int_0^{+\infty}\left(\alpha_t dB_t^\dagger - \alpha_t^* dB_t\right)\right]|vac\rangle \tag{224}$$

with $\alpha_t \in \mathbb{C}$ and $\int_0^{+\infty} dt|\alpha_t|^2 < +\infty$, such that

$$dB_t|\{\alpha\}\rangle = \alpha_t dt|\{\alpha\}\rangle. \tag{225}$$

Note that for the input field prepared in the coherent state, one gets the Markovian evolution for an open system.

I assumed that the initial state of the composed $\mathcal{E}+\mathcal{S}$ system is given by

$$(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle)\otimes|\psi\rangle, \tag{226}$$

where $|\psi\rangle$ is the initial state of $\mathcal{S}$, $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of $\mathcal{H}_\mathcal{E}$, and

$$|c_\alpha|^2 + c_\alpha c_\beta^*\langle\beta|\alpha\rangle + c_\alpha^* c_\beta\langle\alpha|\beta\rangle + |c_\beta|^2 = 1. \tag{227}$$

In order to describe the stochastic evolution corresponding to a counting process, I considered the measurement of the observable

$$\sigma_k^-\sigma_k^+ = |1\rangle_k\langle 1|, \quad k = 0, 1, 2, \ldots. \tag{228}$$

The results then can be formulated in the form of the theorem.

Theorem 9 *The conditional state of $\mathcal{S}$ and the part of the environment which has not interacted with $\mathcal{S}$ up to $j\tau$ for the initial state (226) and the measurement of (228) at the moment $j\tau$ is given by*

$$|\tilde{\Psi}_j\rangle = \frac{|\Psi_j\rangle}{\sqrt{\langle\Psi_j|\Psi_j\rangle}}, \tag{229}$$

where

$$|\Psi_j\rangle = c_\alpha \bigotimes_{k=j}^{+\infty}|\alpha_k\rangle_k \otimes |\psi_j\rangle + c_\beta \bigotimes_{k=j}^{+\infty}|\beta_k\rangle_k \otimes |\varphi_j\rangle. \tag{230}$$

The conditional vectors $|\psi_j\rangle$, $|\varphi_j\rangle$ from $\mathcal{H}_\mathcal{S}$ in (230) are given by the recurrence formulae

$$|\psi_{j+1}\rangle = M_{\eta_{j+1}}^{\alpha_j}|\psi_j\rangle, \;\; |\varphi_{j+1}\rangle = M_{\eta_{j+1}}^{\beta_j}|\varphi_j\rangle, \tag{231}$$

where $\eta_{j+1} = 0, 1$ stands for a random variable describing the $(j+1)$-th output of (228), and

$$M_0^{\alpha_j} = \mathbb{1}_S - \left(iH_\mathcal{S} + \frac{1}{2}L^\dagger L + L^\dagger\alpha_j + \frac{|\alpha_j|^2}{2}\right)\tau + O(\tau^2), \tag{232}$$

$$M_0^{\beta_j} = \mathbb{1}_S - \left(iH_\mathcal{S} + \frac{1}{2}L^\dagger L + L^\dagger\beta_j + \frac{|\beta_j|^2}{2}\right)\tau + O(\tau^2), \tag{233}$$

$$M_1^{\alpha_j} = (L + \alpha_j)\sqrt{\tau} + O(\tau^{3/2}), \tag{234}$$

$$M_1^{\beta_j} = (L + \beta_j)\sqrt{\tau} + O(\tau^{3/2}), \tag{235}$$

and initially we have $|\psi_0\rangle = |\varphi_0\rangle = |\psi\rangle$.

Note that the normalized vector $|\tilde{\Psi}_j\rangle$, from the Hilbert space $\bigotimes_{k=j}^{+\infty} \mathcal{H}_{\mathcal{E},k} \otimes \mathcal{H}_S$, is the entangled state of qubits of the input field and the system $\mathcal{S}$. The conditional vectors $|\psi_j\rangle$ and $|\phi_j\rangle$ depend on all results of the measurements performed on the bath qubits up to time $j\tau$. In order to obtain *a posteriori* state of $\mathcal{S}$ one has to take the partial trace of $|\tilde{\Psi}_j\rangle\langle\tilde{\Psi}_j|$ over the environment. Thus the *a posteriori* state of $\mathcal{S}$ at the time $j\tau$ has the form

$$\tilde{\rho}_j = \frac{\rho_j}{\mathrm{Tr}\rho_j}, \tag{236}$$

where

$$\rho_j = |c_\alpha|^2 |\psi_j\rangle\langle\psi_j| + c_\alpha c_\beta^* \prod_{k=j}^{+\infty} \langle\beta_k|\alpha_k\rangle |\psi_j\rangle\langle\varphi_j| + c_\alpha^* c_\beta \prod_{k=j}^{+\infty} \langle\alpha_k|\beta_k\rangle |\varphi_j\rangle\langle\psi_j| + |c_\beta|^2 |\varphi_j\rangle\langle\varphi_j|. \tag{237}$$

The quantity $\mathrm{Tr}\rho_j$ is the probability of a particular trajectory. To derive the set of recurrence equations describing the stochastic evolution of $\mathcal{S}$, I wrote down the conditional state of $\mathcal{S}$ at $j\tau$ in the form

$$\tilde{\rho}_j = |c_\alpha|^2 \tilde{\rho}_j^{\alpha\alpha} + c_\alpha c_\beta^* \tilde{\rho}_j^{\alpha\beta} + c_\alpha^* c_\beta \tilde{\rho}_j^{\beta\alpha} + |c_\beta|^2 \tilde{\rho}_j^{\beta\beta}, \tag{238}$$

For the discrete evolution, I obtained the set of difference filtering equations of the form

$$\begin{aligned} \tilde{\rho}_{j+1}^{\alpha\alpha} - \tilde{\rho}_j^{\alpha\alpha} &= \mathcal{L}\tilde{\rho}_j^{\alpha\alpha}\tau + [\tilde{\rho}_j^{\alpha\alpha}, L^\dagger]\alpha_j\tau + [L, \tilde{\rho}_j^{\alpha\alpha}]\alpha_j^*\tau \\ &+ \left\{ \frac{1}{\nu_j}\left(L\tilde{\rho}_j^{\alpha\alpha}L^\dagger + \tilde{\rho}_j^{\alpha\alpha}L^\dagger\alpha_j + L\tilde{\rho}_j^{\alpha\alpha}\alpha_j^* + \tilde{\rho}_j^{\alpha\alpha}|\alpha_j|^2 \right) - \tilde{\rho}_j^{\alpha\alpha} \right\} (\Delta n_j - \nu_j\tau), \end{aligned} \tag{239}$$

$$\begin{aligned} \tilde{\rho}_{j+1}^{\alpha\beta} - \tilde{\rho}_j^{\alpha\beta} &= \mathcal{L}\tilde{\rho}_j^{\alpha\beta}\tau + [\tilde{\rho}_j^{\alpha\beta}, L^\dagger]\alpha_j\tau + [L, \tilde{\rho}_j^{\alpha\beta}]\beta_j^*\tau \\ &+ \left\{ \frac{1}{\nu_j}\left(L\tilde{\rho}_j^{\alpha\beta}L^\dagger + \tilde{\rho}_j^{\alpha\beta}L^\dagger\alpha_j + L\tilde{\rho}_j^{\alpha\beta}\beta_j^* + \tilde{\rho}_j^{\alpha\beta}\beta_j^*\alpha_j \right) - \tilde{\rho}_j^{\alpha\beta} \right\} (\Delta n_j - \nu_j\tau), \end{aligned} \tag{240}$$

$$\begin{aligned} \tilde{\rho}_{j+1}^{\beta\beta} - \tilde{\rho}_j^{\beta\beta} &= \mathcal{L}\rho_j^{\beta\alpha}\tau + [\tilde{\rho}_j^{\beta\beta}, L^\dagger]\beta_j\tau + [L, \tilde{\rho}_j^{\beta\beta}]\beta_j^*\tau \\ &+ \left\{ \frac{1}{\nu_j}\left(L\tilde{\rho}_j^{\beta\beta}L^\dagger + \tilde{\rho}_j^{\beta\beta}L^\dagger\beta_j + L\tilde{\rho}_j^{\beta\beta}\beta_j^* + \tilde{\rho}_j^{\beta\beta}|\beta_t|^2 \right) - \tilde{\rho}_j^{\beta\beta} \right\} (\Delta n_j - \nu_j\tau), \end{aligned} \tag{241}$$

where

$$\mathcal{L}\rho = -i[H_\mathcal{S}, \rho] - \frac{1}{2}\left\{ L^\dagger L, \rho \right\} + L\rho L^\dagger \tag{242}$$

with the initial condition $\tilde{\rho}_0^{\alpha\alpha} = \tilde{\rho}_0^{\beta\beta} = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{\alpha\beta} = \langle\beta|\alpha\rangle|\psi\rangle\langle\psi|$. Here n_j is the stochastic process with the conditional mean value

$$\mathbb{E}[\Delta n_j|\tilde{\rho}_j] = \nu_j\tau + O(\tau^2), \tag{243}$$

where

$$\nu_j = |c_\alpha|^2 \nu_j^{\alpha\alpha} + c_\alpha c_\beta^* \nu_j^{\alpha\beta} + c_\alpha^* c_\beta \nu_j^{\beta\alpha} + |c_\beta|^2 \nu_j^{\beta\beta}, \tag{244}$$

and

$$\nu_j^{\alpha\alpha} = \mathrm{Tr}\left[\left(L^\dagger L + L\alpha_j^* + L^\dagger \alpha_j + |\alpha_j|^2\right)\tilde{\rho}_j^{\alpha\alpha}\right], \quad \nu_j^{\alpha\beta} = \mathrm{Tr}\left[\left(L^\dagger L + L\beta_j^* + L^\dagger \alpha_j + \alpha_j \beta_j^*\right)\tilde{\rho}_j^{\alpha\beta}\right],$$

$$\nu_j^{\beta\alpha} = \mathrm{Tr}\left[\left(L^\dagger L + L\alpha_j^* + L^\dagger \beta_j + \alpha_j^* \beta_j\right)\tilde{\rho}_j^{\beta\alpha}\right], \quad \nu_j^{\beta\beta} = \mathrm{Tr}\left[\left(L^\dagger L + L\beta_j^* + L^\dagger \beta_j + |\beta_j|^2\right)\tilde{\rho}_j^{\beta\beta}\right].$$

