Summary of Professional Accomplishments

1. Name

Rafał Lutowski

2. Diplomas, degrees conferred in specific areas of science or arts, including the name of the institution which conferred the degree, year of degree conferment, title of the PhD dissertation

2010 PhD in Mathematics, University of Gdańsk, Faculty of Mathematics, Physics and Informatics, title: *Symmetries of flat manifolds*

2005 MSc in Mathematics, Kazimierz Wielki Academy of Bydgoszcz, Faculty of Mathematics, Technology, and Natural Sciences

3. Information on employment in research institutes or faculties/departments or school of arts

since 02.2011 assistant professor at the Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk

10.2008-02.2011 assistant at the Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk

4. Description of the achievements, set out in art. 219 para 1 point 2 of the Act

4.1. Description of the main achievement

List of publications included in the series of thematically related scientific articles entitled

Application of representation theory and computational methods in the study of flat manifolds and related structures

along with the determination of individual contributions to their creation, in the case of co-authored works, is as follows:

[H1] R. Lutowski. *On symmetry of flat manifolds*. Exp. Math. 18.2 (2009), pp. 201–204. DOI: 10.1080/10586458.2009.10128892.

[H2] R. Lutowski. *Finite outer automorphism groups of crystallographic groups*. Exp. Math. 22.4 (2013), pp. 456–464. DOI: 10.1080/10586458.2013.838719.

[H3] R. Lutowski and A. Szczepański. *Holonomy groups of flat manifolds with the R_∞ property*. Fund. Math. 223.3 (2013), pp. 195–205. DOI: 10.4064/fm223-3-1.

I am the author of the main theorem of the paper – Theorem A. Corollaries of the main theorem (Theorems 1.4 and 1.5) as well as the manuscript are the results of a joint work.

[H4] R. Lutowski and B. Putrycz. *Spin structures on flat manifolds*. J. Algebra 436 (2015), pp. 277–291. DOI: 10.1016/j.jalgebra.2015.03.037.

My contribution to the article is Corollary 2 and its application to the construction of the algorithm described in Sections 5 and 6. I verified the calculations of the coauthor, partially presented in Section 8, and added some statistics to it. I prepared the manuscript.

[H5] A. Gąsior, R. Lutowski, and A. Szczepański. *A short note about diffuse Bieberbach groups*. J. Algebra 494 (2018), pp. 237–245. DOI: 10.1016/j.jalgebra.2017.08.033.

I am the author of the computations of low-dimensional (non)diffuse Bieberbach groups, presented in Theorem 1. I have also constructed the example in the proof of Proposition 1. The beginning of Section 2, which lays the groundwork of the algorithmic determination of diffuseness of Bieberbach groups, is an outcome of joint discussions. The manuscript is a result of a joint work.

[H6] R. Lutowski and A. Szczepański. *Crystallographic groups with trivial center and outer automorphism group*. Math. Proc. Cambridge Philos. Soc. 164.2 (2018), pp. 363–368. DOI: 10.1017/S0305004117000251.

My individual contribution to the main result of the paper (Theorem 2.2) is proving Proposition 2.1 and Lemmas 2.6 and 2.7. The manuscript is an outcome of a joint work.

[H7] G. Hiss, R. Lutowski, and A. Szczepański. *Flat manifolds with holonomy representation of quaternionic type*. Comm. Algebra 49.3 (2021), pp. 1286–1294. DOI: 10.1080/00927872.2020.1834568.

I am the author of the computational results described in Section 3 and Remark 4.9. I verified the results of coauthors. The criteria presented in Proposition 4.3 and the manuscript are the outcome of a joint work.

[H8] R. Lutowski. *Flat manifolds with homogeneous holonomy representation*. Publ. Math. Debrecen 99.1-2 (2021), pp. 117–122. DOI: 10.5486/pmd.2021.8881.

[H9] R. Lutowski and A. Szczepański. *Minimal Nonsolvable Bieberbach Groups*. Exp. Math. (Oct. 2024). Published online. DOI: 10.1080/10586458.2024.2414311.

Definitions, lemmas, and corollaries that start Sections 2 and 3 are the outcome of discussions of both authors. The rest of the results are of my authorship. The manuscript is an outcome of a joint work.

In the following summary, in addition to the main achievement, I will briefly describe my other publications, listed as [O1]-[O5], in Section 4.2.

Outline of the research topic

The Online Dictionary of Crystallography defines it as *the branch of science devoted to the study of molecular and crystalline structure and properties, with far-reaching applications in mineralogy, chemistry, physics, mathematics, biology, metallurgy, and materials science* (see [ODC]). The study of the symmetries of the before-mentioned structures is an important contribution of the group theory to this field. These symmetries are described by crystallographic groups:

Definition. Let $n \in \mathbb{N}$. A *crystallographic group* of dimension n is a discrete and cocompact subgroup of the group $\operatorname{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of isometries of the n-dimensional Euclidean space.

Low-dimensional crystallographic groups, in addition to the ones indicated above, have applications in art and architecture (see, for example, [CBG08], [Wey89]). There is a reason for calling the 2-dimensional ones wallpaper, and some of them – frieze groups. The importance of this family of groups was recognized already back in the 19th century. Among the descriptions of the problems posed by Hilbert, in connection with his speech at the Mathematical Congress in Paris in 1900, there was the following one:

Question ([Hil02, p. 467]). *Is there in n-dimensional Euclidean space only a finite number of essentially different kinds of groups of motions with a (compact) fundamental region?*

In fact, the question concerned crystallographic groups, and at the time the answer was known for $n \leq 3$. A few years later, Bieberbach, in articles [Bie11] and [Bie12], solved the Hilbert's problem, extending the result with information about structures and isomorphisms between crystallographic groups. The results presented by him are now known as Bieberbach's three theorems:

Theorem 1 ([Szc12, Theorem 2.1]). *Let $n \in \mathbb{N}$.*

1. *If $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ is a crystallographic group, then the set of translations $\Gamma \cap \mathbb{R}^n$ is a lattice in $\mathbb{R}^n$, of finite index in Γ. Moreover, it is the maximal normal abelian subgroup of Γ.*
2. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugated in $\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$.*
3. *There are only finitely many isomorphism classes of crystallographic groups of dimension n.*

Note that we identify every vector $a \in \mathbb{R}^n$ with the translation in $\mathbb{R}^n$, given by the formula $x \mapsto x + a$. A *lattice* is a subgroup isomorphic to $\mathbb{Z}^n$, which spans $\mathbb{R}^n$.

Let Γ be a crystallographic group of dimension n. By the first Bieberbach theorem, it fits into the following short exact sequence

$$0 \longrightarrow L \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1, \tag{1}$$

where L is a free abelian group of rank n and it is the maximal normal abelian subgroup Γ, and G – the *holonomy group* of Γ – is finite. In 1957, Auslander and Kuranishi showed that the above – purely algebraic – description defines, up to isomorphism, crystallographic groups (see [Cha86, Theorem III.1.1]).

