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**Fizyczne ograniczenia
na powstawanie i obserwacje korelacji
o charakterze ściśle kwantowym**

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Streszczenie

Zrozumienie zarówno praktycznych, jak i fundamentalnych ograniczeń dotyczących powstawania oraz obserwacji ściśle kwantowych efektów stanowi jedno z kluczowych zagadnień współczesnej fizyki oraz rozwijających się technologii kwantowych. Korelacje pomiędzy wynikami pomiarów wykonywanych na poszczególnych układach kwantowych mogą ujawniać obecność tych efektów i umożliwiać charakteryzację ich właściwości. Liczne badania prowadzone od drugiej połowy dwudziestego wieku dostarczyły wielu rezultatów dotyczących wykrywania splątania kwantowego oraz możliwości łamania nierówności Bella. Wraz z rozwojem technologii kwantowych i ugruntowaniem kwantowej teorii informacji badania te zaczęły koncentrować się między innymi na fizycznych ograniczeniach związanych z generowaniem i obserwacją korelacji kwantowych.

Niniejsza rozprawa poświęcona jest analizie tych zagadnień z kilku perspektyw. Pierwsza z nich dotyczy ograniczeń wynikających ze skończonej liczby kopii stanu dostępnych w eksperymencie. Problem ten jest szczególnie istotny w przypadku stanów wielofotonowych generowanych za pomocą spontanicznej parametrycznej konwersji, gdzie niskie prawdopodobieństwa koincydencji uniemożliwiają uzyskanie dużej statystyki pomiarowej w krótkim czasie. Zbliżony charakter ma problem łamania nierówności Bella z uwzględnieniem wszystkich możliwych wyników pomiarów, w tym przypadków, gdy nie zarejestrowano żadnego sygnału. Połączenie tego podejścia z ograniczeniami wynikającymi z braku kalibracji odseparowanych układów odniesienia pozwala na analizę bardziej przyjaznych eksperymentalnie scenariuszy łamania nierówności Bella opartych na losowych pomiarach. Pojęcie układu odniesienia w mechanice kwantowej można rozszerzyć poprzez utożsamienie go z wybraną cząstką kwantową, tworząc tzw. kwantowy układ odniesienia. Ze względu na możliwość jego skorelowania z innymi cząstkami wchodzącymi w skład badanego układu, istotnym problemem staje się analiza zmiany splątania i superpozycji przy transformacjach pomiędzy takimi układami odniesienia. Kolejnym zagadnieniem badanym w pracy jest możliwość obserwacji łamania nierówności Bella na kilku przekrywających się podukładach jednocześnie. Podczas gdy zjawisko to jest niemożliwe dla układów trzech cząstek z dwoma ustawieniami pomiarowymi na obserwatora, w przypadku układów wielocząstkowych staje się ono możliwe. Wszystkie wymienione scenariusze dotyczą ograniczeń na obserwację ściśle kwantowych korelacji. Należy jednak pamiętać, że zanim możliwa stanie się ich analiza, konieczne jest wygenerowanie odpowiedniego stanu kwantowego. Interesującym zagadnieniem, omawianym w tej rozprawie, jest szybkość generowania silnie splątanych stanów kwantowych przy oddziaływaniach o strukturze dwuciałowej. Naturalnym rozszerzeniem tego problemu jest analiza możliwości detekcji oddziaływań wyższego rzędu, które mogą przyspieszyć ten proces. Każde z opisanych zagadnień stanowi odrębny wkład do niniejszej rozprawy składającej się cyklu sześciu publikacji.

Abstract

Understanding both the practical and fundamental limitations on the generation and observation of strictly quantum effects constitutes a key challenge in modern physics and in the development of modern quantum technologies. Correlations between the outcomes of measurements performed on individual quantum systems can reveal the presence of such effects and enable the characterisation of their properties. Numerous studies conducted since the second half of the twentieth century have produced significant results concerning the detection of quantum entanglement and the violation of Bell inequalities. With the advancement of quantum technologies and the consolidation of quantum information theory, researchers have increasingly focused on the physical constraints governing the generation and observation of quantum correlations.