In the limit $\tau \to dt$, I obtained the set of the stochastic differential equations:

$$\begin{aligned} d\tilde{\rho}_t^{\alpha\alpha} &= \mathcal{L}\rho_t^{\alpha\alpha} dt + [\tilde{\rho}_t^{\alpha\alpha}, L^\dagger]\alpha_t dt + [L, \tilde{\rho}_t^{\alpha\alpha}]\alpha_t^* dt \\ &+ \left\{\frac{1}{\nu_j}\left(L\tilde{\rho}_t^{\alpha\alpha}L^\dagger + \tilde{\rho}_t^{\alpha\alpha}L^\dagger\alpha_t + L\tilde{\rho}_t^{\alpha\alpha}\alpha_t^* + \tilde{\rho}_t^{\alpha\alpha}|\alpha_t|^2\right) - \tilde{\rho}_t^{\alpha\alpha}\right\}(dn_t - \nu_t dt), \end{aligned} \tag{245}$$

$$\begin{aligned} d\tilde{\rho}_t^{\alpha\beta} &= \mathcal{L}\rho_t^{\alpha\beta} dt + [\tilde{\rho}_t^{\alpha\beta}, L^\dagger]\alpha_t dt + [L, \tilde{\rho}_t^{\alpha\beta}]\beta_t^* dt \\ &+ \left\{\frac{1}{\nu_t}\left(L\tilde{\rho}_t^{\alpha\beta}L^\dagger + \tilde{\rho}_t^{\alpha\beta}L^\dagger\alpha_t + L\tilde{\rho}_t^{\alpha\beta}\beta_t^* + \tilde{\rho}_t^{\alpha\beta}\beta_t^*\alpha_t\right) - \tilde{\rho}_t^{\alpha\beta}\right\}(dn_t - \nu_t dt), \end{aligned} \tag{246}$$

$$\begin{aligned} d\tilde{\rho}_t^{\beta\beta} &= \mathcal{L}\rho_t^{\beta\beta} dt + [\tilde{\rho}_t^{\beta\beta}, L^\dagger]\beta_t dt + [L, \tilde{\rho}_t^{\beta\beta}]\beta_t^* dt \\ &+ \left\{\frac{1}{\nu_t}\left(L\tilde{\rho}_t^{\beta\beta}L^\dagger + \tilde{\rho}_t^{\beta\beta}L^\dagger\beta_t + L\tilde{\rho}_t^{\beta\beta}\beta_j^* + \tilde{\rho}_t^{\beta\beta}|\beta_t|^2\right) - \tilde{\rho}_t^{\beta\beta}\right\}(dn_t - \nu_t dt), \end{aligned} \tag{247}$$

with the initial condition $\tilde{\rho}_0^{\alpha\alpha} = \tilde{\rho}_0^{\beta\beta} = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{\alpha\beta} = \langle\beta|\alpha\rangle|\psi\rangle\langle\psi|$. The *a posteriori* state of $\mathcal{S}$ is given as

$$\tilde{\rho}_t = |c_\alpha|^2 \tilde{\rho}_t^{\alpha\alpha} + c_\alpha c_\beta^* \tilde{\rho}_t^{\alpha\beta} + c_\alpha^* c_\beta \tilde{\rho}_t^{\beta\alpha} + |c_\beta|^2 \tilde{\rho}_t^{\beta\beta}, \tag{248}$$

where the conditional operators $\tilde{\rho}_t^{\alpha\alpha}$, $\tilde{\rho}_t^{\alpha\beta}$, $\tilde{\rho}_t^{\beta\beta}$ satisfy Eqs. (245)-(247), and $\tilde{\rho}_t^{\beta\alpha} = \left(\tilde{\rho}_t^{\alpha\beta}\right)^\dagger$. Here n_t is the counting process with the Itô table $(dn_t)^2 = dn_t$. This means that one can measure at most one photon in the interval of length dt. The conditional mean value

$$\mathbb{E}\left[dn_t|\tilde{\rho}_t\right] = \nu_t dt, \tag{249}$$

where

$$\nu_t = |c_\alpha|^2 \nu_t^{\alpha\alpha} + c_\alpha c_\beta^* \nu_t^{\alpha\beta} + c_\alpha^* c_\beta \nu_t^{\beta\alpha} + |c_\beta|^2 \nu_t^{\beta\beta} \tag{250}$$

and

$$\nu_t^{\alpha\alpha} = \mathrm{Tr}\left[\left(L^\dagger L + L\alpha_t^* + L^\dagger \alpha_t + |\alpha_t|^2\right)\tilde{\rho}_t^{\alpha\alpha}\right], \quad \nu_t^{\alpha\beta} = \mathrm{Tr}\left[\left(L^\dagger L + L\beta_t^* + L^\dagger \alpha_t + \alpha_t \beta_t^*\right)\tilde{\rho}_t^{\alpha\beta}\right],$$

$$\nu_t^{\beta\alpha} = \mathrm{Tr}\left[\left(L^\dagger L + L\alpha_t^* + L^\dagger \beta_t + \alpha_t^* \beta_t\right)\tilde{\rho}_t^{\beta\alpha}\right], \quad \nu_t^{\beta\beta} = \mathrm{Tr}\left[\left(L^\dagger L + L\beta_t^* + L^\dagger \beta_t + |\beta_t|^2\right)\tilde{\rho}_t^{\beta\beta}\right].$$

For the coherent states with the amplitude α_t and β_t, that satisfy the conditions

$$\int_0^{+\infty} |\alpha_t|^2 dt < +\infty, \qquad \int_0^{+\infty} |\beta_t|^2 dt < +\infty, \tag{251}$$

one gets

$$\langle\beta|\alpha\rangle = \exp\left\{-\frac{1}{2}\int_0^{+\infty}\left(|\alpha_t|^2 + |\beta_t|^2 - 2\alpha_t\beta_t^*\right)dt\right\}. \tag{252}$$

Let us notice that $\nu_t dt$ is the conditional mean number of photons detected in the time interval $[t, t+dt)$. From a physical point of view, the condition (251) means that the average number of photons in the coherent state is finite.

In the paper [H4] I also studied the stochastic evolution of the system for the measurement of observable

$$\sigma_k^x = \sigma_k^+ + \sigma_k^- = |+\rangle_k\langle+| - |-\rangle_k\langle-|, \quad k = 0, 1, 2, \ldots, \tag{253}$$

where

$$|+\rangle_k = \frac{1}{\sqrt{2}}\left(|0\rangle_k + |1\rangle_k\right), \quad |-\rangle_k = \frac{1}{\sqrt{2}}\left(|0\rangle_k - |1\rangle_k\right), \tag{254}$$

are vectors from the Hilbert space $\mathcal{H}_{\mathcal{E},k}$.
The results of the analyses can be summarised by the following theorem.

Theorem 10 *The conditional state of $\mathcal{S}$ and the part of the environment which has not interacted with $\mathcal{S}$ up to $j\tau$ for the initial state (226) and the measurement of (253) at the moment $j\tau$ is given by*

$$|\tilde{\Psi}_j\rangle = \frac{|\Psi_j\rangle}{\sqrt{\langle\Psi_j|\Psi_j\rangle}}, \tag{255}$$

where

$$|\Psi_j\rangle = c_\alpha \bigotimes_{k=j}^{+\infty} |\alpha_k\rangle_k \otimes |\psi_j\rangle + c_\beta \bigotimes_{k=j}^{+\infty} |\beta_k\rangle_k \otimes |\varphi_j\rangle. \tag{256}$$

The conditional vectors $|\psi_j\rangle$, $|\varphi_j\rangle$ from $\mathcal{H}_\mathcal{S}$ in (255) are given by the recurrence formulae

$$|\psi_{j+1}\rangle = R^{\alpha_j}_{\zeta_{j+1}}|\psi_j\rangle, \quad |\varphi_{j+1}\rangle = R^{\beta_j}_{\zeta_{j+1}}|\varphi_j\rangle, \tag{257}$$

where ζ_{j+1} stands for a random variable describing the $(j+1)$-th output of (253), and

$$R^{\alpha_j}_{\zeta_{j+1}} = \frac{1}{\sqrt{2}}\left[\mathbb{1}_S - \left(iH_\mathcal{S} + \frac{1}{2}L^\dagger L + L^\dagger\alpha_j + \frac{|\alpha_j|^2}{2}\right)\tau + (L+\alpha_j)\zeta_{j+1}\sqrt{\tau} + O\left(\tau^{3/2}\right)\right], \tag{258}$$

$$R^{\beta_j}_{\zeta_{j+1}} = \frac{1}{\sqrt{2}}\left[\mathbb{1}_S - \left(iH_\mathcal{S} + \frac{1}{2}L^\dagger L + L^\dagger\beta_j + \frac{|\beta_j|^2}{2}\right)\tau + (L+\beta_j)\zeta_{j+1}\sqrt{\tau} + O\left(\tau^{3/2}\right)\right], \tag{259}$$

and initially we have $|\psi_0\rangle = |\varphi_0\rangle = |\psi\rangle$.

In this case the stochastic evolution is given by the difference equations of the form

$$\begin{aligned}\tilde{\rho}^{\alpha\alpha}_{j+1} - \tilde{\rho}^{\alpha\alpha}_j &= \mathcal{L}\tilde{\rho}^{\alpha\alpha}_j\tau + [\tilde{\rho}^{\alpha\alpha}_j, L^\dagger]\alpha_j\tau + [L, \tilde{\rho}^{\alpha\alpha}_j]\alpha_j^*\tau \\ &+ \left[(L+\alpha_j)\,\tilde{\rho}^{\alpha\alpha}_j + \tilde{\rho}^{\alpha\alpha}_j(L^\dagger + \alpha_j^*) - \mu_j\tilde{\rho}^{\alpha\alpha}_j\right](\Delta q_j - \mu_j\tau),\end{aligned} \tag{260}$$

$$\begin{aligned}\tilde{\rho}^{\alpha\beta}_{j+1} - \tilde{\rho}^{\alpha\beta}_j &= \mathcal{L}\tilde{\rho}^{\alpha\beta}_j\tau + [\tilde{\rho}^{\alpha\beta}_j, L^\dagger]\alpha_j\tau + [L, \tilde{\rho}^{\alpha\beta}_j]\beta_j^*\tau \\ &+ \left[(L+\alpha_j)\,\tilde{\rho}^{\alpha\beta}_j + \tilde{\rho}^{\alpha\beta}_j(L^\dagger + \beta_j^*) - \mu_j\tilde{\rho}^{\alpha\beta}_j\right](\Delta q_j - \mu_j\tau),\end{aligned} \tag{261}$$

$$\begin{aligned}\tilde{\rho}^{\beta\beta}_{j+1} - \tilde{\rho}^{\beta\beta}_j &= \mathcal{L}\tilde{\rho}^{\beta\beta}_j\tau + [\tilde{\rho}^{\beta\beta}_j, L^\dagger]\beta_j\tau + [L, \tilde{\rho}^{\beta\beta}_j]\beta_j^*\tau \\ &+ \left[(L+\beta_j)\,\tilde{\rho}^{\beta\beta}_j + \tilde{\rho}^{\beta\beta}_j\left(L^\dagger + \beta_j^*\right) - \mu_j\tilde{\rho}^{\beta\beta}_j\right](\Delta q_j - \mu_j\tau)\end{aligned} \tag{262}$$

with the initial conditions $\tilde{\rho}_0^{\alpha\alpha} = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{\beta\beta} = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{\alpha\beta} = \langle\beta|\alpha\rangle|\psi\rangle\langle\psi|$. Here we deal with the stochastic process

$$q_j = \sqrt{\tau}\sum_{k=1}^{j}\zeta_k, \tag{263}$$

such that

$$\mathbb{E}[\Delta q_j = q_{j+1} - q_j|\tilde{\rho}_j] = \mu_j\tau + O(\tau^{3/2}), \tag{264}$$

where

$$\mu_j = |c_\alpha|^2\mu_j^{\alpha\alpha} + c_\alpha c_\beta^*\mu_j^{\alpha\beta} + c_\alpha^* c_\beta\mu_j^{\beta\alpha} + |c_\beta|^2\mu_j^{\beta\beta} \tag{265}$$

and

$$\mu_j^{\alpha\alpha} = \mathrm{Tr}\left[\left(L + L^\dagger + \alpha_j + \alpha_j^*\right)\tilde{\rho}_j^{\alpha\alpha}\right], \quad \mu_j^{\alpha\beta} = \mathrm{Tr}\left[\left(L + L^\dagger + \alpha_j + \beta_j^*\right)\tilde{\rho}_j^{\alpha\beta}\right], \tag{266}$$

$$\mu_j^{\beta\alpha} = \mathrm{Tr}\left[\left(L + L^\dagger + \beta_j + \alpha_j^*\right)\tilde{\rho}_j^{\beta\alpha}\right], \quad \mu_j^{\beta\beta} = \mathrm{Tr}\left[\left(L + L^\dagger + \beta_j + \beta_j^*\right)\tilde{\rho}_j^{\beta\beta}\right]. \tag{267}$$