Let a map $\varphi\colon G \to \mathrm{GL}(L)$ be given by $g \mapsto \varphi_g$, where

$$\varphi_g(l) = \gamma l \gamma^{-1}, \tag{2}$$

for $g \in G, l \in L$ and $\gamma \in \pi^{-1}(g) \subset \Gamma$ (with the identification of L and $\iota(L)$). The function φ is a group homomorphism, called the *integral holonomy representation*, or simply the *holonomy representation*, of Γ. The maximality of L implies, that φ is faithful (see also [Cha86, Proposition I.6.1]). This representation endows L with the structure of a left faithful G-module. To underline the fact, that the module is a lattice in $\mathbb{R}^n$, we call it a *G-lattice*. By analogy, G-lattices are those G-modules, which are free $\mathbb{Z}$-modules of finite rank.

Suppose that the crystallographic group Γ is torsion-free. In this case, we call it a *Bieberbach group*, and the orbit space $X = \mathbb{R}^n/\Gamma$ is a compact Riemannian manifold with zero sectional curvature (a *flat manifold*). Moreover, every compact flat manifold, up to isometry, is such an orbit space (see [Wol11, Theorem 3.3.2]). The universal cover of X equals $\mathbb{R}^n$ and $\pi_1(X) = \Gamma$, and hence X is an Eilenberg-MacLane space of type $K(\Gamma, 1)$ (see [Cha86, p. 52] and [Lüc10, Lemma 1.2]). Therefore, the topology of X is fully described by its fundamental group. It turns out that the differential structure of X is determined by Γ as well. Indeed, the geometric version of Bieberbach's theorems has the following form:

Theorem 2 ([Cha86, Theorems 5.3-5.5], [Wol11, Theorem 3.3.1]).

1. *Let X be a compact flat manifold. Then X is covered by a flat torus, and the covering map is a local isometry. Moreover, the holonomy group of X is finite.*
2. *Let X and Y be compact flat manifolds. Suppose that $\pi_1(X) \cong \pi_1(Y)$. Then X and Y are affinely equivalent.*
3. *There are only finitely many affine equivalence classes of compact flat manifolds of a given dimension.*

Let's go back to formula (1). Maps ι and π are in fact restrictions of the homomorphisms

$$\iota\colon \mathbb{R}^n \to \mathrm{Aff}(\mathbb{R}^n), \pi\colon \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{GL}_n(\mathbb{R}), \tag{3}$$

given by

$$\iota(a) := (I, a), \pi(A, a) := A,$$

for $A \in \mathrm{GL}_n(\mathbb{R}), a \in \mathbb{R}^n$. Then, $L = \iota^{-1}(\Gamma \cap \mathbb{R}^n)$ and (2) takes the following form:

$$\varphi_g(l) = g \cdot l.$$

Suppose that Γ is torsion-free. We get that every automorphism φ_g of L is given by the left multiplication by the element g of the holonomy group $G = \pi(\Gamma)$ of the flat manifold X (see [Cha86, p. 52]). Hence, the holonomy representation $\varphi\colon G \to \mathrm{GL}(L)$ of the Bieberbach group Γ and the holonomy representation of the manifold X, seen as the embedding $G \hookrightarrow O(n)$, are equivalent over the field of real numbers. This fact gives another connection between algebraic properties of Bieberbach groups and geometric properties of compact flat manifolds.

Essential, especially when it comes to constructions of Bieberbach groups, is a cohomology class

$$\alpha \in H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} L/L) \cong H^2(G, L) \tag{4}$$

which corresponds to the extension (1). Let

$$*\colon N_{\mathrm{GL}(L)}(\varphi(G)) \times H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} L/L) \to H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} L/L) \tag{5}$$

be the action defined in [Szc12, s. 65]. Then Γ is torsion-free if and only if

$$\operatorname{res}_{\langle x \rangle} \alpha \neq 0$$

for $x \in \mathcal{X}$, where $\mathcal{X}$ is a set of generators of representatives of conjugacy classes of subgroups of G of prime order. In this case, α is called a *special element*.

In the more than one hundred years since Bieberbach's articles appeared, the theory of crystallographic groups, and in particular the associated geometry, has been systematically developed. Among the investigated topics, we can find:

1. *Homology and cohomology theory.* As an example, in [Szc83] the author presents a family of flat manifolds, elements of which are rational homology spheres. Cohomological rigidity is investigated in [KM09], [PS16]. Issues of characteristic classes one can find in [CMR10], [Vas70], in particular spin and spin^c structures are considered in [DSS06], [GS13], [GS14], [Gąs17], [HS08], [Pfä00], [PS10].

2. *Study of holonomy groups.* In [AK57] Auslander and Kuranishi showed that every finite group is a holonomy group of some compact flat manifold. Over 30 years later, Cliff and Weiss gave a new proof of this fact and proved that every finite group is a holonomy group of some compact flat manifold with the first Betti number equal to 1 (see [CW89, Theorem 5]). Connections between holonomy groups and the first Betti number were also considered in [Hil+87], [HS86].

3. *Study of holonomy representation and its connection to the properties of a manifold.* From the perspective of the study of compact flat manifolds, one of the more important is the main result of [HS91] which states that the holonomy representation of a Bieberbach group is reducible. Connections of the holonomy representations with the symmetries or the fixed point theory of compact flat manifolds and, even more general – of crystallographic groups, have been investigated (see [DDP09], [DKT19], [HS97], [HS95]).

4. *Study of algebraic properties of Bieberbach groups.* Bieberbach groups have their place in the theory of the Yang-Baxter equation as so-called structural groups of a certain family of solutions. Moreover, some of their properties affect the properties of the solutions. From this, though not only, perspective, the objects of interest are diffuse Bieberbach groups, or groups with the unique product property (see [Bow00], [Gar21a], [Gar23], [GV98], [KR16], [LV19], [Pro88]).

5. *Classification problem.* There are three known methods of constructing flat manifolds in the literature (see [Szc12, Chapter 3]):
   - Calabi's method, who presented in 1957 some sort of decomposition of flat manifolds with a positive first Betti number (see [Cal57], [Wol11, Theorem 3.6.3]);
   - Auslander-Vasquez method, which classifies manifolds with the given holonomy group and is based on Vasquez number (see [Szc97], [Vas70]);
   - Zassenhaus algorithm, based on the algebraic structure of crystallographic groups (see [CS01], [Zas48]).

   It is worth noticing that the Zassenhaus algorithm has been used to classify all crystallographic groups of degree up to 6. Calculations have been performed with the usage of the computer package CARAT [OPS08].

6. *Study of certain families of compact flat manifolds.* Here I would like to put attention on the (general) Hantzsche-Wendt manifolds (see [HW35], [MR99], [PS16], [Put07], [RS05], [Zim90]) and, on a larger family of flat manifolds of diagonal type, which consists, for example, of real Bott manifolds, investigated in [CMO17], [GS14], [GL24], [KM09]. Except for tools of geometry and topology, their properties can be studied using combinatorics, which makes them a nice object for the application of computational methods.
7. *Study of deformations of flat manifolds* in the sense of Gromov-Haussdorff limit. These can be treated as a more topological or geometrical approach to their study. In general, for Riemannian manifolds, such a *collapsing* is described in [Fuk06]. It is known that the limits of flat manifolds don't have to be smooth. Constructions and properties of these limits are considered in [Bet+22], [BDP18], [DP19].
8. *Study of flat manifolds with additional structures.* In particular, I'm thinking here about manifolds with Kaehler structures and problems connected to the structure of their fundamental groups, symmetries, classification, or deformation of their complex structures. These topics can be found in [BR11], [CC17], [DHS09], [DG23], [GL23], [Joh90].