This dissertation is devoted to the analysis of these issues from several complementary perspectives. The first one concerns the limitations imposed by the finite number of state copies available in an experiment. This problem is particularly relevant for multiphoton states generated via spontaneous parametric down conversion, where low coincidence rates make it difficult to obtain large measurement statistics within a short time. A similar issue arises when testing Bell inequalities while accounting for all possible measurement outcomes, including cases in which no detection event occurs. Combining this approach with the constraints resulting from the lack of calibration between separated reference frames enables the analysis of more experimentally friendly scenarios of Bell inequalities violation based on random measurements. The notion of a reference frame in quantum mechanics can be extended by associating it with a specific quantum particle, forming a so-called quantum reference frame. Since such a frame can be correlated with other particles in the composite system, analysing changes in entanglement and superposition under transformations between the quantum reference frames becomes an important problem. Another topic explored in this work concerns the possibility of observing Bell inequalities violations simultaneously across several overlapping subsystems. While this phenomenon is impossible for three-particle systems with two measurement settings per party, it becomes feasible in multi-particle systems. All of the aforementioned scenarios address the limitations in observing strictly quantum correlations. However, it should be emphasised that before such correlations can be analysed, the appropriate quantum states must first be generated. An additional issue discussed in this dissertation concerns the speed at which strongly entangled quantum states can be produced through two-body interactions. A natural extension of this idea is the investigation of higher-order interactions verification that may accelerate this process. Each of the described problems contributes to this dissertation, which consists of a series of six publications.

Rozdział 1

Wstęp

Niniejsza rozprawa doktorska została przygotowana w formie cyklu sześciu publikacji naukowych, które składają się na spójną całość tematyczną. W jej skład wchodzi następujące prace, oznaczone kolejnymi wielkimi literami alfabetu:

- [A] Paweł Cieśliński, Jan Dziewior, Lukas Knips, Waldemar Kłobus, Jasmin Meinecke, Tomasz Paterek, Harald Weinfurter, Wiesław Laskowski, *Valid and efficient entanglement verification with finite copies of a quantum state*, npj Quantum Information 10 (1), 14 (2024).
- [B] Paweł Cieśliński, Mateusz Kowalczyk, Wiesław Laskowski, Tamás Vértesi, *How likely are you to observe non-locality with imperfect detection efficiency and random measurement settings?* New Journal of Physics, 27 074504 (2025).
- [C] Carlo Cepollaro, Ali Akil, Paweł Cieśliński, Anne-Catherine de la Hamette, Časlav Brukner, *Sum of Entanglement and Subsystem Coherence Is Invariant under Quantum Reference Frame Transformations*, Physical Review Letters 135, 010201 (2025).
- [D] Paweł Cieśliński, Lukas Knips, Mateusz Kowalczyk, Wiesław Laskowski, Tomasz Paterek, Tamás Vértesi, Harald Weinfurter, *Unmasking the Polygamous Nature of Quantum Nonlocality*, Proceedings of the National Academy of Sciences of the United States of America, 121 (44) e2404455121 (2024).
- [E] Paweł Cieśliński, Waldemar Kłobus, Paweł Kurzyński, Tomasz Paterek, Wiesław Laskowski, *The fastest generation of multipartite entanglement with natural interactions*, New Journal of Physics 25 093040 (2023).
- [F] Paweł Cieśliński, Paweł Kurzyński, Tomasz Sowiński, Waldemar Kłobus, Wiesław Laskowski, *Exploring many-body interactions through quantum Fisher information*, Physical Review A 110, 012407 (2024).

W trakcie realizacji doktoratu uzyskano również dodatkowe wyniki, które nie zostały włączone do rozprawy. Do tej grupy należy:

- [1] Paweł Cieśliński, Marcin Markiewicz, Konrad Schlichtholz, Jan-Åke Larsson, Marek Żukowski, *Unquestionable Bell theorem for interwoven frustrated down conversion processes*, arXiv:2508.19207 [quant-ph] (2025), przyjęte do druku w Physical Review Letters.

Współtworzyłem także artykuł przeglądowy dotyczący analizy stanów kwantowych z wykorzystaniem losowych pomiarów:

- [2] Paweł Cieśliński, Satoya Imai, Jan Dziewior, Otfried Gühne, Lukas Knips, Wiesław Lasowski, Jasmin Meinecke, Tomasz Paterek, Tamás Vértesi, *Analysing quantum systems with randomised measurements*, Phys. Rep. 1095 (1), 1-48 (2024).

Motywacja i rezultaty zawarte w rozprawie

Korelacje kwantowe, których nie sposób wyjaśnić w ramach klasycznego opisu rzeczywistości, stanowią jedną z najbardziej fundamentalnych i zarazem intrygujących cech teorii kwantowej. Choć odgrywają one kluczową rolę w badaniach podstawowych, ich znaczenie wykracza daleko poza ten obszar. Korelacje te leżą bowiem u podstaw licznych zastosowań praktycznych, w szczególności komunikacji kwantowej [3, 4, 5] oraz obliczeń kwantowych [6, 7, 8]. Wraz z dynamicznym rozwojem teorii informacji kwantowej oraz technologii wykorzystujących ściśle kwantowe zjawiska, takie jak nielokalność Bella czy splątanie, coraz wyraźniej zarysowuje się potrzeba systematycznego badania fizycznych ograniczeń wpływających na możliwość ich wytwarzania i obserwacji.