In the continuous in time observation, the stochastic evolution of the quantum system is given by the set of differential filtering equations

$$\begin{aligned} d\tilde{\rho}_t^{\alpha\alpha} &= \mathcal{L}\rho_t^{\alpha\alpha}dt + [\tilde{\rho}_t^{\alpha\alpha}, L^\dagger]\alpha_t dt + [L, \tilde{\rho}_t^{\alpha\alpha}]\alpha_t^* dt \\ &\quad + \left[(L+\alpha_t)\,\tilde{\rho}_t^{\alpha\alpha} + \tilde{\rho}_t^{\alpha\alpha}\left(L^\dagger + \alpha_t^*\right) - \mu_t\tilde{\rho}_t^{\alpha\alpha}\right](dq_t - \mu_t dt)\,, \end{aligned} \tag{268}$$

$$\begin{aligned} d\tilde{\rho}_t^{\alpha\beta} &= \mathcal{L}\rho_t^{\alpha\beta}dt + [\tilde{\rho}_t^{\alpha\beta}, L^\dagger]\alpha_t dt + [L, \tilde{\rho}_t^{\alpha\beta}]\beta_t^* dt \\ &\quad + \left[(L+\alpha_t)\,\tilde{\rho}_t^{\alpha\beta} + \tilde{\rho}_t^{\alpha\beta}\left(L^\dagger + \beta_t^*\right) - \mu_t\tilde{\rho}_t^{\alpha\beta}\right](dq_t - \mu_t dt)\,, \end{aligned} \tag{269}$$

$$\begin{aligned} d\tilde{\rho}_t^{\beta\beta} &= \mathcal{L}\rho_t^{\beta\beta}dt + [\tilde{\rho}_t^{\beta\beta}, L^\dagger]\beta_t dt + [L, \tilde{\rho}_t^{\beta\beta}]\beta_t^* dt \\ &\quad + \left[(L+\beta_t)\,\tilde{\rho}_t^{\beta\beta} + \tilde{\rho}_t^{\beta\beta}\left(L^\dagger + \beta_t^*\right) - \mu_t\tilde{\rho}_t^{\beta\beta}\right](dq_t - \mu_t dt)\,, \end{aligned} \tag{270}$$

where

$$\mu_t = |c_\alpha|^2\mu_t^{\alpha\alpha} + c_\alpha c_\beta^*\mu_t^{\alpha\beta} + c_\alpha^* c_\beta\mu_t^{\beta\alpha} + |c_\beta|^2\mu_t^{\beta\beta} \tag{271}$$

and

$$\mu_t^{\alpha\alpha} = \mathrm{Tr}\left[\left(L + L^\dagger + \alpha_t + \alpha_t^*\right)\tilde{\rho}_t^{\alpha\alpha}\right], \quad \mu_t^{\alpha\beta} = \mathrm{Tr}\left[\left(L + L^\dagger + \alpha_t + \beta_t^*\right)\tilde{\rho}_t^{\alpha\beta}\right], \tag{272}$$

$$\mu_t^{\beta\alpha} = \mathrm{Tr}\left[\left(L + L^\dagger + \beta_t + \alpha_t^*\right)\tilde{\rho}_t^{\beta\alpha}\right], \quad \mu_t^{\beta\beta} = \mathrm{Tr}\left[\left(L + L^\dagger + \beta_t + \beta_t^*\right)\tilde{\rho}_t^{\beta\beta}\right]. \tag{273}$$

The process q_j in the limit $\tau \to 0$ converges to the stochastic process q_t with the conditional probability $\mathbb{E}[dq_t = q_{t+dt} - q_t|\tilde{\rho}_t] = \mu_t dt$. One can check that that the stochastic process $w_t = q_t - \int_0^t \mu_s ds$ is the Wiener process.

In paper [H6], I also determined the set of differential equations governing the unconditional evolution of the system. Moreover, I presented in [H6] analytical results for *a priori* and *a posteriori* dynamics of a cavity mode for both counting and diffusive observations. The cavity mode was initially prepared in a coherent state.

4.2.8 Filtering equations and quantum trajectories for an open system interacting with a field in the squeezed Fock state

In the paper [**H7**], the filtering and master equations for a quantum system interacting with wave-packet of light in a continuous-mode squeezed number state were derived. The problem of conditional evolution of a quantum system was formulated by making use of the model of repeated interactions and measurements. In this approach, the quantum system undergoes a sequence of interactions with an environment defined by a chain of harmonic oscillators. The harmonics oscillators do not interact with each other but they interact with the system one by one and they are subsequently monitored. Random results of the measurements lead to a random sequence of the system states. In this paper, it is considered a photon-counting measurement scheme. It is assumed that the environment is prepared in an entangled state being a discrete analogue of a continuous-mode squeezed number state.

The set of stochastic recurrence equations describing the conditional evolution of the system was derived and the analytical solution related to different realization of the stochastic process was displayed. In the limit of continuous-time observation, the set of stochastic differential equations describing the *a posteriori* evolution of the quantum system was obtained. In the case of the input field prepared in a squeezed Fock state, we obtain the set of infinitely many equations governing the evolution of the open system. In addition to deriving the differential equations for the conditional and unconditional evolution of the quantum system, the paper also presents analytical formulae for quantum trajectories associated with continuous-time photon detection. It is shown how to use these formulae for quantum trajectories to determine the whole photon counting statistics in the output field.

In the final part of the work, it is presented how to use knowledge of the formulas for the conditional vectors to solve the problem of optimal cavity excitation. The time profile of the photons, which ensures optimal conditions for the transfer of photons from the wave packet to the resonant cavity, was determined.

In this paper, the notation used differs from that in [H5]. The Hilbert space of the environment is the same as in [H5]. We introduce the creation wave-packet operator acting in the Hilbert space $\mathcal{H}_{\mathcal{E}}^{[j}$:

$$\hat{B}_{[j}^{\dagger}[\xi] = \sum_{k=j}^{M-1} \sqrt{\tau}\xi_k \hat{\hat{b}}_k^{\dagger}, \tag{274}$$

where

$$\hat{\hat{b}}_k^{\dagger} = \hat{\mathbb{1}}_{\mathcal{E}}^{k-1]} \otimes \hat{b}_k^{\dagger} \otimes \hat{\mathbb{1}}_{\mathcal{E}}^{[k+1}, \tag{275}$$

$\xi_k \in \mathbb{C}$, and $\sum_{k=0}^{M-1} \tau|\xi_k|^2 = 1$. The commutator of $\hat{B}_{[j}^{\dagger}[\xi]$ and its Hermitian-conjugate operator $\hat{B}_{[j}[\xi]$ has the form

$$[\hat{B}_{[j}[\xi], \hat{B}_{[j}^{\dagger}[\xi]] = \sum_{k=j}^{M-1} \tau|\xi_k|^2. \tag{276}$$

The creation operator (274) can be used to construct the number vectors

$$|m_{\xi}\rangle_{[j} = \frac{1}{\sqrt{m!}} \left(\hat{B}_{[j}^{\dagger}[\xi]\right)^m |vac\rangle_{[j}\,, \tag{277}$$

where $|vac\rangle_{[j} = |0\rangle_j \otimes |0\rangle_{j+1} \otimes \ldots |0\rangle_{M-1}$ is the vacuum vector in $\mathcal{H}_{\mathcal{E}}^{[j}$ and $m = 0, 1, \ldots$. We define the squeezed number state in $\mathcal{H}_{\mathcal{E}}$ by the formula

$$|n_{\gamma,\xi}\rangle = \hat{S}[\gamma,\xi]\,|n_\xi\rangle \tag{278}$$

with the squeeze operator

$$\hat{S}[\gamma,\xi] = \exp\left(\gamma \hat{B}^2[\xi] - \gamma^* \hat{B}^{\dagger 2}[\xi]\right), \tag{279}$$

where $\gamma = \frac{r}{2}e^{-2i\phi}$. One can check that the mean number of photons for the wave-packet prepared in the squeezed number state $|n_{\gamma,\xi}\rangle$ is given by

$$\langle n_{\gamma,\xi}|\,\hat{n}\,|n_{\gamma,\xi}\rangle = c^2 n + s^2(n+1), \tag{280}$$

where $c = \cosh r$ and $s = \sinh r$. Note that any squeezed Fock state $|n_{\gamma,\xi}\rangle$ can be expanded into the number states with profiles ξ,

$$|n_{\gamma,\xi}\rangle = \sum_{m=0}^{+\infty} a_m(n_\gamma)\,|m_\xi\rangle\,. \tag{281}$$

We assume that the initial state of the composed $\mathcal{E}+\mathcal{S}$ system has the form

$$|\Psi_0\rangle = |n_{\gamma,\xi}\rangle \otimes |\psi_0\rangle\,, \tag{282}$$

where $|\psi_0\rangle$ is the initial state of $\mathcal{S}$.

After each interaction, a measurement is performed on the last harmonic oscillator which has interacted with the system $\mathcal{S}$. We consider the measurement of the field observable

$$\hat{n}_k = \hat{b}_k^\dagger \hat{b}_k = |n\rangle_k \langle n|_k\,, \quad k = 0, 1, \ldots. \tag{283}$$

It is assumed that a detector is perfect and it works instantaneously. To represent the results of measurements performed up to time $j\tau$ the stochastic vector $\boldsymbol{\eta}_j = (\eta_j, \eta_{j-1}, \ldots, \eta_1)$ is used.