Please note that the above list illustrates some – not all – directions of current or possible study of the widely understood topic of compact flat manifolds. Furthermore, the articles authored by me are restricted to preprints only, since the published ones will be discussed in this summary.

Remark. Unless stated otherwise, all manifolds in this summary are *Riemannian*, and all flat manifolds are assumed to be *compact*.

Outer automorphisms of crystallographic groups. Articles [H1], [H2], [H6].

Let Γ be a crystallographic group of dimension n, given by the short exact sequence (1), which corresponds to the cohomology class α as in (4). Let N_α be the stabilizer of α under the action (5), and let $\mathrm{Aut}^0(\Gamma)$ be the subgroup of those automorphisms of Γ, which induce identities on both L and G (see [Cha86, Definition V.1.1]). Then Diagram 1 commutes, and its columns and rows are exact (see [Cha86, Theorem V.1.1], [Szc12, Theorem 5.1]). Moreover, if Γ is a Bieberbach group, the diagram provides essential information on the structure of the

$$
\begin{array}{ccccccccc}
 & & 0 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/L^G & \longrightarrow & \mathrm{Inn}(\Gamma) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Aut}^0(\Gamma) & \longrightarrow & \mathrm{Aut}(\Gamma) & \longrightarrow & N_\alpha & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(G,L) & \longrightarrow & \mathrm{Out}(\Gamma) & \longrightarrow & N_\alpha/G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 1 & & 1 & &
\end{array}
$$

Diagram 1. (Outer) Automorphisms of crystallographic groups.

group of affine equivalences of the flat manifold $X = \mathbb{R}^n/\Gamma$. We have a short exact sequence

$$1 \longrightarrow \mathrm{Aff}_0(X) \longrightarrow \mathrm{Aff}(X) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow 1,$$

where $\mathrm{Aff}_0(X)$ is a torus of the dimension equal to the first Betti number $b_1(X)$ of the manifold X (see [Cha86, Theorem V.6.1], [Szc12, Theorem 5.2]). This shows a significant role played by the flat manifolds with finite first homology groups. The question, which finite groups are their groups of symmetries, is stated as [Szc06, Problem 6].

Remark. The before-mentioned Calabi method is another indicator of the importance of flat manifolds with the first Betti number equal to zero. Let X be a flat manifold. If $q = b_1(X) > 0$, then there exists a flat manifold Y and a finite abelian group $H < \text{Aff}(Y)$, which acts freely on the q-dimensional torus $T^q = (\mathbb{R}/\mathbb{Z})^q$, such that

$$X = (Y \times T^q)/H$$

(see [Wol11, Theorem 3.6.3]). If $b_1(X) = 0$, then, in the sense of this construction, X is an indecomposable object.

The main goal of the short paper **[H1]** was an answer to the question about the existence of a flat manifold with a nontrivial, odd-order group of symmetries. Such an example has not been known before, hence – referring to the mentioned above [Szc06, Problem 6] – the question was natural and important. The answer was "Yes", and it was given with the usage of the example of the Bieberbach group with a trivial outer automorphism group, presented by Waldmüller in [Wal03]. Modifying his construction, I showed that there exists a Bieberbach group with a trivial center and an outer automorphism group of order 3. I would like to draw attention to the experimental nature of not only finding a G-lattice, denoted in the paper as M_2, but also determining the group $C_{\text{Aut}(M_2)}(G)$ of the invertible elements of the ring $\text{End}_{\mathbb{Z}G}(M_2)$. Finding a nontrivial generator B of this ring alone was a linear problem. Determining the relation of B with one of the ring was the result of trials and errors. For the sake of completeness, there exist algorithms, implemented for example in CARAT [OPS08], for calculating the centralizers of finite subgroups of $\text{GL}_n(\mathbb{Z})$. However, in the case of M_2 the $\mathbb{Z}$-rank n was equal to 32, which was too big a computational challenge (I dare to say this is still the case).

In the article, I also proved Theorem 3.4 on the structure of the outer automorphism group of a direct product of Bieberbach groups with trivial centers. In particular, this gave a result that for every n the symmetric group S_n can be realized as a group of symmetries of a flat manifold. Therefore, by the Cayley theorem, every finite group is a subgroup of finite index of the group of symmetries of some flat manifold with the first Betti number equal to zero.

The paper **[H2]** deals with an algorithmic approach to the calculation of outer automorphism groups of crystallographic groups in the case, when they are known to be finite. The criterion of the finiteness can be found in [Szc96, Theorem A] (note that the assumption on the group to be torsion-free is not really needed here).

Let Γ be a crystallographic group as above. In the computational theory of crystallographic groups, it is convenient to fix a basis of L, identifying it with $\mathbb{Z}^n$. Since the holonomy representation $\varphi\colon G \to \text{GL}_n(\mathbb{Z})$ is faithful, hence G can be identified with $\varphi(G)$.

The key to calculating $\text{Out}(\Gamma)$ is to know two elements of Diagram 1: N_α and $H^1(G, \mathbb{Z}^n)$. In turn, the main tool to determine those objects is solving the system of equations of the form

$$AX = B, \tag{6}$$

where $A \in M_{n\times m}(\mathbb{Z}), X, B \in M_{n\times 1}(\mathbb{Q}/\mathbb{Z}) = (\mathbb{Q}/\mathbb{Z})^n$. By $M_{k\times l}(R)$ I denote the set of matrices with k rows and l columns, with coefficients in R. We solve the system (6) by transforming A into a diagonal form (a Smith normal form, for example). Note that we use the same matrix A in the calculation of both N_α and $H^1(G, \mathbb{Z}^n)$. Using, in addition, a long cohomology sequence, [H2, Theorem 4.5] determines generators of the group in the middle row of Diagram 1. In particular, we get that the representatives of cohomology classes in $H^1(G, \mathbb{Z}^n)$ are elements from $\mathbb{Q}^n$, and that N_α/G acts on them simply by matrix multiplication. By [Joh97, Proposition 10.1] we can give the presentation $\text{Out}(\Gamma)$ as an extension of $H^1(G, \mathbb{Z}^n)$ by N_α/G.

Continuing the topic of symmetries, the paper **[H6]** deals with *complete* crystallographic groups. Recall that a group is complete if both its center and outer automorphism group are trivial. In [Wal03] Waldmüller constructed a complete Bieberbach group, in [BL05] the authors proved that for every $n \geq 3$ there exists a compact hyperbolic n-manifold with a complete fundamental group.

The basic ingredient in the proof of the main result is a sequence (Γ_i), where

$$\Gamma_0 := \Gamma \text{ and } \Gamma_{i+1} := N_{\mathrm{Aff}(\mathbb{R}^n)}(\Gamma_i) \text{ for } i \geq 0. \tag{7}$$

It turns out that if

$$Z(\Gamma) = 1, |\operatorname{Out}(\Gamma)| < \infty \text{ and } H^1(G, L) = 0, \tag{8}$$

then the above sequence stabilizes, and it consists of crystallographic groups. This means in particular that for some $N \in \mathbb{N}$, the Γ_N is complete.