Istnieje wiele potencjalnych ograniczeń wpływających na nasze zdolności do wytwarzania i obserwacji korelacji kwantowych. Z perspektywy eksperymentalnej jednym z podstawowych czynników są ograniczone zasoby czasowe niezbędne do przeprowadzenia eksperymentu. Problem ten jest szczególnie istotny w przypadku stanów wielofotonowych wytwarzanych za pomocą spontanicznej parametrycznej konwersji. Częstość zliczeń koincydencyjnych jest wówczas na tyle niska, że uzyskanie statystycznie istotnej liczby wyników, rzędu 10^4 dla pojedynczej obserwacji, w zaawansowanych eksperymentach (zob. na przykład [9]) dla stanów dziesięciofotonowych może zająć nawet 35 dni. W pracy *Valid and efficient entanglement verification with finite copies of a quantum state* [A] zaproponowano rozwiązanie tego problemu poprzez wprowadzenie modelu pomiaru korelacji na skończonej liczbie kopii stanów kwantowych oraz ścisłą analizę statystyczną obserwowanych danych.

Pozostając w obszarze problemów o charakterze eksperymentalnym, jednym z najistotniejszych czynników ograniczających możliwość obserwacji nieklasycznych korelacji jest wydajność detekcji. Gdy prawdopodobieństwo detekcji danej cząstki η jest mniejsze od wartości krytycznej η_{crit} , możliwe staje się wyjaśnienie obserwowanych korelacji za pomocą modeli lokalnych zmiennych ukrytych [3]. W sytuacji, gdy poszczególne stacje pomiarowe są niezależne i przestrzennie rozdzielone, w standardowym scenariuszu Bella konieczna jest również kalibracja lokalnych układów odniesienia. Przykładowo, w przypadku nierówności Clausera–Horna–Shimony’ego–Holtza (CHSH) [10] optymalne pomiary dla dowolnego maksymalnie splątanego stanu dwóch kubitów zadane są poprzez precyzyjnie określone kierunki. Jeśli

jednak oś z lokalnego układu odniesienia Boba jest obrócona o 45° wokół osi y , wspólnej z osią Alicji, wówczas ich kierunki pomiarowe stają się identyczne, a wartość wyrażenia CHSH wynosi zero. Problem kalibracji układów odniesienia może zostać rozwiązany poprzez zastosowanie jednorodnie losowych kierunków pomiarowych [2]. Okazuje się, że podejście to pozwala całkowicie wyeliminować konieczność kalibracji kosztem skończonego prawdopodobieństwa złamania nierówności Bella w danym scenariuszu. Dla dwóch stacji pomiarowych, dwóch ustawień oraz dwóch wyników na stronę prawdopodobieństwo to, dla dowolnego stanu maksymalnie splątanego dwóch kubitów, wynosi $\mathcal{P}_V \approx 28\%$. Zwiększając liczbę ustawień pomiarowych, można je podnieść do wartości bliskiej 100% [11, 12]. W pracy *How likely are you to observe non-locality with imperfect detection efficiency and random measurement settings?* [B] przeanalizowano wpływ ograniczonej wydajności detekcji na powyższe prawdopodobieństwo łącząc obydwa ograniczenia eksperymentalne, uzyskując jednocześnie głębszy wgląd w zależność pomiędzy splątaniem kwantowym a możliwymi kierunkami pomiarowymi w testach Bella wykorzystujących pełną statystykę pomiarową.

Koncepcja układu odniesienia jako abstrakcyjnego narzędzia umożliwiającego definiowanie kierunków pomiarowych może zostać rozszerzona na układy o charakterze ściśle kwantowym [13]. W standardowym podejściu układ odniesienia utożsamiany jest z makroskopowym obiektem, punktem lub kierunkiem. Dla cząstki o spinie $1/2$ pomiar obserwabli odpowiadającej macierzy Pauliego Z realizowany jest poprzez makroskopową orientację magnesu w doświadczeniu Sterna–Gerlacha. Możliwe jest jednak utożsamienie układu odniesienia z inną cząstką o spinie $1/2$, definiując w ten sposób kwantowy układ odniesienia. W takiej sytuacji opis pozostałych cząstek dokonywany jest z jej perspektywy. Ze względu na kwantowy charakter układu odniesienia może on znajdować się w superpozycji lub stanowić część większego stanu splątanego. Zmieniając perspektywę pomiędzy pozostałymi cząstkami poprzez transformacje kwantowych układów odniesienia własności te mogą ulec zmianie. Ścisłe relacje pomiędzy ilością splątania w stanie czystym dwóch cząstek a ilością superpozycji w jednym z podukładów zostały wyprowadzone w pracy *Sum of Entanglement and Subsystem Coherence Is Invariant under Quantum Reference Frame Transformations* [C].