Theorem 11 *The a posteriori state vector of the system $\mathcal{S}$ and the input part of the environment for the initial state (282) and the measurement of the observable (283) at time $j\tau$ is given by*

$$|\tilde{\Psi}^n_{j|\boldsymbol{\eta}_j}\rangle = \frac{|\Psi^n_{j|\boldsymbol{\eta}_j}\rangle}{\sqrt{\langle\Psi^n_{j|\boldsymbol{\eta}_j}|\Psi^n_{j|\boldsymbol{\eta}_j}\rangle}}, \tag{284}$$

where $|\Psi^n_{j|\boldsymbol{\eta}_j}\rangle$ is the unnormalized conditional vector from $\mathcal{H}_{\mathcal{E}}^{[j} \otimes \mathcal{H}_{\mathcal{S}}$ having the form

$$|\Psi^n_{j|\boldsymbol{\eta}_j}\rangle = \sum_{m=0}^{+\infty} |m_\xi\rangle_{[j} \otimes |\psi^n_{j|\boldsymbol{\eta}_j}(m)\rangle\,, \tag{285}$$

where $\{|\psi^n_{j|\boldsymbol{\eta}_j}(m)\rangle\}$, $m = 0, 1, \ldots$ is the set of conditional vectors from $\mathcal{H}_{\mathcal{S}}$ which satisfy the set of recurrence equations

$$|\psi^n_{j+1|\boldsymbol{\eta}_{j+1}}(m)\rangle = \sum_{m'=0}^{+\infty} \sqrt{\binom{m+m'}{m'}}\left(\sqrt{\tau}\xi_j\right)^{m'} \hat{V}_{\eta_{j+1}m'}\,|\psi^n_{j|\boldsymbol{\eta}_j}(m+m')\rangle\,, \tag{286}$$

The operators $\hat{V}_{\eta_{j+1}r} \in \mathcal{B}(\mathcal{H}_S)$ are defined by (175), and initially we have

$$|\psi^n_{j=0}(m)\rangle = a_m(n_\gamma)\,|\psi_0\rangle\,. \tag{287}$$

The infinite set of conditional vectors $\{\,|\psi^{n}_{j|\boldsymbol{\eta}_j}(m)\rangle\}$ with $m=0,1,\ldots$ depends on the initial state of the composed system and all results of the measurements up to $j\tau$.

It was shown in the paper that the probability of detecting more than one photon in the output field in the time interval $[k\tau,(k+1)\tau)$ is an expression of order $\overline{O}(\tau^2)$. The probability of such detection is equal to zero in the continuous-time limit and we ignore such cases. Now by neglecting in (286) all terms of order more than one in τ and the terms associated with the processes of probability of $\overline{O}(\tau^2)$, we obtain from (286) the following the set of difference equations

$$|\psi^{n}_{j+1|\boldsymbol{\eta}_{j+1}}(m)\rangle = \hat{V}_{\eta_{j+1}0}\,|\psi^{n}_{j|\boldsymbol{\eta}_j}(m)\rangle + \sqrt{(m+1)\tau}\xi_j\hat{V}_{\eta_{j+1}1}\,|\psi^{n}_{j|\boldsymbol{\eta}_j}(m+1)\rangle \tag{288}$$

with the system operators (191). The paper presents the general solution to this set of equations.

The reduced state of the system $\mathcal{S}$ at the time $j\tau$ has the form

$$\tilde{\rho}_{j|\boldsymbol{\eta}_j} = \frac{\rho_{j|\boldsymbol{\eta}_j}}{\mathrm{Tr}_{\mathcal{S}}\rho_{j|\boldsymbol{\eta}_j}}, \tag{289}$$

where

$$\rho_{j|\boldsymbol{\eta}_j} = \sum_{m=0}^{+\infty} |\psi^{n}_{j|\boldsymbol{\eta}_j}(m)\rangle\,\langle\psi^{n}_{j|\boldsymbol{\eta}_j}(m)| \left(\sum_{k=j}^{N-1}\tau|\xi_k|^2\right)^{m}. \tag{290}$$

Initially $\rho_{j=0} = |\psi_0\rangle\,\langle\psi_0|$. The operator $\tilde{\rho}_{j|\boldsymbol{\eta}_j}$ is the conditional state of $\mathcal{S}$ depending on the results of all measurements performed on the output field up to time $j\tau$, and the quantity $\mathrm{Tr}_{\mathcal{S}}\rho_{j|\boldsymbol{\eta}_j}$ is the probability of a given trajectory.

This paper presents the derivation of the set of discrete filtering equations for conditional operators.

Theorem 12 *The a posteriori evolution of the system $\mathcal{S}$ interacting with the environment prepared in the state (278) for the measurement of (283) is given by an infinite set of the coupled difference stochastic equations of the form*

$$\begin{aligned}
\tilde{\rho}^{n',n''}_{j+1} \;=\;& \tilde{\rho}^{n',n''}_{j} + \mathcal{L}\tilde{\rho}^{n',n''}_{j}\tau + [\sqrt{n'}c\tilde{\rho}^{n'-1,n''}_{j} - \sqrt{n'+1}se^{2i\phi}\tilde{\rho}^{n'+1,n''}_{j},\hat{L}^\dagger]\xi_j\tau \\
&+[\hat{L},\sqrt{n''}c\tilde{\rho}^{n',n''-1}_{j} - \sqrt{n''+1}se^{-2i\phi}\tilde{\rho}^{n',n''+1}_{j}]\xi_j^*\tau \\
&+\Bigg\{\frac{1}{k_j}\Big[\hat{L}\tilde{\rho}^{n',n''}_{j}\hat{L}^\dagger + \xi_j^*\hat{L}\left(\sqrt{n''}c\tilde{\rho}^{n',n''-1}_{j} - \sqrt{n''+1}se^{-2i\phi}\tilde{\rho}^{n',n''+1}_{j}\right) \\
&+\xi_j\left(\sqrt{n'}c\tilde{\rho}^{n'-1,n''}_{j} - \sqrt{n'+1}se^{2i\phi}\tilde{\rho}^{n'+1,n''}_{j}\right)\hat{L}^\dagger \\
&+|\xi_j|^2\left(\sqrt{n'n''}c^2\tilde{\rho}^{n'-1,n''-1}_{j} + \sqrt{(n'+1)(n''+1)}s^2\tilde{\rho}^{n'+1,n''+1}_{j}\right) \\
&-|\xi_j|^2cs\sqrt{n'(n''+1)}e^{-2i\phi}\tilde{\rho}^{n'-1,n''+1}_{j} \\
&-|\xi_j|^2cs\sqrt{(n'+1)n''}e^{2i\phi}\tilde{\rho}^{n'+1,n''-1}_{j}\Big] - \tilde{\rho}^{n',n''}_{j}\Bigg\}(\Delta N_j - k_j\tau),
\end{aligned} \tag{291}$$

where

$$\mathcal{L}\tilde{\rho} = -i[\hat{H}_{\mathcal{S}},\tilde{\rho}] - \frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\tilde{\rho}\right\} + \hat{L}\tilde{\rho}\hat{L}^\dagger, \tag{292}$$

$$\begin{aligned}
k_j &= \mathrm{Tr}_{\mathcal{S}}\Big\{\hat{L}^\dagger\hat{L}\tilde{\rho}_j^{n,n} + \xi_j\hat{L}^\dagger\left(\sqrt{n}c\tilde{\rho}_j^{n-1,n} - \sqrt{n+1}se^{2i\phi}\tilde{\rho}_j^{n+1,n}\right) \\
&+\xi_j^*\hat{L}\left(\sqrt{n}c\tilde{\rho}_j^{n,n-1} - \sqrt{n+1}se^{-2i\phi}\tilde{\rho}_j^{n,n+1}\right) \\
&+|\xi_j|^2\left(nc^2\tilde{\rho}_j^{n-1,n-1} + (n+1)s^2\tilde{\rho}_j^{n+1,n+1}\right) \\
&-|\xi_j|^2\sqrt{n(n+1)}cs\left(e^{2i\phi}\tilde{\rho}_j^{n+1,n-1} + e^{2i\phi}\tilde{\rho}_j^{n-1,n+1}\right)\Big\}, \qquad (293)
\end{aligned}$$

and the initial conditions: $\tilde{\rho}_{j=0}^{n',n''} = \delta_{n',n''}\left|\psi_0\right\rangle\left\langle\psi_0\right|$ *for* $n', n'' \in \mathbb{N}$. *The a posteriori state of* $\mathcal{S}$ *at time* $j\tau$ *is given by* $\tilde{\rho}_j^{n,n}$. *The discrete stochastic process* N_j *characterizing photon counts defined by the observable (283) is uniquely determined by the conditional probabilities (given in the paper).*

Taking the average over all trajectories one obtains from (291) the unconditional (*a priori*) dynamics of the system $\mathcal{S}$:

$$\sigma_j^{n,n} = \langle\tilde{\rho}_j^{n,n}\rangle_{st}. \qquad (294)$$

Theorem 13 *The a priori dynamics of* $\mathcal{S}$ *is given by the infinite set of difference master equations*

$$\begin{aligned}
\sigma_{j+1}^{n',n''} &= \sigma_j^{n',n''} + \mathcal{L}\sigma_j^{n',n''}\tau + [\sqrt{n'}c\sigma_j^{n'-1,n''} - \sqrt{n'+1}se^{2i\phi}\sigma_j^{n'+1,n''}, \hat{L}^\dagger]\xi_j\tau \\
&+ [\hat{L}, \sqrt{n''}c\sigma_j^{n',n''-1} - \sqrt{n''+1}se^{-2i\phi}\sigma_j^{n',n''+1}]\xi_j^*\tau, \qquad (295)
\end{aligned}$$

where

$$\sigma_j^{n',n''} = \langle\tilde{\rho}_j^{n',n''}\rangle_{st} \qquad (296)$$

with the initial condition $\sigma_{j=0}^{n',n''} = \delta_{n',n''}\left|\psi_0\right\rangle\left\langle\psi_0\right|$ *for* $n', n'' \in \mathbb{N}$.