Using [H1, Theorem 3.4] for a direct product of some copies of certain crystallographic groups of dimensions 2 and 3, we get the main result of [H6]:

Theorem 3 ([H6, Theorem 2.2]). *For every natural number $n \geq 2$ there exists n-dimensional complete crystallographic group.*

Remark. If the conditions given in (8) are met, then $\Gamma_{i+1} \cong \operatorname{Aut}(\Gamma_i)$, for $i \geq 0$. Hence, the above considerations give a sufficient condition for the *automorphism tower* (see[Tho85]) of a crystallographic group to end after a finite number of steps. Moreover, in every dimension greater than 1, there exists a crystallographic group with the finite automorphism tower.

Fixed point theory. Article [H3].

Finding fixed points of a self-map of a topological space has been in the area of interest of mathematicians since the beginning of the topology itself (see [Bro99]). Some time later, a more general question, which has been a starting point of the fixed point theory until today, was stated: what is the minimal number of fixed points for maps in the homotopy class of a given function? It is enough for our purposes to assume that $f: X \to X$ is a continuous self-map of a closed smooth n-dimensional manifold X. In the first half of the twentieth century, some invariants of the homotopy class of f were defined, which provide certain information about fixed points of maps homotopic to f. These are: Lefschetz number $L(f)$, given by maps induced by f on rational homologies of X, Nielsen number $N(f)$, and Reidemeister number $R(f)$. The latter ones can be defined by covering maps of f in the universal cover $\widetilde{X}$. Of great usefulness is the Nielsen number, since:

a) it gives a lower bound for the number of fixed points in the whole homotopy class of f; in particular, $N(f)$ is a nonnegative integer (see [Jia83, Theorem 4.3]);
b) if $n \geq 3$ then there exists a map homotopic to f, and which has $N(f)$ of fixed points (see [Jia83, Theorem 6.3], [Jia81]).

The inconvenience of using the Nielsen number lies in the fact that it is difficult to compute, especially when compared to the other two invariants. In general, we have an inequality $N(f) \leq R(f)$, and one of the directions of the research in the theory is looking for conditions, under which it becomes an equality. Therefore, a sort of extreme independence between Nielsen and Reidemeister numbers is the case when they are not equal for a certain class of maps. One of these situations is given by the R_∞ property:

Definition. A manifold X has the R_∞ property if $R(f) = \infty$ for every homeomorphism $f: X \to X$.

Reidemeister number can be defined at the level of the fundamental group of X. If

$$f_\sharp: \pi_1(X) \to \pi_1(X)$$

is the homomorphism induced by f, then $R(f)$ counts so-called *$f_\sharp$-conjugacy classes*, also called *twisted conjugacy classes* or *Reidemeister classes of f* (see [Jia83, Sections I.1 and II.1]). Obviously, the Reidemeister number can be defined for any group, and in this summary we speak about the R_∞ property for flat manifolds and Bieberbach groups as well.

The article **[H3]** deals with connections of holonomy representation and the R_∞ property for Bieberbach groups. By [DDP09, Theorem 5.9] we know that the holonomy representation itself does not determine the property in general. However, under conditions given by [DDP09, Theorem 4.7], it does. Assumptions of the theorem imply certain properties of the holonomy representation, which justify stating the following conjecture.

Conjecture 1 ([DDP09, Conjecture 4.8]). *Let $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation of a finite group $G \neq 1$. Suppose that n is an odd integer. Then for every element $D \in N_{\mathrm{GL}_n(\mathbb{Z})}(\varphi(G))$ there exists $a \in G$ such that $\varphi(a)D$ has an eigenvalue* 1.

Note that [DDP09, Example 4.9] shows that the assumption of n being odd is necessary. The main goal of [H3] is the proof of the conjecture for a certain family of groups:

Theorem 4 ([H3, Theorem A]). *Conjecture 1 is true if G has a nontrivial abelian normal subgroup.*

As a result, we get a sufficient condition for flat manifolds with solvable fundamental groups to have the R_∞ property:

Theorem 5 ([H3, Theorem 1.4]). *Let Γ be a Bieberbach group with the holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$, where G is solvable. Suppose d is odd and $\varphi'\colon G \to \mathrm{GL}_d(\mathbb{Z})$ is an $\mathbb{R}$-irreducible of φ of multiplicity* 1. *Assume moreover that if a subrepresentation $\tilde{\varphi}\colon G \to \mathrm{GL}_d(\mathbb{Z})$ of φ is not equivalent to φ', then the groups $\varphi'(G)$ and $\tilde{\varphi}(G)$ are not conjugated in $\mathrm{GL}_d(\mathbb{Q})$. Then Γ has the R_∞ property.*

Spin structures on flat manifolds. Article [H4].

Let $n \in \mathbb{N}$. The group $\mathrm{Spin}(n)$ is a subgroup of invertible elements of the Clifford algebra C_n. It is a double cover, and for $n \geq 3$ it is also the universal cover, of the orthogonal group $\mathrm{SO}(n)$. Denote by $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ the covering homomorphism. Assume that X is an orientable closed n-dimensional manifold, and Q is the $\mathrm{SO}(n)$-principal frame bundle over X. A *spin structure* on X is an equivariant lift of Q via λ to a $\mathrm{Spin}(n)$-principal bundle. Spin structures and, connected to them, Dirac operators are an important object of study not only from the perspective of geometry, but also physics.

In the case when X is flat, [Pfä00, Proposition 3.2] and [HS08, Proposition 2.1] give an algebraic tool for the classification of spin structures. In fact, they imply [H4, Corollary 1] which states that we are interested in the commutativity of the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon} & \mathrm{Spin}(n) \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\lambda} \\ G & \xrightarrow{\rho} & \mathrm{SO}(n) \end{array} \tag{9}$$

where $\rho\colon G \to \mathrm{SO}(n)$ is any representation equivalent to the identity $id\colon G \to G \subset \mathrm{SO}(n)$. Knowing the right representation ρ allows one to overcome the greatest difficulty in application of the above criterion in finding spin structures on X, that is – determination of $\lambda^{-1}(\rho(G))$. In general, we have to use ad hoc methods – the article [PS10] is a good example here. In the case of larger dimensions or the number of manifolds under consideration, this approach is at least problematic. We know, however, that X is spin if and only if the flat manifold $\mathbb{R}^n/\pi^{-1}(\mathrm{Syl}_2 G)$ is spin, where $\mathrm{Syl}_2 G$ denotes a 2 Sylow subgroup of G (see [DSS06, Proposition 1]). Suppose that G is a 2-group. In this case, we can construct ρ in such a way that finding spin structures on X becomes an algorithmic process. For 2-groups [EM79, Theorem 1.10] gives us the following corollary.

Corollary ([H4, Corollary 2]). *Every irreducible rational representation of a* 2*-group is induced from a rational representation of degree* 1.

We get that there exists a representation $\rho\colon G \to \mathrm{SO}(n,\mathbb{Z})$, equivalent to the identity, where

$$\mathrm{SO}(n,\mathbb{Z}) := \mathrm{SL}(n,\mathbb{Z}) \cap \mathrm{SO}(n).$$

As a consequence, in diagram (9) we can replace $\mathrm{SO}(n)$ with $\mathrm{SO}(n,\mathbb{Z})$. This change seems minor at first sight, but by [H4, Lemma 7] we get formulas for the preimages of the generators, and eventually – all the elements of the group $\mathrm{SO}(n, \mathbb{Z})$, via λ. Therefore, we know all the possible forms of the homomorphism ε restricted to a generating set of Γ. Since Γ is finitely presented, we can determine all possible maps ε. Finally, we know how to determine all spin structures for flat manifolds with holonomy being 2-groups.