Standardowe sformułowanie mechaniki kwantowej wprowadza również ograniczenia na możliwość obserwacji zjawisk kwantowych z różnych perspektyw. W kontekście łamania nierówności Bella jednym z takich ograniczeń jest monogamia nielokalności. Rozważając układ trzech kubitów A , B i C , wyniki Scaraniego i Gisina [14], rozwinięte później przez Toner i Verstraete [15], pokazują, że niemożliwe jest jednoczesne złamanie dwóch dwuustawieniowych nierówności Bella dla par AB oraz AC . Złamanie jednej z tych nierówności oznacza natychmiastowo brak możliwości złamania jej przez pozostałą parę obserwatorów. Wyniki przedstawione w pracy *Unmasking the Polygamous Nature of Quantum Nonlocality* [D] pokazują, że niemożność złamania wielu przekrywających się dwuustawieniowych nierówności Bella jest zarezerwowana wyłącznie dla układu trzech kubitów i nierówności mierzonych parami. Wykazano tam, że dla liczby kubitów $N > 3$ istnieje zawsze taka nierówność Bella oraz stan kwantowy, że wszystkie odpowiednie $N - 1$ nierówności łamane są jednocześnie. Przykładowo, dla $N = 4$ istnieje dwuustawieniowa nierówność Bella I , mierzona pomiędzy trzema obserwatorami, oraz stan czterech kubitów, dla którego wszystkie możliwe nierówności trójcialowe (I_{ABC} , I_{ABD} , I_{ACD} , I_{BCD}) są łamane jednocześnie.

Możliwość obserwacji korelacji kwantowych jest nierozzerwalnie związana z procesem przygotowania odpowiedniego stanu kwantowego. Najbardziej powszechną techniką ich genero-

wania jest stosowanie odpowiednich jedno- i dwukubitowych operacji unitarnych w sekwencji poprzez kwantowe bramki logiczne, na przykład w układach nadprzewodzących kubitowych lub przy użyciu elementów optycznych w przypadku stanów fotonowych. Metoda ta ma charakter uniwersalny, gdyż pozwala na wytworzenie dowolnego stanu kwantowego. Jednocześnie nie umożliwia ona bezpośredniego badania fizycznych ograniczeń na przetwarzanie informacji kwantowej i generowanie silnie splątanych stanów wynikających ze struktury oddziaływań. Układy wielu cząstek mogą znajdować się w nich naturalnie, a techniki inżynierii stanów kwantowych pozwalają na manipulację i projektowanie oddziaływań w taki sposób, aby oczekiwany stan był stanem podstawowym zadanego Hamiltonianu. Podstawowe oddziaływania wykorzystywane w przetwarzaniu informacji kwantowej mają najczęściej charakter fundamentalnie dwuciałowy – odpowiadające im Hamiltoniany sprzęgają cząstki co najwyżej parami. Abstrahując od struktury oddziaływań, fundamentalne ograniczenia na szybkość generowania dowolnego stanu kwantowego, znane jako kwantowe granice szybkości (QSL, *ang.* quantum speed limits), zostały wyprowadzone przez Mandelstama i Tamma [16] oraz Margolusa i Levitina [17]. Okazuje się, że Hamiltoniany prowadzące do nasycenia tych ograniczeń zawierają zazwyczaj oddziaływania wielociałowe. Naturalnym problemem staje się zatem zbadanie wpływu struktury oddziaływań na możliwość wytwarzania silnie splątanych stanów kwantowych. Problem szybkości generowania wielocząstkowego splątania przy ewolucjach generowanych przez oddziaływania dwuciałowe został zbadany w pracy *The fastest generation of multipartite entanglement with natural interactions* [E].

Poza optymalną ewolucją prowadzącą do przygotowania dowolnego stanu kwantowego, oddziaływania wielociałowe stanowią również naturalny mechanizm umożliwiający realizację silnie splątanych stanów kwantowych jako stanów podstawowych układów wielu cząstek [18]. Hamiltoniany o takich własnościach nie występują jednak bezpośrednio w żadnym z obecnie kontrolowanych układów kwantowych dostępnych w laboratoriach. Choć oddziaływania wielociałowe mogą być symulowane i szczegółowo projektowane (zob. np. [19]), techniki te są bardzo kosztowne, co wskazuje na użyteczność metod umożliwiających ich weryfikację. Problem ten został podjęty w pracy *Exploring many-body interactions through quantum Fisher information* [F], w której zaproponowano technikę weryfikacji obecności oddziaływań wielociałowych za pomocą kwantowej informacji Fishera. Przedstawiona metoda znajduje zastosowanie dla uogólnionych Hamiltonianów typu Isinga oraz w sytuacjach, w których struktura oddziaływań jedno- i dwuciałowych jest już znana.