Finally, taking the limit of $\tau \to 0$ and $M \to \infty$ such that $T = M\tau$ is fixed one obtains from (291) the infinite set of the coupled differential stochastic equations of the form

$$\begin{aligned}
d\tilde{\rho}_t^{n',n''} &= \mathcal{L}\tilde{\rho}_t^{n',n''}dt + [\sqrt{n'}c\tilde{\rho}_t^{n'-1,n''} - \sqrt{n'+1}se^{2i\phi}\tilde{\rho}_t^{n'+1,n''}, \hat{L}^\dagger]\xi_t\tau \\
&+[\hat{L}, \sqrt{n''}c\tilde{\rho}_t^{n',n''-1} - \sqrt{n''+1}se^{-2i\phi}\tilde{\rho}_t^{n',n''+1}]\xi_t^*\tau \\
&+\Big\{\frac{1}{k_t}\Big[\hat{L}\tilde{\rho}_t^{n',n''}\hat{L}^\dagger + \xi_t^*\hat{L}\left(\sqrt{n''}c\tilde{\rho}_t^{n',n''-1} - \sqrt{n''+1}se^{-2i\phi}\tilde{\rho}_t^{n',n''+1}\right) \\
&+\xi_t\left(\sqrt{n'}c\tilde{\rho}_t^{n'-1,n''} - \sqrt{n'+1}se^{2i\phi}\tilde{\rho}_t^{n'+1,n''}\right)\hat{L}^\dagger \\
&+|\xi_t|^2\left(\sqrt{n'n''}c^2\tilde{\rho}_t^{n'-1,n''-1} + \sqrt{(n'+1)(n''+1)}s^2\tilde{\rho}_t^{n'+1,n''+1}\right) \\
&-|\xi_t|^2cs\sqrt{n'(n''+1)}e^{-2i\phi}\tilde{\rho}_t^{n'-1,n''+1} \\
&-|\xi_t|^2cs\sqrt{(n'+1)n''}e^{2i\phi}\tilde{\rho}_t^{n'+1,n''-1}\Big] - \tilde{\rho}_t^{n',n''}\Big\}\left(dN_t - k_tdt\right), \qquad (297)
\end{aligned}$$

where

$$\begin{aligned}
k_t &= \mathrm{Tr}_{\mathcal{S}}\Big\{\hat{L}^\dagger\hat{L}\tilde{\rho}_t^{n,n} + \xi_t^*\hat{L}\left(\sqrt{n}c\tilde{\rho}_t^{n,n-1} - \sqrt{n+1}se^{-2i\phi}\tilde{\rho}_t^{n,n+1}\right) \\
&+\xi_t\hat{L}^\dagger\left(\sqrt{n}c\tilde{\rho}_t^{n-1,n} - \sqrt{n+1}se^{2i\phi}\tilde{\rho}_t^{n+1,n}\right) \\
&+|\xi_j|^2\left(nc^2\tilde{\rho}_t^{n-1,n-1} + (n+1)s^2\tilde{\rho}_t^{n+1,n+1}\right) \\
&-|\xi_t|^2\sqrt{n(n+1)}cs\left(e^{2i\phi}\tilde{\rho}_t^{n+1,n-1} + e^{2i\phi}\tilde{\rho}_t^{n-1,n+1}\right)\Big\} \qquad (298)
\end{aligned}$$

and the initial condition of the form $\tilde{\rho}_{t=0}^{n',n''} = \delta_{n',n''} |\psi_0\rangle \langle\psi_0|$. Here N_t is the stochastic counting process with the increment $dN_t = N_{t+dt} - N_t$ having the conditional mean value

$$\mathbb{E}[dN_t|\tilde{\rho}_t] = k_t dt. \tag{299}$$

For the process N_t one obtains the relation $(dN_t)^2 = dN_t$. Note that k_t is the intensity of N_t and $k_t dt$ is the conditional mean number of photons that could be detected from t to $t + dt$. Clearly, taking finally the limit of $T \to +\infty$, one gets the amplitude $\xi \in L^2([0,\infty))$ with the normalization condition

$$\int_0^\infty |\xi_t|^2 dt = 1. \tag{300}$$

The reduced dynamics of $\mathcal{S}$ is given by the infinite set of differential equations

$$\begin{aligned} \frac{d}{dt}\sigma_t^{n',n''} &= \mathcal{L}\sigma_t^{n',n''} + [\sqrt{n'}c\sigma_t^{n'-1,n''} - \sqrt{n'+1}se^{2i\phi}\sigma_t^{n'+1,n''}, \hat{L}^\dagger]\xi_t \\ &+ [\hat{L}, \sqrt{n''}c\sigma_t^{n',n''-1} - \sqrt{n''+1}se^{-2i\phi}\sigma_t^{n',n''+1}]\xi_t^* \end{aligned} \tag{301}$$

with the initial condition $\sigma_{t=0}^{n',n''} = \delta_{n',n''} |\psi_0\rangle \langle\psi_0|$. The *a priori* state of $\mathcal{S}$ is given by $\sigma_t = \sigma_t^{n,n}$.

Let us notice that all realization of the counting stochastic process N_t may be divided into disjoint sectors: $\mathcal{C}_s$ containing trajectories with exactly s detected photons in the nonoverlapping intervals $[t_1, t_1 + dt_1)$, $[t_2, t_2 + dt_2)$, …, $[t_s, t_s + dt_s)$ lying in the interval from zero to t, where $t_1 < t_2 < \ldots < t_s$.

The *a priori* state of $\mathcal{S}$ at time t can be express by the stochastic representation as

$$\sigma_t = \rho_{t|0} + \sum_{s=1}^{+\infty} \int_0^t dt_s \int_0^{t_s} dt_{s-1} \ldots \int_0^{t_2} dt_1 \rho_{t|t_s,t_{s-1},\ldots,t_2,t_1}, \tag{302}$$

where

$$\rho_{t|0} = \sum_{m=0}^{+\infty} u_t^m |\psi_{t|0}^n(m)\rangle \langle\psi_{t|0}^n(m)| \tag{303}$$

with $u_t = \int_t^{+\infty} dt' |\xi_{t'}|^2$ and

$$dt_s dt_{s-1} \ldots dt_1 \rho_{t|t_s,t_{s-1},\ldots,t_2,t_1} = \sum_{m=0}^{+\infty} u_t^m |\psi_{t|t_s,t_{s-1},\ldots,t_2,t_1}^n(m)\rangle \langle\psi_{t|t_s,t_{s-1}\ldots,t_2,t_1}^n(m)|.$$

We have here the sum over all photons detection pathways that might take place from 0 to time t. They could involve s detections where s could change from 0 to ∞. The operators under integrals are interpreted as the unnormalized conditioned density operator of $\mathcal{S}$ associated with different scenarios of photon detections. For instance, $\rho_{t|0}$ refers to the situation when we do not observe any photons up to t while $\rho_{t|t_s,t_{s-1},\ldots,t_2,t_1}$ to the case when s photons were registered in the intervals $[t_1, t_1 + dt_1)$, $[t_2, t_2 + dt_2)$, …, $[t_s, t_s + dt_s)$, where $t_1 < t_2 < \ldots < t_s$, and no other photons in the interval from zero to t.

The paper contains a complete characterization of conditional vectors $\{\psi_{t|t_s,t_{s-1},\ldots,t_2,t_1}^n(m)\}$, $m = 0, 1, 2, \ldots$. The form of the conditional vectors is given for any quantum system. Along with the solution, the physical interpretation of the formulae for them was provided.

In addition to determining the *a posteriori* and *a priori* evolution of an open system, the conditional vectors can be used to find the statistics of photon counts in the output field. The probability of not observing any photons up to time t is given as

$$P_0^t(0) = \sum_{m=0}^{+\infty} \langle \psi_{t|0}^n(m)|\psi_{t|0}^n(m)\rangle\, u_t^m. \tag{304}$$

The exclusive probability density $p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1)$ for a trajectory corresponding to s detections in the interval from 0 to t in the intervals $[t_1, t_1 + dt_1)$, $[t_2, t_2 + dt_2)$, …, $[t_s, t_s + dt_s)$, where $0 < t_1 < t_2 < \ldots < t_s$ is given by

$$p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1)dt_s dt_{s-1}\ldots dt_1 = \sum_{m=0}^{+\infty} \langle \psi_{t|t_s,t_{s-1},\ldots,t_2,t_1}^n(m)|\psi_{t|t_s,t_{s-1},\ldots,t_2,t_1}^n(m)\rangle\, u_t^m. \tag{305}$$

Hence the probability of registering exactly s photons up to time t is

$$P_0^t(s) = \int_0^t dt_s \int_0^{t_s} dt_{s-1}\ldots \int_0^{t_2} dt_1 p_0^t(t_s, t_{s-1}, \ldots, t_2, t_1). \tag{306}$$

In the paper, a cavity mode is considered as an example of a system driven by light in the squeezed Fock state. In this case, the Hamiltonian of the system is of the form

$$\hat{H}_{\mathcal{S}} = \Delta \hat{a}^\dagger \hat{a}, \tag{307}$$

where $\Delta = \omega_0 - \omega_c$, where ω_0 is the frequency of the cavity mode and ω_c stands for the central frequency of the input wave packet. The coupling operator is defined as

$$\hat{L} = \sqrt{\Gamma}\hat{a}, \tag{308}$$

where Γ is a positive coupling constant. It is assumed that the harmonic oscillator is initially in the vacuum state.

Using the conditional vectors it was shown that if the input field is prepared in the squeezed number state $|n_{\gamma,\xi}\rangle$, then the probability that the mean number of photons inside the cavity at time t is equal to $c^2 n + s^2(n+1)$ is given by

$$P_{n_{\gamma,\xi}}(t) = \sum_{k=0}^{+\infty} \Gamma^k k! e^{-k\Gamma t} \left| \int_{t_0}^t ds \xi_s e^{\left(i\Delta + \frac{\Gamma}{2}\right)s} \right|^{2k} |a_k(n_\gamma)|^2. \tag{309}$$

Using this formula, the condition for optimal cavity excitation was determined. The result summarizes the theorem.

Theorem 14 *The maximum value of the probability of the transfer of the wave packet photons into the cavity at time $t > t_0$ for the cavity mode prepared in the vacuum state and the input field in $|n_{\gamma,\xi}\rangle$ is*

$$P_{n_{\gamma,\xi}}^{\max}(t) := \max_{\xi} P_{n_{\gamma,\xi}}(t) = \sum_{k=0}^{+\infty} e^{-k\Gamma t} \left(e^{\Gamma t} - e^{\Gamma t_0}\right)^k |a_k(n_\gamma)|^2, \tag{310}$$

and is realized only at the resonance (i.e. $\Delta = 0$) by the pulse of the profile

$$\xi_s = \sqrt{\frac{\Gamma}{e^{\Gamma t} - e^{\Gamma t_0}}} e^{\frac{\Gamma}{2}s} \tag{311}$$

for $s \in [t_0, t]$, and $\xi_s = 0$ elsewhere.

Thus one obtains perfect transfer, i.e. $P_{n_{\gamma,\xi}}(t) = 1$, for the squeezed Fock state with photons having exponential rising temporal profiles in the interval $(-\infty, t]$. The obtained formula is a generalization of the result for the optimal excitation of a two-level atom by a sinle-photon field [106].

4.3 The other achievements

The papers [P4, P5] consider the stochastic evolution of an open system in the Markov regime. The papers [P4, P5] describe a quantum filtration using light in the squeezed state. The numbering is according to the List of scientific or artistic achievements which present a major contribution to the development of a specific discipline.

Article [P4] **Anita Dąbrowska**, John Gough. Belavkin filtering with squeezed light sources. *Russian Journal of Mathematical Physics*, 23(2), 172-184, 2016.

Summary:

The paper addresses the problem of quantum filtering using the Bose field in a squeezed state. For the field in the squeezed state, calculations are preformed in the Araki-Woods representation. It should be stress that the counting process can not be defined for the field in such state.