Now, consider a situation when the holonomy G of the manifold X is arbitrary. In this case, we cannot use our algorithm to classify spin structures on X. We are able, however, to check if X is spin or not. When it is, we can employ the fact that the spin structures are in bijection with the elements of the cohomology group $H^1(X, \mathbb{Z}/2) \cong H^1(\Gamma, \mathbb{Z}/2)$, and – using for example the GAP package HAPcryst [RG22] – give their number.

In the last section of the article, we give detailed information on the dimension 5 along with some statistics on dimensions both 5 and 6. These and some other data on low-dimensional flat manifolds can be found on the website [Lut21].

Properties of the holonomy representation. Articles [H7], [H8].

A Bieberbach group Γ is given by the short exact sequence (1), and so it is defined by a G-lattice L and a special element α. Obviously, those two objects are not mutually independent. Only when we have a proper G-module, we can construct a Bieberbach group, for which the module defines the holonomy representation. Moreover, as stated earlier in the summary, some properties of flat manifolds depend directly on certain properties of their holonomy representations. Therefore, the study of the latter is an important component of the theory of flat manifolds.

The article **[H7]** deals with the decomposition of the holonomy representation over the field of the real numbers. It is well known, that the endomorphism algebra of every irreducible component is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the paper, we are interested in the case when all irreducible summands have isomorphic endomorphism algebras. We can say then that the manifold is of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ type. In the complex and quaternionic cases we get, in a way nontrivial, examples of flat manifolds with Kaehler and hyperkaehler structures, respectively. The non-triviality has the following meaning. By Propositions 7.1 and 7.2 from [Szc12] one can conclude that if φ is a holonomy representation of a flat manifold, then using $\varphi \oplus \varphi$ we can construct a flat Kaehler manifold. By the same method, we get flat hyperkaehler manifolds from the ones with Kaehler structures.

In the literature, there have been examples of flat manifolds of type $\mathbb{R}$ and $\mathbb{C}$, with holonomy groups being elementary 2 and 3 groups, respectively. It has not been known whether there exists a flat manifold X of type $\mathbb{H}$ at all. An essential part of the investigation was stating the necessary conditions for the holonomy group and representation of such a possible X. In particular, it turned out that 2 groups may be important. Having quite a big set of information about the potential holonomy group, one could start a systematic approach by looking through the finite group database of GAP [20].

It turns out that, up to order 64, there exists only one finite group G that fulfills all the conditions given by [H7, Proposition 4.3]. It is denoted in GAP as [64, 245], and it is a central extension of $C_2^2 = \langle a, b \rangle$ by C_2^4, where C_2 is a cyclic group of order 2. The automorphism group of G acts transitively on the set $\mathcal{X} := \{a, b, ab\}$ of the nontrivial elements of the center $Z(G)$ of G. Moreover, $\mathcal{X}$ consists of all elements of order 2 in G. Therefore, finding a special element is equivalent to finding such a G-lattice L_a that there exists $\alpha_a \in H^2(G, L_a)$, for which the restriction $\mathrm{res}_{\langle a \rangle} \alpha_a$ to $\langle a \rangle$ is nonzero. For a flat manifold constructed this way to be of type $\mathbb{H}$, we require in addition that the character of $\mathbb{C} \otimes_{\mathbb{Z}} L_a$ is a sum of characters with the Frobenius-Schur indicators equal to -1. A good choice for L_a is a *homogeneous* component of $\mathrm{ind}_{\langle a \rangle}^G \mathbb{Z}$, which is a summand of the so-called *standard lattice* (see [Ser77, Chapter I] and [CW89] respectively). The rank of $\mathbb{Z}$-lattice L_a equals 16, and hence there exists a 48-dimensional flat manifold of quaternionic type.

In the context of computations connected with the above result, I would like to put attention on a certain fact. The module $\mathbb{Q} \otimes_{\mathbb{Z}} L_a$ has two isomorphic components of dimension 8. This means in particular that there exists a G-lattice L' of $\mathbb{Z}$-rank 8 such that $\mathbb{R} \otimes_{\mathbb{Z}} L'$ is irreducible, and $\mathrm{End}_{\mathbb{R}G}(\mathbb{R} \otimes_{\mathbb{Z}} L') \cong \mathbb{H}$. During the research, we have constructed such a lattice L', however $\mathrm{res}_{\langle a \rangle} \left(H^2(G, L')\right) = 0$, hence L' was not suitable for a construction of a special element described above. The next – a natural – step was investigation of all submodules of L', up to isomorphism. To find them, an application from the package CARAT was started. After a few weeks, computations did not finish. Despite the passage of over twenty years since the before mentioned package was developed and a giant growth of computer performance, matrix groups of not so big dimension 8 can still be a challenge.

The above article makes a good example of the difficulties one encounters when trying to find flat manifolds with certain properties. The basis, and at the same time the greatest challenge, is finding a holonomy representation. By [HS91] we know that over rationals it has at least two components. The article **[H8]** continues the topic addressed by Hiss and Szczepański. It turns out that their result can be generalized, so that in the decomposition of the holonomy representation over $\mathbb{Q}$ there are not only at least two components, but at least two of them are pairwise nonisomorphic. The idea of the generalization is as follows. Let M and L be G-lattices such that $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is irreducible, and $\mathbb{Q} \otimes_{\mathbb{Z}} L$ is a homogeneous $\mathbb{Q}G$-module containing $\mathbb{Q} \otimes_{\mathbb{Z}} M$, i.e.,

$$\mathbb{Q} \otimes_{\mathbb{Z}} L = \mathbb{Q} \otimes_{\mathbb{Z}} M \oplus \ldots \oplus \mathbb{Q} \otimes_{\mathbb{Z}} M.$$

In this case, the *sets* of simple components of $\mathbb{C} \otimes_{\mathbb{Z}} M$ and $\mathbb{C} \otimes_{\mathbb{Z}} L$ coincide. This fact leads to the conclusion that the necessary conditions for the existence of a special element in both $H^2(G, M)$ and $H^2(G, L)$ are the same. Assume that $G \neq 1$ and denote by $\operatorname{Soc} G$ the socle of G, i.e., the subgroup generated by the minimal subgroups of G. Corollary 3.4 of [H8] states that if we can use the G-lattice L to define a Bieberbach group (1), then $\operatorname{Soc} G$ is a product of elementary abelian groups. Here, our assumption of the existence of a special element implies, that L is not a faithful G-module. Finally, the only possibility of using L in a construction of a torsion-free extension (1) is the case when $G = 1$ (see [H8, Section 4]). We get the main result of the paper, which according to our notation has the following form.

Theorem 6 ([H8, Theorem 1]). *If Γ is a Bieberbach group given by the short exact sequence* (1)*, where $\mathbb{Q} \otimes_{\mathbb{Z}} L$ is homogeneous, then $\Gamma \cong L$.*

The above theorem provides us substantial information on compact flat Kaehler manifolds. If X is such a manifold, of complex dimension n, and $\Gamma = \pi_1(X)$ defines the short exact sequence (1), then Γ can be realized as a discrete, cocompact and torsion-free subgroup of $U(n) \ltimes \mathbb{C}^n$. In this case, the holonomy representation is of the form $\varphi_c \colon G \to U(n)$ (see [Szc12, Proposition 7.1]). It turns out that the homogeneity of φ_c implies the homogeneity of the $\mathbb{Q}G$-module $\mathbb{Q} \otimes_{\mathbb{Z}} L$. It is possible only if $G = 1$ and, therefore, X is a compact flat complex torus. We get in particular the complex analog of the main result of [HS91]. To be more precise – the representation φ_c is irreducible.