Rozdział 2

Weryfikacja splątania przy ograniczonej liczbie kopii stanów [A]

Generowanie wielu kopii stanów splątanych złożonych z N kubitów stanowi wymagające zadanie eksperymentalne. Nowoczesne techniki wykorzystywane na platformach opartych o fotony korzystają najczęściej ze zjawiska spontanicznej parametrycznej konwersji, gdzie problem ten jest niezwykle istotny. Przykładowo, w eksperymencie [20] rejestrowano koincydencje dziesięciofotonowe ze średnią częstością jednego zdarzenia co około pięć minut. Takie ograniczenia, zarówno w czasie trwania eksperymentu, jak i w liczbie dostępnych kopii stanu, doprowadziły do opracowania nowych metod detekcji splątania opartych na skończonej statystyce [21, 22, 23, 24, 25, 26, 27, 28]. Zagadnienie to zostało podjęte również w pracy [A], w której zaproponowano model pomiaru korelacji na skończonej liczbie kopii stanu kwantowego oraz wykorzystano go do formułowania wniosków na temat splątania w oparciu o rygorystyczną analizę częstościową i bayesowską, uwzględniającą zarówno wiarygodność, jak i efektywność testu. Wprowadzona technika jest łatwa do zastosowania, gdyż nie wymaga modyfikacji istniejącej aparatury eksperymentalnej i umożliwia weryfikację splątania bezpośrednio na podstawie surowych danych pomiarowych, przy jednoczesnej kontroli błędów statystycznych pierwszego i drugiego rodzaju.

2.1 Model pomiaru korelacji

Dowolny stan kwantowy N kubitów można przedstawić jako

$$\rho = \frac{1}{2^N} \sum_{\mu_1, \dots, \mu_N=0}^3 T_{\mu_1 \dots \mu_N} \sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N}, \quad (2.1)$$

gdzie σ_{μ_i} to macierze Pauliego, natomiast $T_{\mu_1 \dots \mu_N} \in [-1, 1]$ to elementy tensora korelacji, będące wartością oczekiwaną $\langle \sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N} \rangle$ w stanie ρ . Formalizm tensora korelacji stanowi użyteczny opis korelacji pomiędzy poszczególnymi kubitami i pozwala na sformułowanie licznych kryteriów splątania kwantowego. Dowolny liniowy świadek splątania [29] można zapisać w postaci

$$E_W = \sum_{\mu_1, \dots, \mu_N=0}^3 \alpha_{\mu_1 \dots \mu_N} T_{\mu_1 \dots \mu_N} \leq 0. \quad (2.2)$$

W powyższym $\alpha_{\mu_1 \dots \mu_N}$ są liczbami rzeczywistymi dobranymi w taki sposób, aby $\langle E_W \rangle$ w dowolnym stanie separowalnym (wyznaczone przy *nieskończenie* wielu pomiarach) była mniejsza lub równa zero. Innym przykładem kryterium splątania, określanym tutaj jako kwadratowy świadek splątania, jest [30, 31, 32]

$$S = \sum_{\mu_1, \dots, \mu_N=1}^3 T_{\mu_1 \dots \mu_N}^2 > 1 \Rightarrow \rho \text{ jest splątany} \quad (2.3)$$

Warunek ten jest konieczny i wystarczający dla stanów czystych [32]. Wygodnie jest również wprowadzić pojęcie świadka S_M , definiowanego poprzez sumę kwadratów M wybranych członów korelacyjnych. Ograniczenie się do dowolnej liczby korelacji $M > 1$ jest uzasadnione, ponieważ każda z tych wielkości jest nieujemna, a już skończona liczba elementów sumy może prowadzić do przekroczenia granicy separowalności.

W pojedynczym eksperymencie, tj. przy pomiarze pojedynczej kopii stanu, możliwe wyniki pomiaru $T_{\mu_1 \dots \mu_N} = T_i$ przyjmują wartości ± 1 . Dla uproszczenia notacji wprowadzono globalny indeks i . Każda z takich koincydencji stanowi zatem próbę Bernoulliego. Ponieważ $T_i = p_+ - p_-$, gdzie $p_{\pm}$ oznaczają prawdopodobieństwa zaobserwowania odpowiednio korelacji i antykorelacji, rozkład prawdopodobieństwa wyników pomiaru obserwabli związanej z daną korelacją, przy liczbie prób n_i , można zapisać jako

$$P(\tau_i) = \text{Binom} \left(\frac{n_i}{2} (1 + T_i) \right), \quad (2.4)$$

gdzie $\text{Binom}(\cdot)$ oznacza rozkład dwumianowy, a τ_i reprezentuje estymowaną wartość elementu tensora korelacji T_i na podstawie n_i pomiarów. Dodatkowo spełnione są zależności

$$\langle \tau_i \rangle = T_i, \quad \text{Var}(\tau_i) = \frac{\sigma_i}{n_i}, \quad (2.5)$$

gdzie wariancja $\sigma_i = 1 - T_i^2$ odpowiada wariancji kwantowej danej obserwabli. Zgodnie z oczekiwaniem, wariancja estymowanej wielkości maleje proporcjonalnie do $1/n_i$.