The first part of the paper considers the situation when the Bose field, interacting with an open system, is prepared in a coherent state. It is assumed that the output field is sent through a beam splitter. At the other input of the beam splitter there is the field prepared a squeezed state. The filtering equation for the *a posteriori* state dependent on the results of simultaneous imperfect measurement of the two optical quadratures is determined. Thus in this paper the stochastic equation with a two-dimensional diffusion process was obtained.

In the second part of this paper, the stochastic equation for an open quantum system is determined for the case when the input field is in a squeezed state and the evolution of the system is conditioned by the results of measurement of the optical quadrature of the output field. As an example, the paper considers the situation where the quantum system interacting with the environment is a cavity mode. It is shown that the stochastic evolution preserves the Gaussian state in the considered cases.

My contribution to this paper.

- Participation in the determination of solutions for the stochastic equations.
- Participation in the preparation of the manuscript.

Article [P5] **Anita Dąbrowska**, John Gough. Quantum trajectories for squeezed input processes: explicit solutions. *Open Systems & Information Dynamics*, 23(1), 1650004-1-1650004-16, 2016.

Summary:

The paper contains a detail discussion on the solutions of two filtration equations. It is assumed that the input Bose field is prepared in a Gaussian state. This can be a vacuum state, a coherent state, a thermal state or a squeezed state. The Bose field interacts with a cavity mode (a harmonic oscillator).

In the first part of the paper, the evolution of the open system dependent on the results of the homodyne observation of the output field is considered. The paper describes the stochastic evolution of a cavity mode that is initially prepared in a Gaussian state. It is shown that the *a posteriori* state is also a Gaussian state in this case. In this paper, the differential equations for three coefficients that define the state of open system are determined. Only one of these coefficients depends on the measurement results. It was shown that for the other two, the inhomogeneous Riccati matrix differential equation can be determined. The solution of this equation is given in the paper. It allows, for example, to determine the exact formulae for the time-dependent variances of the optical quadratures of the field in the cavity. This paper also gives the condition when the filtration equation preserves the purity of state of the open system. The results obtained for the stochastic evolution are compared with the solution of the master equation.

The second part of the paper contains a discussion of the stochastic evolution for a double homodyne observation, i.e. a simultaneous imperfect measurement of two optical quadratures of the output field. It was assumed that the output field is mixed with another Bose field prepared in a squeezed state. Also in this case the Gaussian state of the open system is preserved. For the two coefficients defining the *a posteriori* state, the homogeneous Riccati matrix differential equation is determined. The paper contains a generalisation of the results of [P7].

My contribution to this paper.

- Participation in the determination of solutions for the stochastic equations.
- Participation in the preparation of the manuscript.

I was a corresponding author of this article.

5 Other post-doctoral achievements

Description of results published in the papers [P1-P3] and [P6-P8]

In the articles [P6-P8] the stochastic evolution of an open system in the Markov regime was considered. The environment was in a vacuum state [P8] and a coherent state [P6, P7]. The papers are discussed in order from oldest to newest.

Article [P8]: **Anita Dąbrowska**, Przemysław Staszewski. Filtering equation for measurement of a coherent channel. *Journal of the Optical Society of America B*, 28(5), 1238-1244, 2011.

Summary:

This paper presents the derivations of the filtering equations for the continuous observation of the output field in the case of counting and heterodyne detection for an environment defined as a unidirectional electromagnetic field prepared in a coherent state. The derivation presented here is based on the use of the quantum stochastic calculus of the Itô type. We defined in this paper a generating map for the output counting process corresponding to the direct photon counting of the output field. The differential equation for the generating map was determined and its solution was presented. The solution, having the form of a von Neumann-Dyson series, allowed us to obtain the formula for the stochastic propagator associated with the output counting process. Using this formula, the equation for the stochastic state vector for the open system was obtained for the observation counting photons of the output field. The filtering equation for heterodyne observation was derived in a similar way. In this case, the output field falls on the beam splitter. At the other input of the beam splitter is the field in the coherent state. The starting point here is also the counting process. We first obtain the filtering equation for the counting process, to finally go to the diffusion observation limit. It is worth noting that both determined filtration equations preserve the purity of state of the open system. In this paper, both linear and non-linear versions of the filtration equations are given.

There was a previous paper in the literature [102] in which the filtration equation for the field in the coherent state was determined, but the method used there was different than proposed in the papers in [P8] and [P6]. The filtering equations determined in the paper [5] correspond to the situation where the input field is taken a vacuum state. In the papers [17, 103, 104] one can find a description of the output field statistics for the input field taken in the coherent state, but the equation for the *a posteriori* state for the observation performed on the field prepared initially in the coherent state was not determined there.

My contribution to this paper.

- Defining the problem.
- Determination of the filtration equations.
- Preparation of the manuscript.

I was a corresponding author of this article.

Article [P7]: **Anita Dąbrowska**, Przemysław Staszewski. Squeezed coherent state undergoing a continuous nondemolition observation. *Physics Letters A*, 375(45), 3950-3955. 2011.

Summary:

This paper considers the stochastic evolution of a cavity mode (a harmonic oscillator) dependent on the results of two types of observations: single and double heterodyne measurements. The input field is taken in the vacuum state. It is assumed that initially the harmonic oscillator is in a squeezed coherent state [105]. We proved that in this case the *a posteriori* state, that is the state of the system depending on the measurement results, remains the squeezed coherent state. The differential equations for all coefficients defining the conditional squeezed coherent state were determined. The squeeze parameter satisfies the Riccati differential equation. It was shown that the squeeze parameter decreases in time and the open system goes to the vacuum state. It is worth noting that the mean values of the optical quadratures of the system depend on the results of measurement whereas their variances are deterministic. The master equation does not preserved the squeezed coherent state.

My contribution to this paper.

- Defining the problem.
- Determination of solutions.
- Preparation of the manuscript.

I was a corresponding author of this article.

Article [P6] **Anita Dąbrowska**, Przemysław Staszewski. Posterior quantum dynamics for a continuous diffusion observation of a coherent channel. *Journal of the Optical Society of America B*, 29(11), 3072-3077, 2012.

Summary:

This paper includes the derivation of both a linear and a non-linear filtering equation for the case where an environment of an open quantum system is the Bose field in a coherent state. The filtering equations are determined for a balance heterodyne observation [93, 94]. Unlike in paper [P8], the filtering equation for the diffusion process is not obtained here by taking the limit of the stochastic equation for the counting process. In this paper one can find the stochastic differential equation for the generating map defined for the optical quadrature of the output field. Using this equation, we obtain a differential equation for the stochastic propagator and hence the equation for the stochastic state vector of the open system.

As an example, the paper considers the situation where the open quantum system is a cavity mode (harmonic oscillator) prepared initially in a squeezed coherent state. It is shown that the stochastic evolution preserves the squeezed coherent state. Any initial coherent state becomes the squeezed coherent state. The coefficients defining the squeezed coherent state depend on the results of continuous observation. The squeeze parameter decreases in time to zero and the system approaches a coherent state with an amplitude independent of the initial state of the system. The results are a generalisation of those published in papers [P7] and [P8].

My contribution to this paper.

- Defining the problem.
- Derivation of the filtering equation.
- Determination of an example of solution to the filtering equation.
- Preparation of the manuscript.

I was a corresponding author of this article.

Article [P3] **Anita Dąbrowska**. Quantum filtering equation for system driven by field in a mixture of vacuum and coherent state. *Acta Physica Polonica A*, 132(1), 112-114, 2017.

Summary:

The paper contains a discussion on the optimal state estimation of a quantum system interacting with the unidirectional Bose field prepared in a mixture of vacuum and coherent state.

I considered the evolution of the open system conditioned by the results of the continuous in time observation of the optical quadrature of the output field. In this case, instead of a single equation describing the stochastic evolution of the quantum system, I obtained the set of two differential equations. In order to determine the set of stochastic equations, I used a method based on cascaded open systems. I assumed that the input field for an auxiliary system is prepared in a vacuum state and chose the auxiliary system so that the state of the output field was the mixture of vacuum and coherent states. This output field then interacts with another quantum system being the system of interest. In the first step, the filtering equation for a system consisting of the auxiliary system and the main system was given. Then, the stochastic equation for the main system was determined by taking a partial trace over the auxiliary system.

Article [P2] **Anita Dąbrowska**. Quantum filtering equations for system driven by non-classical fields. *Open Systems & Information Dynamics*, 25(2), 1850007-1-1850007-23, 2018.

Summary:

This paper contains an analysis of a behaviour of an open quantum system interacting with the unidirectional Bose field in the superposition state of the vacuum and single-photon state. In order to determine the stochastic evolution of the quantum system in this case, the Hilbert space of this system was enlarged by the space of the auxiliary system, which acts as the generator of the Bose field in the chosen state. The quantum system is cascaded with the auxiliary system in such a way that the Bose field after interaction with the auxiliary system is the input field of the main system. The field before interaction with the auxiliary system is in a vacuum state. The filtering equations corresponding to the two types of output field measurement were derived, for photon counting and optical field quadrature observation. In the both cases, the stochastic evolution is given by the set of four coupled differential equations.

The paper also gives formulae for unconditional and conditional mean values for the increments of the stochastic processes: Λ_t^{out} and $B_t^{out} + B_t^{outdagger}$. The paper is not limited to describing the stochastic evolution of the system, but one can also find there a description of the photon statistics of the field after interacting with the system. The general recipe for the probability of no photon counts in the time interval from 0 to t is given, as well as the formula for the probability density of photon counts at times t_1, t_2, ..., t_n such that $0 < t_1 < t_2 < ... < t_n < t$ and no other counts in this interval.

As an example of a system interacting with the Bose field in the superposition state of vacuum and single-photon states, a two-level atom was considered. The coupling operator is of the form $L = \sqrt{\kappa}\sigma_-$, where σ_- is the lowering operator and $\kappa > 0$. The single-photon part was assumed to have a Gaussian amplitude. The plots for the probability of being in the excited state and the probability of zero counting until a given moment are given. The paper [P2] also contains a description of the stochastic evolution of a quantum system interacting with the Bose field prepared in a mixture of coherent states. The results obtained in the paper are correct, but the mentioned mixture is an example of classical not non-classical state.

Article [P1] **Anita Dąbrowska**, Sylwia M. Kolenderska, Jakub Szlachetka, Karolina Słowik, Piotr Kolenderski. Quantum-inspired optical coherence tomography using classical light in a single-photon counting regime. *Optics Letters*, 49(2), 363-366, (2024).

Summary:

This paper is devoted to Quantum Optical Coherence Tomography (Q-OCT), which presents many advantages over its classical counterpart, Optical Coherence Tomography (OCT). Specifically, it provides increased axial resolution and is immune to even orders of dispersion. The core of Q-OCT is the quantum interference of negatively correlated entangled photon pairs, which, in the Fourier domain, are observed by means of a joint spectrum measurement. In this paper, the use of a spectral approach in a novel configuration where classical light pulses are employed instead of entangled photons is explored. The intensity of these light pulses is reduced to the single-photon level. The paper reports a theoretical analysis along with its experimental validation to show that although such a classical light is much easier to launch into an experimental system, it offers limited benefits compared to Q-OCT based on entangled light. The paper includes an analysis of the differences in the characteristics of the joint spectrum obtained with the entangled photons and with classical optical pulses and points out the source of these differences.