Diffuse and solvable Bieberbach groups. Prace [H5], [H9].

Let me put the attention on those aspects, connected with Bieberbach groups, which – at least at first sight – are more of algebraic than geometric nature. Papers [H5], [H9] restrict the case considered to the one where the groups are solvable. Although any diffuse Bieberbach group is solvable, these notions are connected not only by these simple implications. Enough to say, both diffuse and solvable groups are in the area of interest of researchers working with Yang-Baxter equations and connected to them theories of braces, structure and adjoint groups (cf. [BCV18], [LV19], [Rum24]). Note that diffuseness implies unique product property, which gives us the Kaplansky unit conjecture on invertible elements in group rings. This is a place, where examples presented in [Gar21b], [Gar23] should be mentioned. They show that in general the conjecture does not hold, and it fails for group rings over Γ, where Γ is one of the three-dimensional Bieberbach groups, so called Hantzsche-Wendt group.

The before-mentioned Hantzsche-Wendt group is also known in the literature under the name *Promislow group* Δ_P. The name comes from the author of [Pro88], where he showed that it does not have the unique product property. In a way, it has been a generic case of such a group until today – all known examples of Bieberbach groups without the unique product property contain Δ_P. One of the motivations for writing **[H5]** was the answer to the similar question for diffuse groups. To be more precise, the authors of [KR16] asked about the minimal dimension d of a nondiffuse Bieberbach group, which does not contain the Promislow group. Classifying diffuse groups of dimension up to four and, using an example with holonomy C_3^2, they got $5 \leq d \leq 8$. One of the aspects of the research presented in the article [H5] was the construction of an algorithm that classifies diffuse Bieberbach groups. The idea of the algorithm lays in the Calabi construction. For any Bieberbach group Γ, it allows building a subnormal series of Bieberbach groups

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \ldots \rhd \Gamma_k,$$

for which every factor $\Gamma_0/\Gamma_1, \ldots, \Gamma_{k-1}/\Gamma_k$ is a nontrivial free abelian group. Then Γ is diffuse if and only if Γ_k is free abelian. We have used our algorithm for the groups of dimension less than or equal to 6. The summary of this application is given by [H5, Theorem 1], and the details can be found on the website [Lut21]. We found out that the list in [KR16] omits one nondiffuse group in dimension 4. However, since the group contains Δ_P, hence the conclusion of the paper on d remains correct. Finally, it turned out that in dimension 5 there exists a nondiffuse Bieberbach group that does not contain Δ_P.

The problem solved in **[H9]** is connected to a general question of the minimal Hirsch length of torsion-free virtual solvable groups, posed by Jonathan Hillman. He asked a coauthor of the paper if there is something known in the case of torsion-free virtually abelian groups, i.e., Bieberbach groups. Obviously, Hirsch length here is simply the dimension. Although the topic seems purely algebraic, it has some topological background. Finding a nonsolvable Bieberbach group of minimal dimension allows, in particular, to find a perfect Bieberbach group and therefore – a flat manifold with the trivial first homology group (see [Bel]).

The starting point of our considerations consisted of

a) a simple fact, that solvability of any Bieberbach group is equivalent to the solvability of its holonomy group;
b) [Ple89, Theorem V.1], which shows the existence of a Bieberbach group with a simple nonabelian holonomy group in dimension 15.

It seemed natural to conjecture that up to dimension 15 all Bieberbach groups are solvable. To show it, by Bieberbach theorems, it was enough to check only finitely many groups. Unfortunately, Bieberbach groups are classified up to dimension 6 and, as mentioned before, problems with calculations already in dimension 8 suggest that the classification will not expand in the near future. As a result, we could not focus on the dimension alone. Looking for possible holonomy groups, assuming only their nonsolvability, could be beyond our computational capabilities. There was a need to minimize holonomy groups as well.

The idea above lays the foundations of the definition of a *minimal nonsolvable Bieberbach group*. Assume that Γ, given by the extension (1), is such a group. This implies that G is a minimal nonsolvable finite group. By the investigation of the decomposition of the holonomy representation over the rationals (of the $\mathbb{Q}G$-module $\mathbb{Q} \otimes_{\mathbb{Z}} L$), we get that G must have at least two irreducible, not equivalent over $\mathbb{Q}$ representation of the form $G \to \mathrm{GL}_n(\mathbb{Z})$, where $4 \leq n \leq 10$. It is the first key information. Suppose that $4 \leq n \leq 10$. By the results of Dade [Dad65], Plesken and Pohst [PP77a], [PP77b], [PP80a], [PP80b], [PP80c] and Souvignier [Sou94] we know what are the maximal irreducible subgroups of $\mathrm{GL}_n(\mathbb{Z})$ and we can look for their minimal nonsolvable subgroups. This task turns out to be computationally expensive, especially for the groups of large order and $n \in \{9, 10\}$. We have, however, the second key information – G is a perfect group, and thanks to the classification by Plesken and Holt [HP89], recently completed by Hulpke [Hul22], we know perfect groups up to the order $2 \cdot 10^6$. With the usage of the mentioned databases and representation theory, one can prove:

Proposition ([H9, Proposition 4.6]). *If $\rho\colon G \to \mathrm{GL}_n(\mathbb{Z})$ is an irreducible representation of a minimal nonsolvable group G, where $4 \leq n \leq 10$, then $\rho(G)$ is:*

a) *a simple group H, for $H \in \{A_5, L_3(2), L_2(8)\}$ or*
b) *$H2^1$ – a nontrivial central extension of C_2 by a simple group H, for $H \in \{A_5, L_3(2)\}$ or*
c) *$L_3(2)N2^3$ – a nontrivial extension of C_2^3 by $L_3(2)$.*

Considering all possible cases of a decomposition of the holonomy representation of Γ, we can determine the isomorphism type of G. Analyzing possible dimensions of Γ, these up to 12 are excluded by the before-mentioned [Ple89, Theorem V.1] and Theorem 6. Dimensions 13 and 14 require looking into arguments lying foundations of the results of [HS91], and which illustrate an important contribution of the local representation theory to the theory of Bieberbach groups. Excluding those dimensions, we get the main result of the paper:

Theorem 7 ([H9, Theorem 1]). *The minimal dimension of a nonsolvable Bieberbach group is* 15.

4.2. Other achievements

The list of articles not contained in the cycle described above is the following:

[O1] R. Lutowski and Z. Marciniak. *Affine representations of Fibonacci groups and flat manifolds*. Comm. Algebra 46.6 (2018), pp. 2738–2741. DOI: `10.1080/00927872.2017.1399412`.

[O2] R. Lutowski, N. Petrosyan, and A. Szczepański. *Classification of spin structures on four-dimensional almost-flat manifolds*. Mathematika 64.1 (2018), pp. 253–266. DOI: `10.1112/S0025579317000560`.