Dowolna wielkość oparta o M korelacji, służąca do wykrywania splątania, $Q = f(T_1, \dots, T_M)$, posiada rozkład prawdopodobieństwa dany przez

$$P(Q) = \sum_{\substack{\tau_{j_1}, \dots, \tau_{j_M} \\ f(\tau_{j_1}, \dots, \tau_{j_M}) = Q}} \prod_{k=1}^M P(\tau_{j_k}). \quad (2.6)$$

W przypadku świadków liniowych oraz kwadratowego świadka splątania, przy założeniu równej liczby kopii stanu użytych do każdego pomiaru ($n_i = n$), otrzymujemy zależności

$$\langle E_W \rangle = E_W^\infty, \quad \text{Var}(E_W) = \sum_{i=1}^M \alpha_i^2 \text{Var}(\tau_i), \quad (2.7)$$

$$\langle S_M \rangle = \frac{(n-1)S_M^\infty + M}{n}, \quad \text{Var}(S_M) = \sum_{i=1}^M \text{Var}(\tau_i^2), \quad (2.8)$$

gdzie górny indeks ∞ oznacza wartość oczekiwaną w granicy nieskończonej liczby kopii stanu. W dalszej części rozdziału wielkości pozbawione indeksu będą odnosić się do przypadku ograniczonej statystyki.

2.2 Analiza świadków splątania

Przy niskiej statystyce pomiarowej należy zachować wysoką ostrożność we wnioskach dotyczących obserwowanych danych. W tym przypadku bowiem rozkłady prawdopodobieństwa przyjmują bardzo nieintuicyjny kształt, a prawdopodobieństwo przekroczenia granicy separowalności wyznaczonej dla nieskończonej statystyki może pozostać istotne dla niemal każdego możliwego wyniku pomiaru. Nawet w skrajnych przypadkach, w których obserwuje się ekstremalne wartości świadka, prawdopodobieństwa uzyskania tego wyniku dla stanu splątanego i separowalnego mogą być porównywalne. Zatem przy niewielkiej liczbie kopii stanu nie wystarczy jedynie przeskalować granicy separowalności odpowiadającej nieskończonej statystyce. Nie wystarczy także rozważać wyłącznie błędów pierwszego rodzaju, tj. prawdopodobieństwa, że zaobserwowany sygnał pochodzi od stanu separowalnego. Należy również uwzględnić błędy drugiego rodzaju, określające prawdopodobieństwo, że dany wynik pojawi się także dla stanu splątanego. Możliwe są bowiem sytuacje, w których oba te prawdopodobieństwa są porównywalne, a w skrajnym przypadku niemal identyczne.

W podejściu częstościowym wiarygodność testu określa poziom ufności $q_F = 1 - \alpha$, gdzie α jest poziomem istotności. Aby zapewnić określony poziom ufności q_F , zbiór akceptacji wyników pomiaru O_a (nazywany także przedziałem ufności) należy dobrać tak, aby prawdopodobieństwo uzyskania wyniku z tego zbioru dla stanu separowalnego nie przekraczało $1 - q_F$:

$$\sum_{Q \in O_a} P(Q|\text{sep}) \leq q_F. \quad (2.9)$$

Warto zauważyć, że rozkład $P(Q|\text{ent})$ nie wpływa bezpośrednio na poziom istotności testu, lecz ma kluczowe znaczenie dla tzw. *mocy testu* r , czyli zdolności procedury do prawidłowej identyfikacji stanów splątanych (efektywności detekcji). Moc testu definiuje się jako:

$$r = 1 - \epsilon_2 = \sum_{Q \in O_a} P(Q|\text{ent}), \quad (2.10)$$

gdzie ϵ_2 oznacza błąd drugiego rodzaju. Zatem skuteczny test weryfikacji splątania powinien nie tylko minimalizować prawdopodobieństwo błędu pierwszego rodzaju, lecz także maksymalizować moc, co nabiera szczególnego znaczenia w warunkach ograniczonej statystyki.