My contribution to this paper.

- Conducting a theoretical analysis of the problem.
- Preparation of the supplement material.
- Participation in the preparation of the manuscript.

Participation in the work on the articles [P9-P22] and [R1-R5]

A part of my scientific activity after obtaining my doctoral degree involved the application of statistical methods in scientific research. After obtaining the doctoral degree, I participated in many research projects carried out in five units of the Collegium Medicum NCU. I participated in the processes of analyzing and interpreting research results as well as designing research experiments. I would like to point out that some of the projects were carried out in cooperation with other research centers in Poland and abroad.

My involvement in the projects consisted of selecting appropriate statistical models to represent the studied processes and phenomena, performing statistical analyses, and interpreting the results. The aim of my research was to define the structure of the analyzed data and identify significant relationships between the studied variables. In my work, I used methods of multivariate analysis. A particularly important aspect of my research has been the issue of adjustment in comparisons, which involves identifying and accounting for the potential influence of confounding factors on the studied variables. Only by considering the impact of these variables is it possible to interpret the results accurately and avoid drawing misleading conclusions caused by the effects of confounding factors. The obtained models contributed to a better understanding of the studied phenomena, and they are used for making diagnostic and therapeutic decisions, forecasting future outcomes, and identifying potential issues during therapy.

In 2010, I started working with Dr habil. Grzegorz Przybylski, head of the Department of Lung Diseases, Cancer, and Tuberculosis, Faculty of Medicine, Collegium Medicum of the

Nicolaus Copernicus University. The earliest papers written with Dr habil. Przybylski, [P22, R3, R4], concern the influence of environmental risk factors on the development of allergy and the assessment of asthma knowledge among people with asthma, and the relationship between knowledge of the disease and quality of life of chronically ill patients. Four further papers [P18, P19, P21, R2] were based on a retrospective study of more than 2000 of patients treated for tuberculosis at the Regional Center of Pulmonology in Bydgoszcz between 2001 and 2010. The papers [P18, P19, R2] analysed the role of social and demographic factors in the incidence of tuberculosis and examined the impact of these factors on the treatment of tuberculosis. In the paper [P21], the smoking among the people with tuberculosis was analysed. In [R1] the report on the relationship between selected elements of a health-promoting lifestyle and quality of life and asthma control in bronchial asthma patients was presented. The papers [P10, P11, P14] are concerned with assessing the levels of selected inflammatory and immunological parameters, including C-reactive protein (CRP) and circulating immune complexes (CIC) and the CRP/CIC ratio, in people with obstructive lung disease and those treated for lung cancer. The results of the mentioned work were presented at three international conferences organised by the European Respiratory Society.

My collaboration with Dr habil. Magdalena Pasinska, from the Department of Clinical Genetics, Faculty of Medicine, Collegium Medicum NCU, resulted in 3 publications [P12, P20, R5]. These papers deal with genetic causes of pregnancy failure as well as the influence of environmental factors on fetal and neonatal well-being. The results of the collaboration were presented, among others, at the Congress of the Polish Society of Human Genetics in Bydgoszcz in 2014.

From 2015 to 2019, I was part of a research team led by Prof. Dr habil. Jacek Kubica, who heads the Department of Cardiology and Internal Medicine at the Faculty of Medicine, Collegium Medicum NCU. I was a member of the multidisciplinary international team that carried out the randomised study of name „IMPRESSION". The aim of this study was to evaluate the effect of morphine on the pharmacokinetics and pharmacodynamics of ticagrelor, a drug platelet aggregation inhibitor, in patients with myocardial infarction. The study showed the existence of negative interactions between ticagrelor and morphine, the primary drugs used to treat patients with myocardial infarction. It confirmed, that morphine attenuates and diminishes the effect of ticagrelor. Based on the results of the „IMPRESSION" study, the European Society of Cardiology in 2017 revised its recommendations for the use of morphine in patients with myocardial infarction. I am a co-author of a review article on the effect of morphine on the delay and attenuation of oral P2Y12 receptor inhibitors in patients with myocardial infarction [P16].

I also participated in the multicenter „UNICORN" project, which involved comparing esophageal and bladder temperature measurements in comatose patients after cardiac arrest who underwent mild therapeutic hypothermia. The collaboration resulted in papers [P15, P9].

I also participated in the work of the team of Prof. Dr habil. Michał Marszałł, who heads the Department of Medicinal Chemistry at the Collegium Medicum NCU. The research conducted by the team focused on the search for new biomarkers of cancerous changes in the prostate gland. It was part of an international project. The collaboration resulted in the paper [P13].

The paper [P17] was written in collaboration with Dr habil. Dariusz Nowak, and concerned the results of a pilot study on the effects of chokeberry juice consumption on lipid profile and endothelial function in healthy people.

6 Current research

- I continue to work on developing stochastic methods for analyzing the interaction of quantum systems with light in non-classical states. In papers [H5, H7], I considered a unidirectional continuous-mode field with uncorrelated photons of the same time profiles. I generalized these results to two-photon states of the form

$$\frac{1}{\sqrt{N}}\int_0^{+\infty}\int_0^{+\infty}\phi(t_2)\xi(t_1)dB^\dagger(t_2)dB^\dagger(t_1)|vac\rangle, \tag{312}$$

where $N = 1 + |\langle\xi|\phi\rangle|^2$ and

$$\langle\xi|\phi\rangle = \int_0^{+\infty} dt\xi(t)^*\phi(t), \quad \int_0^{+\infty} dt|\xi(t)|^2 = \int_0^{+\infty} dt|\phi(t)|^2 = 1, \tag{313}$$

In the next step, I considered an arbitrary pure two-photon state of the form

$$\int_0^{+\infty}\int_0^{+\infty}\Phi(t_2, t_1)dB^\dagger(t_2)dB^\dagger(t_1)|vac\rangle \tag{314}$$

with the amplitude $\Phi(t_2, t_1)$, which, in general, defines photons that are time-correlated. I have also obtained results for a bidirectional field in a two-photon state

$$\int_0^{+\infty}\int_0^{+\infty}\phi(t_2)\xi(t_1)dB_2^\dagger(t_2)dB_1^\dagger(t_1)|vac, vac\rangle \tag{315}$$

with the field operators that satisfy the commutation relations

$$[B_i(t), B_j(t')] = [B_i^\dagger(t), B_j^\dagger(t')] = 0, \;\; [B_i(t), B_j^\dagger(t')] = \delta_{ij}\, t\wedge t', \tag{316}$$

where $t \wedge t' = \min(t, t')$. Using quantum trajectories for the input field in the state (315), I obtained results for the state

$$\int_0^{+\infty}\int_0^{+\infty}\Phi(t_2, t_1)dB_2^\dagger(t_2)dB_1^\dagger(t_1)|vac, vac\rangle, \tag{317}$$

which, in general, can be an entangled state of two photons. For the given field states, I derived stochastic equations for conditional vectors for one- and two-dimensional counting processes, respectively. These results were published in arXiv:2409.07428, and the paper is currently under review. The co-author of the paper is Dr habil. Gniewomir Sarbicki. The paper derives formulas for two-photon absorption in a three-level atom excited by light in a two-photon state. Formulas for states optimally exciting the atomic system are also presented. Part of the results was presented this year at two scientific conferences. Dr habil. Gniewomir Sarbicki contributed to the work on determining the two-photon states optimally exciting the three-level atom.

  Preprint available: **Anita Dąbrowska**, Gniewomir Sarbicki. Quantum trajectories and output field properties for systems driven by two-photon input field. arXiv:2409.07428

- Together with Dr habil. Karolina Słowik, Dr habil. Gniewomir Sarbicki, and M.Sc. Masood Valipour from Nicolaus Copernicus University, I conduct research on the optimal excitation of multilevel atoms using wave packets with a fixed photon number. A theoretical description of the excitation of a two-level atom by a wave packet in a single-photon and N-photon state can be found, for instance, in [44, 45, 47, 95, 106]. The maximum value of the excitation probability depends not only on the type of light state but also on the shape of the temporal profile. For a two-level atom, the optimal choice is a single-photon state with an exponentially rising time profile. In the case of multilevel atoms, the optimization of excitation depends on more parameters and is therefore more complex. In our work, we extend the theoretical model, based on the Wigner-Weisskopf approximation, published in arXiv:2409.07428. We investigate the effect of temporal photon entanglement on the excitation probability of a three-level atom. Two publications are in preparation. Some of the obtained results were already presented this year at four scientific conferences.

  Preprint available: Masood Valipour, Gniewomir Sarbicki, Karolina Slowik, **Anita Dąbrowska**. Optimization of two-photon absorption for three-level atoms. arXiv:2411.13274

- I continue to collaborate with Dr habil. Piotr Kolenderski from Nicolaus Copernicus University. The collaboration focuses on optical coherence tomography (OCT) and its quantum version (Q-OCT). In experiments conducted under the supervision of Dr habil. Kolenderski, non-classical light with temporally entangled photons is used in Q-OCT to image objects. The results for non-classical light are tested and compared with those obtained by traditional OCT methods. My contribution involves developing the theoretical foundations for the methods being tested.

7 Bibliography

Other articles by the postdoctoral researcher not listed in 4.1

Indications:

P stands for a publication that appeared in a scientific journal.

R stands for a chapter from a scientific book.

A stands for preprints that appeared in the ArXiv archive.

[P1] **Anita Dąbrowska**, Sylwia M. Kolenderska, Jakub Szlachetka, Karolina Słowik, Piotr Kolenderski. Quantum-inspired optical coherence tomography using classical light in a single-photon counting regime. *Optics Letters*, 49(2), 363-366, (2024).

[P2] **Anita Dąbrowska**. Quantum filtering equations for system driven by non-classical fields. *Open Systems & Information Dynamics*, 25(2), 1850007-1-1850007-23, 2018.

[P3] **Anita Dąbrowska**. Quantum filtering equation for system driven by field in a mixture of vacuum and coherent state. *Acta Physica Polonica A*, 132(1), 112-114, 2017.

[P4] **Anita Dąbrowska**, John Gough. Belavkin filtering with squeezed light sources. *Russian Journal of Mathematical Physics*, 23(2), 172-184, 2016.

[P5] **Anita Dąbrowska**, John Gough. Quantum trajectories for squeezed input processes: explicit solutions. *Open Systems & Information Dynamics*, 23(1), 1650004-1-1650004-16, 2016.

[P6] **Anita Dąbrowska**, Przemysław Staszewski. Posterior quantum dynamics for a continuous diffusion observation of a coherent channel. *Journal of the Optical Society of America B*, 29(11), 3072-3077, 2012.

[P7] **Anita Dąbrowska**, Przemysław Staszewski. Squeezed coherent state undergoing a continuous nondemolition observation. *Physics Letters A*, 375(45), 3950-3955, 2011.