[O3] R. Lutowski, N. Petrosyan, J. Popko, and A. Szczepański. *Spin structures of flat manifolds of diagonal type*. Homology Homotopy Appl. 21.2 (2019), pp. 333–344. DOI: `10.4310/HHA.2019.v21.n2.a18`.

[O4] E. Acri, R. Lutowski, and L. Vendramin. *Retractability of solutions to the Yang-Baxter equation and p-nilpotency of skew braces*. Internat. J. Algebra Comput. 30.1 (2020), pp. 91–115. DOI: `10.1142/S0218196719500656`.

[O5] R. Lutowski, J. Popko, and A. Szczepański. *Spin^c structures on Hantzsche-Wendt manifolds*. J. Geom. Phys. 171 (2022), Paper No. 104394, 17. DOI: `10.1016/j.geomphys.2021.104394`.

Spin structures on almost flat manifolds. Article [O2].

The article **[O2]** deals with the problem of classification of spin structures on low-dimensional almost flat manifolds. The family of almost flat manifolds is, in fact, a generalization of the flat manifolds to the case where $\mathbb{R}^n$ is replaced by a connected, simply-connected nilpotent Lie group N (see [Gro78], [Ruh82]). We get an *almost Bieberbach* group, i.e., a torsion-free and discrete subgroup of $C \ltimes N$, for which the orbit space – *infra-nilmanifold* $X = N/\Gamma$ – is compact. By C, we denote a maximal compact subgroup of the group $\operatorname{Aut}(N)$ of Lie automorphisms of N. Bieberbach theorems, up to a certain extent, are generalized to this family of groups (see [Dek96, Chapters 2.2, 2.3]). In particular, Γ fits into the short exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \tag{10}$$

where Λ is the maximal normal nilpotent subgroup of Γ, and finite group G is the holonomy group of the manifold X. Let $n = \dim X$. By a modification of the lower central series of Λ, we can construct an integral representation $G \to \mathrm{GL}_n(\mathbb{Z})$, equivalent to the holonomy representation of X. Classification of spin structures, and in particular their existence, is given by analogs of the conditions for flat manifolds. We have that the following are important: commutativity of the correspondent of diagram (9) and 2-Sylow subgroups of G.

The goal of the paper is the determination of almost flat spin 4-manifolds. Although there are infinitely many of them, the classification given in [Dek96] gives an idea. Suppose that an almost Bieberbach group Γ, of dimension 4, defines the short exact sequence (10). Assume that S is a set of generators Λ. Let Λ^2 be the normal closure of the subgroup of generated by squares of the generators of Λ:

$$\Lambda^2 := \langle\!\langle s^2 \mid s \in S \rangle\!\rangle.$$

The before-mentioned analog of the diagram (9) and the corresponding classification of spin structures can be modified in such a way, that we use Γ/Λ^2 instead of Γ. What we get, and what is the key from the perspective of an infinite number of infra-nilmanifolds of dimension 4, is that there are only finitely many groups of the form Γ/Λ^2.

Using tools of [H4] allows one to find preimages of the holonomy groups of almost flat 4-manifolds via the covering map $\lambda\colon \mathrm{Spin}(4) \to \mathrm{SO}(4)$, which finally leads to the classification of spin structures on those manifolds. Additionally, in this case, we also get information whether they are parallelizable.

Structure groups of multipermutation solutions of the Yang-Baxter equation. Article [O4].

A *set-theoretic solution to the Yang-Baxter equation* is a pair (X, R), where X is a set, and $r\colon X \times X \to X \times X$ is such a bijective map, that

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r).$$

Using the notation $r(x, y) = (\sigma_x(y), \tau_y(x))$, the solution is *non-degenerate*, if σ_x is a permutation for every $x \in X$. *Involutivity* means that $r^2 = id$. There exists a certain equivalence relation on (X, r), which can be used to generate a sequence of solutions. If there is a solution of size one in this sequence, then (X, r) is called *multipermutation*.

Gateva-Ivanowa and Van den Bergh in [GV98] proved, that the structure group

$$G(X, r) := \langle X \mid xy = uv \text{ if } r(x, y) = (u, v)\rangle$$

of a non-degenerate, involutive solution (X, r) is a Bieberbach group. Afterward, other researchers showed that the structure of the group determines certain properties of the solution itself. For example, (X, r) is multipermutation if and only if $G(X, r)$ is diffuse. Having this, one can also ask about the unique product property of $G(X, r)$. As I wrote in the description of the article [H5], the Promislow group is an essential example here. From this perspective, an important part of the article consists of the presentation of the algorithm that checks whether a Bieberbach group contains Δ_P.

In the investigation of non-involutive solutions, algebraic structures called skew left braces play a significant role. They are of the form $(B, +, \circ)$, where $(B, +)$, $(B, \circ)$ are groups, and we have

$$\forall_{a,b,c\in B}\ a \circ (b + c) = a \circ b - a + a \circ c.$$

It is known that every non-degenerate solution (X, r) determines skew left brace $(B, +, \circ)$ and the other way around. Multipermutation of a non-involutive solution (X, r) is connected with so-called right nilpotency of B. In the article, we consider connections of this property with certain local properties of B, as a brace and a group $(B, \circ)$. Furthermore, left nilpotency of B through its local properties is investigated.

Flat manifolds of diagonal type. Articles [O1], [O3], [O5].

We say that a flat n-dimensional manifold X is of *diagonal type*, if there exists a normal abelian subgroup L' of its fundamental group Γ, given by the sequence (1), with the following properties:

a) $G' := \Gamma/L'$ is an elementary abelian 2-group;

b) there exists a basis $l_1, \dots, l_n$ of a $\mathbb{Z}$-module L' such that

$$g \cdot l_i = \pm l_i$$

for $g \in G', 1 \leq i \leq n$.

In the case when $L' = L$, we say that X and Γ are *diagonal*. For our purposes, assume that $G' = \Gamma/L' = C_2^d$ and $\rho\colon \Gamma \to G'$ is the natural homomorphism.

One can say that standard, in a way, examples of diagonal Bieberbach groups are the generalized Hantzsche-Wendt groups, which in the orientable case are simply called Hantzsche-Wendt groups (*HW-groups*). They are defined as Bieberbach groups, with holonomies being elementary abelian 2-groups of the maximal rank. The fact that they are diagonal was proved by Rosetti and Szczepański in [RS05]. Problem 10 of [Szc06] essentially raises the issue of relations in generalized Hantzsche-Wendt groups of given dimension. In [Szc01] the author showed that for every odd dimension n there exists a HW-group, which is an epimorphic image of the Fibonacci group $F(n-1, 2n)$. In the paper **[O1]**, the result has been generalized to all HW-groups. We can say that in a certain sense all Hantzsche-Wendt groups are cyclic:

Theorem 8 ([O1, Theorem]). *Every n-dimensional Hantzsche-Wendt group is an epimorphic image of the Fibonacci group $F(n-1, 2n)$.*

Articles [O3], [O5] have their basis in the description of the cohomology rings of HW-groups, with coefficients in $\mathbb{F}_2$, presented in [PS16]. It turns out that this description can be – up to some extent – applied to flat manifolds of diagonal type. To be more precise, with the notation above, we have

$$X = \mathbb{R}^n/\Gamma = (\mathbb{R}^n/L')/(\Gamma/L') = \left(\prod_{i=1}^{n} \operatorname{span}\{l_i\}/\langle l_i \rangle\right)/G' = (S^1)^n/G' = T^n/G'.$$

This way, the group G' acts on the torus T^n as a subgroup of $\mathcal{D}^n$, where

$$\mathcal{D} = \{z \mapsto z, z \mapsto -z, z \mapsto \bar{z}, z \mapsto -\bar{z}\} \cong C_2 \times C_2$$

is a subgroup of the symmetry group of the unit circle of the complex plane. Having this description, by choosing d generators of G', we can construct a matrix A of dimension $d \times n$ with entries in $\mathcal{D}$. With the help of A, by combinatorial means, we define the polynomials in $\mathbb{F}_2[x_1, \dots, x_d]$:

- of degree 1: $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ and
- of degree 2: $\theta_1, \dots, \theta_n$.