W podejściu bayesowskim prawdopodobieństwa obu hipotez – o separowalności lub splątaniu stanu – obliczane są dla konkretnego wyniku Q z wykorzystaniem $P(\text{ent}|Q)$ oraz $P(\text{sep}|Q) = 1 - P(\text{ent}|Q)$. Wiarygodność testu określa tzw. *bayesowski poziom akceptacji* q_B , czyli minimalną wartość prawdopodobieństwa a posteriori, dla której hipoteza o splątaniu zostaje przyjęta. Splątanie jest zatem potwierdzane, gdy:

$$P(\text{ent}|Q) \geq q_B, \quad (2.11)$$

co oznacza, że zbiór akceptacji O_a musi być tak dobrany, aby powyższy warunek był spełniony dla każdego $Q \in O_a$. Zgodnie z twierdzeniem Bayesa, prawdopodobieństwo $P(\text{ent}|Q)$ zależy od rozkładów $P(Q|\text{sep})$ i $P(Q|\text{ent})$, a także od stosunku prawdopodobieństw początkowych $P(\text{ent})/P(\text{sep})$. Aby sformułować test bayesowski w kategoriach wiarygodności i efektywności, wprowadza się funkcję straty L_Q , opisującą koszt błędu dla danego wyniku Q . Dla wag β

oraz $(1 - \beta)$ przypisanych odpowiednio błędom pierwszego i drugiego rodzaju, funkcja ta przyjmuje postać:

$$L_Q = \begin{cases} \beta P(\text{sep}|Q), & \text{dla } Q \in O_a, \\ (1 - \beta) P(\text{ent}|Q), & \text{dla } Q \in O_o, \end{cases} \quad (2.12)$$

gdzie O_o oznacza zbiór wyników prowadzących do odrzucenia hipotezy o splątaniu. Można wykazać, że strata L_Q jest minimalna, gdy $q_B = \beta$, co pozwala utożsamić ten parametr ze względną wagą błędu fałszywie pozytywnego. Średnia oczekiwana strata, niezależna od konkretnego wyniku pomiaru, ma postać:

$$L = \sum_Q L_Q P(Q) = q_B P(Q \in O_a | \text{sep}) P(\text{sep}) + (1 - q_B) P(Q \in O_o | \text{ent}) P(\text{ent}), \quad (2.13)$$

co odpowiada średniemu kosztowi błędów obu typów. Podobnie jak moc testu w podejściu częstościowym, średnia strata L umożliwia porównanie różnych strategii bayesowskich dla danego poziomu akceptacji. Maksymalna efektywność testu odpowiada zatem minimalnej wartości L .

W granicy dużej liczby kopii stanu ($n \gg 1$) wpływ prawdopodobieństw a priori zanika. Zbiory akceptacji można wówczas dobrać tak, by $P(Q \in O_a | \text{sep}) \approx 0$, co prowadzi do $P(\text{ent}|Q) \approx 1$. W takim przypadku oba podejścia, częstościowe i bayesowskie, prowadzą do równoważnych wyników ($q_F \approx q_B \approx 1$), a ich rozróżnienie traci praktyczne znaczenie.

2.2.1 Strategia optymalizacyjna

Model pomiaru korelacji kwantowych przy ograniczonej statystyce, wraz ze ścisłą analizą statystyczną, umożliwia badanie niewielkich zbiorów danych pomiarowych η oraz ich optymalizację pod względem zadanej wiarygodności i maksymalnej efektywności wnioskowania o splątaniu. Problem można sformułować następująco [A]:

Dla ustalonego minimalnego poziomu wiarygodności $q_{\min}$ oraz całkowitej liczby kopii stanu η , należy znaleźć taki zbiór M obserwacji, z których każda jest estymowana na podstawie n_j kopii stanu, oraz odpowiedni zbiór akceptacji O_a , aby obserwacja $Q \in O_a$ umożliwiała weryfikację splątania z wiarygodnością $q \geq q_{\min}$ przy maksymalnej możliwej efektywności testu.

Prawdopodobieństwa $P(Q | \text{sep})$ potrzebne do realizacji tej procedury obliczane są dla najgorszego możliwego przypadku korelacji zgodnych z separowalnością stanu, tj. takich, które maksymalizują prawdopodobieństwo znalezienia się w zbiorze akceptacji, nie przekraczając jednocześnie granicy separowalności określonej dla nieskończonej statystyki.