[P8] **Anita Dąbrowska**, Przemysław Staszewski. Filtering equation for measurement of a coherent channel. *Journal of the Optical Society of America B*, 28(5), 1238-1244, 2011.

[P9] Julia M. Umińska, Katarzyna Buszko, Jakub Ratajczak, Piotr Łach, Krzysztof Pstrągowski, **Anita Dąbrowska**, Piotr Adamski, Grzegorz Skonieczny, Jacek Manitius, and Kubica Jacek, *Comparison of temperature measurements in esophagus and urinary bladder in comatose patients after cardiac arrest undergoing mild therapeutic hypothermia*, *Cardiology Journal*, **27**(6), 735-741, 2020.

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[P11] Marta Pilaczyńska-Cemel, Ryszard Gołda, **Anita Dąbrowska**, and Grzegorz Przybylski. Analiza poziomu wybranych parametrów stanu zapalnego, krążących kompleksów immunologicznych oraz związanych z nimi wskaźników (neutrofil/limfocyt, płytki/limfocyt, CRP/KKI) w raku płuc. *Państwo i Społeczeństwo*, *Państwo i Społeczeństwo*, 19(4), 9-25, 2019.

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8 Presentation of significant scientific activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

Activity at Polish universities

- University of Gdańsk (UG)

  Papers from postdoctoral achievement having the UG affiliation: H2, H3, H4, and H7.

- Nicolaus Copernicus University in Toruń (NCU)

  Papers from postdoctoral achievement having the NCU affiliation: H1, H5, and H6.

Activity in foreign scientific institutions

- The paper [P4] was partly written during my three-week stay in the UK, funded by the University of Cambridge. Work on the publication was conducted at the Isaac Newton Institute for Mathematical Science. Length of stay: 21.07–15.08.2014.
- In the period 01.03–31.08.2006 I did a pre–doctoral internship under the supervision of Prof. V. P. Belavkin at the School of Mathematical Sciences at the University of Nottingham.

9 Presentation of teaching and organisational achievements as well as achievements in the popularisation of science

9.1 Teaching activity

Lectures and tutorials conducted at the Faculty of Mathematics, Physics, and Informatics of the University of Gdańsk in the years 2019–2024

Lectures:

- Statistical analysis and probability calculus for bioinformaticians, field of study: bioinformatics, bachelor's studies, 30 hours in the academic years: 2021/2022, 2022/2023, 2023/2024, 2024/2025 (in progress)
- Biothermodynamics with elements of statistical physics, field of study: medical physics, bachelor's studies, 30 hours in the academic years: 2019/2020, 2020/2021, 2021/2022, 2022/2023, 2023/2024, 2024/2025 (in progress)

- Mathematical methods of bioinformatics - discrete probability, field of study: bioinformatics, bachelor's studies, 15 hours in the academic years: 2020/2021, 2021/2022, 2022/2023, 2023/2024
- Stochastic processes: fundamentals and applications, field of study: mathematical modeling and data analysis, bachelor's studies, 30 hours in the academic years: 2020/2021, 2021/2022, 2022/2023
- Stochastic processes, fields of study: mathematical modeling and data analysis, mathematics, graduate studies, 30 hours in the academic year 2024/2025 (in progress)
- Probability calculus, field of study: bioinformatics, bachelor's studies, 15 hours in the academic year: 2020/2021
- Probability calculus and elements of statistics, field of study: medical physics, bachelor's studies, 30 hours in the academic years: 2020/2021, 2021/2022, 2022/2023, 2023/2024
- Advanced theoretical mathematics in physics, field of study: physics, bachelor's studies, lecture conducted together with Dr habil. Marcin Marciniak, 12 hours in 2021/2022

Auditorium and laboratory tutorials:

- Algebra, auditorium tutorials, field of study: nuclear safety and radiation protection, bachelor's studies, 30 hours in the academic year 2019/2020
- Linear algebra, auditorium tutorials, field of study: medical physics, bachelor's studies, 30 hours in the academic year 2019/2020
- Linear algebra with geometry, auditorium tutorials, field of study: physics, bachelor's studies, 60 hours in the academic year 2019/2020
- Statistical analysis and probability calculus for bioinformaticians, laboratory tutorials, field of study: bioinformatics, bachelor's studies, 45 hours in the academic years: 2021/2022, 2022/2023, 2023/2024, 2024/2025 (in progress)
- Biothermodynamics with elements of statistical physics, auditorium tutorials, field of study: medical physics, bachelor's studies, 30 hours in the academic years: 2019/2020, 2020/2021, 2021/2022, 2022/2023, 2023/2024, 2024/2025 (in progress)
- Elements of statistics, laboratory tutorials, field of study: medical physics, bachelor's studies, 45 hours in the academic year 2019/2020
- Probability calculus, auditorium tutorials, field of study: bioinformatics, bachelor's studies, 30 hours in the academic year 2020/2021
- Probability calculus and elements of statistics, laboratory tutorials, field of study: medical physics, bachelor's studies, 45 hours in the academic years: 2020/2021, 2021/2022, 2022/2023, 2023/2024
- Fundamentals of the theory of stochastic processes, auditorium tutorials, field of study: mathematical modeling and data analysis, bachelor's studies, 15 hours in the academic year 2019/2020

- Rotation laboratory, field of study: bioinformatics, bachelor's studies, 20 hours in the academic year 2021/2022, 15 hours in the academic year 2023/2024
- Stochastic processes: fundamentals and applications, auditorium tutorials, field of study: mathematical modeling and data analysis, bachelor's studies, 15 hours in the academic year 2020/2021
- Advanced practical mathematics in physics, auditorium tutorials, field of study: physics, bachelor's studies, 12 hours in the academic year 2021/2022

Lectures and tutorials conducted at Collegium Medicum NCU in the years 2000–2019

Lectures:

- Mathematics, field of study: biotechnology, bachelor's studies, 38 hours in the academic year 2017/2018, 60 hours in the academic year 2018/2019
- Mathematics, field of study: pharmacy, master's studies, 10 hours in the academic years: 2016/2017, 2017/2018, 2018/2019
- Mathematical and statistical foundations of biomedical sciences, field of study: cosmetology, the part-time studies, 5 hours in the academic year 2014/2015
- Medical statistics, language: English, field of study: medicine, 10 hours in the academic year 2013/2014
- Medical statistics, field of study: medicine, 10 hours in the academic year 2013/2014
- Fundamentals of medical physics, field of study: electroradiology, bachelor's studies, 30 hours in the academic years: 2013/2014, 2014/2015, 2015/2016, 2016/2017, 2017/2018
- Fundamentals of medical physics, field of study: electroradiology, the part-time studies, 30 hours in the academic years: 2013/2014, 2014/2015
- Medical statistics, a postgraduate program in medical analytics, 5 hours in the academic years 2014/2015, 2015/2016, 2016/2017, 2017/2018

Tutorials:

- Elements of physics, field of study: biotechnology
- Informatics, fields of study: electroradiology (part-time studies), nursing (part-time studies), and medicine
- Mathematics, fields of study: biotechnology, pharmacy, and medical analytics
- Mathematical and statistical foundations of biomedical sciences, field of study: cosmetology (part-time studies)
- Mathematics with elements of statistics, field of study: cosmetology (part-time studies)
- Medical statistics, Language: English, field of study: medicine

- Fundamentals of medical physics, field of study: electroradiology
- Statistics, fields of study: medical analytics, pharmacy, and cosmetology
- Medical statistics, field of study: medicine

Coordinator of the following subjects:

- Mathematics, field of study: biotechnology, in the academic years: 2017/2018, 2018/2019, 60 hours of lectures, 60 hours of exercises
- Mathematics, field of study: pharmacy, in the academic years: 2016/2017, 2017/2018, 2018/2019, 10 hours of lectures, 25 hours of exercises
- Medical statistics, Language: English, field of study: medicine, in the academic year 2013/2014, 10 hours of lectures, 5 hours of exercises
- Fundamentals of medical physics, field of study: electroradiology (full-time and part-time studies) in the academic years: 2013/2014, 2014/2015, 2015/2016, 2016/2017, 2017/2018, 30 hours of lectures and 30 hours of exercises; in the academic year 2013/2014 30 hours of lectures and 60 hours of exercises; in the academic year 2013/2014 the subject was carried out in the amount of 30 hours of lectures and 60 hours of exercises
- Medical statistics, field of study: medicine, in the academic year 2013/2014, 10 hours of lectures, 5 hours of exercises

9.2 Supervision of master's and doctoral theses

- I am an assistant supervisor in the doctoral project of MSc. Masood Valipour, a participant in the doctoral program at the Faculty of Physics, Astronomy and Informatics, NCU; Title of the project: *Theoretical characteristics of two-photon absorption for classical and quantum fields*, main supervisor: Dr habil. Gniewomir Sarbicki, professor at Nicolaus Copernicus University, project in progress
- I supervised the master thesis of Rafał Kluska. Title of thesis: *Relational databases in the analytical laboratory*, Collegium Medicum NCU, 2018

9.3 Organisation work

- currently a member of the program committee for the Medical Physics study program
- currently a member of the program committee for the Nuclear Safety and Radiation Protection study program
- work from 2021 to 2023 in the team implementing the project of four summer schools *GENERATION QI. Next generation of quantum information scientists. Series of international schools for students in Gdańsk* at the Faculty of Mathematics, Physics and Informatics

of the University of Gdańsk, names of schools: *Quantum computation*, *Quantum dynamics and open systems*, *Quantum cryptography*, and *Picturing quantum weirdness*, project website: https://gqi.ug.edu.pl

- work in the team preparing an application for funding of summer schools at the Faculty of Mathematics, Physics and Informatics of the University of Gdańsk submitted within the SPINAKER project — intensive international education programmes 2020
- preparing syllabuses during the curriculum reform of the bioinformatics field of study at the Faculty of Mathematics, Physics and Informatics of the University of Gdańsk
- work in the team preparing the program of postgraduate studies in biostatistics at the Faculty of Pharmacy in Collegium Medicum NCU
- supervisor of the first year of medical analytics at the Collegium Medicum NCU, continuously from the academic year 2004/2005 to 2018/2019

9.4 Popularisation of science

- I participated in the science event entitled *Nauka? Taką — to ja lubię!* prepared at the Faculty of Mathematics, Physics and Informatics of the University of Gdańsk in 2021. I gave the lecture entitled *Is it possible to meet a conditional probability in a hospital, street, or courtroom?* in two secondary schools.
- On March 22, 2019, at the invitation of the Nicolaus Copernicus University Chemistry Student Association, I gave a lecture titled *Regression and linear correlation* at the Faculty of Chemistry Nicolaus Copernicus University. The lecture focused on statistical inference in regression modeling and its implementation in the SPSS software.

10 Awards

- Individual IV degree award of the Rector of the University of Gdańsk for outstanding organisational work, teaching and research in the year 2021
- Individual III degree award of the Rector of Nicolaus Copernicus University in Toruń for achievements in research activities in the year 2018

Signed by / Podpisano przez:

Anita Magdalena Dąbrowska
Uniwersytet Gdański

Date / Data: 2024-11-24 20:37