They have the following properties:

1. The image of
$$\prod_{i=1}^{n}(1 + \alpha_i + \beta_i) \tag{11}$$
under the induced homomorphism $\rho^* \colon H^*(G', \mathbb{F}_2) \to H^*(\Gamma, \mathbb{F}_2) = H^*(X, \mathbb{F}_2)$ is the total Stiefel-Whitney class of the manifold X.
2. If $\rho^*_{(i)} : H^i(G', \mathbb{F}_2) \to H^i(\Gamma, \mathbb{F}_2)$ is the map induced by ρ, then
$$\operatorname{Ker} \rho^*_{(1)} = 0 \text{ and } \operatorname{Ker} \rho^*_{(2)} = \langle \theta_1, \dots, \theta_n \rangle.$$

Therefore, we can describe the first and the second Stiefel-Whitney class of X as elements of the quotient algebra $\mathbb{F}_2[x_1, \dots, x_d]/I$, where I is the ideal generated by $\operatorname{Ker} \rho^*_{(2)}$.

In the above language, the paper **[O3]** gives an example of such a flat manifold $X = \mathbb{R}^n/\Gamma$ of diagonal type, which is not spin, but every proper and finite cover of X has a spin structure. By a *proper*, we understand a cover of the manifold X, with the holonomy group being a proper subgroup of G. The example is in striking contrast to the case of real Bott manifolds, for which the existence of spin structures is implied by their existence in every finite cover with the holonomy group C_2^2 (see [Gąs17, Theorem 1.2]).

In the article **[O5]**, we show that no HW-manifold of dimension greater that 3 admits any spin^c structure. The problem of the existence of spin^c structures arises naturally in the case of those manifolds, which are not spin, and HW-manifolds of dimension greater than or equal to 5 do not have any spin structure (see [MP06, Example 4.6]). In the first place, the knowledge of certain homology and cohomology groups of HW-manifolds allows formulating a criterion of the existence of spin^c structures on those manifolds with the usage of the manifold (11) and the generators of $\operatorname{Ker} \rho^*_{(2)}$. As a consequence, we move our focus to the investigation of properties of the matrix A, which defines a HW-manifold as above. Finally, the condition on the existence of spin^c structures is purely combinatorial. By showing that in dimensions greater than or equal to 5 this condition does not hold, we get the main result of the paper.

5. Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

- **09.2016** One-week visit at the Southampton University, working on the article [O2].
- **12.2018** Few-day visit at the RWTH Aachen, working on the article [H7].
- **07.2019** Two-week visit at the RWTH Aachen: start of the research on the conjecture partly proved in [H3].
- **07.2023** One-week visit at the Maria Curie-Skłodowska University in Lublin, working on the article [GL23] (under review).
- **09.2023** One-week visit at the Kiel University, working on the article [LSW24] (under review).

6. Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

Teaching.

- Promotion of 13 students in master's studies in mathematics.
- Many years of teaching at the University of Gdańsk. The list of courses includes: Linear algebra, Abstract algebra, Calculus, Galois theory, Representation theory, Computational group theory, Databases. For most of the courses, I have prepared presentations or scripts, available to students.
- A short *Modular representation theory course* for participants of the Introductory School to the 21st Andrzej Jankowski Memorial Lecture.
- *GAP – a very brief introduction* – a course for the staff of the Institute of Mathematics of the University of Gdańsk (notes of the course are available online at `www.gap-system.org/doc/learn`).

Organization.

- Member of the Discipline Council for Mathematics at the University of Gdańsk, 2024-2028.
- Member of the Council of the Faculty of Mathematics, Physics, and Informatics of the University of Gdańsk, 2024-2028.
- Member of the working group creating *Mathematical Modeling and Data Analysis* field of study at the Faculty of Mathematics, Physics, and Informatics of the University of Gdańsk, both bachelor's and master's level.
- Cooperation with the staff of the University of Gdańsk on the website of the Institute of Mathematics of the University of Gdańsk.
- Working on the organization of the series of *Andrzej Jankowski Memorial Lectures* (`mat.ug.edu.pl/ajml`) and *Killing-Weierstrass colloquia* (`mat.ug.edu.pl/kwwk`), including creating websites and books of abstracts.
  For the organization of the 20th Andrzej Jankowski Memorial Lecture I was granted, in 2019, with University of Gdańsk Rector's Award (details in pt. 7).

Popularization. This activity has been led for people on many levels of their education or career:

1. For students of elementary and high schools in the areas of topology, geometry, and cryptography. Selected workshops:
   - **2019** *Top secret – what Caesar knew about encryption*,
   - **2020** *Surfaces – what one can and what cannot do with gum, wire and paper*,
   - **2021** *Wallpaper groups – mathematicians decorate walls*.
2. For teachers:
   - **2019** LaTeXand its applications in teaching (preparing presentations, exams, application in different areas of science).

In many events, apart from conducting courses, I played a role in their organization. In particular, I participated in the organization of the Year of Mathematics in Pomerania (Night of Mathematics, 17th Andrzej Jankowski

Memorial Lecture, Mathematics as an important component of culture and civilization).

7. Apart from information set out in 1-6 above, the applicant may include other information about his/her professional career, which he/she deems important.

7.1. Awards

In the field of research and organization, I have been granted with:

2019 University of Gdańsk Rector's Award for the series of two articles on connections between representation theory of finite groups and crystallographic groups, their geometry, and for an exemplary organization of the jubilee 20th Andrzej Jankowski Memorial Lecture and the following mini-conference (team award of the second degree).

2022 University of Gdańsk Rector's Award for research achievements, documented with scientific publications (individual award of the fourth degree).

7.2. Seminar talks

Apart from many talks at the Geometry Seminar of the Institute of Mathematics of my home university, I gave the following ones:

20.05.2014 University of Warsaw (Algebraic topology seminar): *Irreducible Euclidean representations of the Fibonacci groups*

22.01.2019 University of Warsaw (Algebraic topology seminar): *Spin structures on flat manifolds of diagonal type*

12.01.2021 Warsaw University of Technology (Algebra seminar): *Structure groups of multipermutation solutions to the Yang-Baxter equation*

10.11.2021 University of Warsaw (Algebraic topology seminar): *Flat manifolds with homogeneous holonomy representation*

23.05.2023 Warsaw University of Technology (Algebra seminar): *Nonsolvable Bieberbach groups*

28.09.2023 Kiel University (Mathematisches Seminar): *Complex Vasquez invariant*

22.11.2023 University of Warsaw (Algebraic topology seminar): *Complex Vasquez invariant*

23.11.2023 University in Bayreuth (online): *Complex Vasquez invariant*

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