W pracy [A] jako przykład zastosowania wprowadzonej metody rozważono stan N kubitów posiadający M perfekcyjnych (± 1) korelacji, który został poddany szumowi

$$\rho = p|\psi\rangle\langle\psi| + (1 - p)\rho_N, \quad (2.14)$$

gdzie ρ_N opisuje szum eksperymentalny. Model ten jest ogólny, jednak w pracy [A] wykazano, że dla jednorodnego i nieobciążonego (*ang.* unbiased) szumu prawdopodobieństwa istotne dla testowania hipotez zależą wyłącznie od parametru p . Dlatego w typowych przypadkach

można ograniczyć się do białego szumu $\rho_N = \mathbb{1}/2^N$, gdzie $\mathbb{1}$ oznacza macierz jednostkową. W poniższych przykładach przyjęto $q_F = q_B = 97.5\%$ oraz $P(\text{ent}) = P(\text{sep}) = 1/2$. Dla uproszczenia założono także, że każda korelacja jest estymowana na tej samej liczbie kopii stanu n , a całkowita liczba kopii wynosi $\eta = 20$.

2.2.2 Liniowy świadek splątania

Niech zbiór akceptacji określony będzie przez górne ograniczenie $\mathcal{C}$ (tj. $O_a = \{E_W : E_W < \mathcal{C}\}$), a liniowy świadek splątania oparty na M korelacjach ma postać $E_W = T_1 - T_2 - \dots - T_M + 1$. Dla dostatecznie małego progu $\mathcal{C}$ stan separowalny, który maksymalizuje prawdopodobieństwo znalezienia się w zbiorze akceptacji (tzw. najgorszy przypadek), charakteryzuje się korelacjami $T_1 = -1/M$ oraz $T_i = 1/M$ dla $i = 2, \dots, M$. Okazuje się, że optymalizacja $P(E_W \leq \mathcal{C}|\text{sep})$ oraz $P(E_W > \mathcal{C}|\text{ent})$, odpowiadająca odpowiednio wiarygodności i efektywności testu polegająca na jednoczesnym minimalizowaniu obu tych prawdopodobieństw, uzyskuje najlepsze rezultaty dla $M = 5$ oraz $n = 4$.

Osoba preferująca częstościowe wnioskowanie uzna że, konieczne do osiągnięcia minimalnej wiarygodności 97.5%, jest przyjęcie prógu $\mathcal{C}_F = -2.5$, co prowadzi do efektywności $r \approx 76.5\%$. Najwyższy możliwy poziom wiarygodności, jaki można uzyskać przy 20 kopiach stanu, wynosi 99.996% i odpowiada progowi $\mathcal{C}_F = -4$, jednak odbywa się to kosztem znacznego spadku efektywności do $r \approx 6.9\%$.

W podejściu bayesowskim wymaga się, aby dla wszystkich wartości w zbiorze akceptacji zachodziło $P(\text{ent}|E_W) \geq 97.5\%$. Warunek ten spełniony jest przy progowi $\mathcal{C}_B = -3$, dla którego odpowiadająca strata, opisana funkcją kosztu, wynosi $L \approx 0.007$.

2.2.3 Nieliniowy świadek splątania

Dla świadka kwadratowego wprowadzono próg $\mathcal{B}$ definiujący zbiór akceptacji $O_a = \{S : S > \mathcal{B}\}$. Ponownie, z rozważań $P(S \geq \mathcal{B}|\text{sep})$ oraz $P(S < \mathcal{B}|\text{ent})$, wynika, że optymalny rozkład kopii stanu zachodzi dla $n = 4$ oraz $M = 5$. Dla minimalnie wymaganej wiarygodności 97.5% otrzymujemy w podejściu częstościowym próg $\mathcal{B}_F = 4$, natomiast w analizie bayesowskiej $\mathcal{B}_B = 5$. Odpowiadają im efektywność $r \approx 31.4\%$ oraz strata $L \approx 0.013$.

W ogólności badany stan uznaje się za splątany, jeśli $p > 1/(2^{N-1} + 1)$. Dlatego bardziej realistyczne podejście uwzględnia licznosc stanów separowalnych w obrębie badanej rodziny w rozkładzie początkowym. Przykładowo, dla $N = 4$, naturalne prawdopodobieństwa początkowe wynoszą $P(\text{ent}) = 8/9$ i $P(\text{sep}) = 1/9$, co prowadzi do zmiany progu na $\mathcal{B}_B = 4$ i efektywności $L \approx 0.018$. Dla $N = 5$ otrzymujemy $P(\text{ent}) = 16/17$ oraz $P(\text{sep}) = 1/17$, przy czym próg bayesowski wynosi $\mathcal{B}_B = 3.5$, a odpowiadająca mu efektywność $L \approx 0.015$.

Dla równych prawdopodobieństw początkowych, podejście częstościowe daje szerszy zbiór akceptacji. Jednak przy fizycznie uzasadnionych prawdopodobieństwach początkowych sytuacja może być odwrotna i próg $\mathcal{C}_B$ może być niższy niż $\mathcal{C}_F$. Dla rozważanych przypadków, tj., równych prawdopodobieństw a priori oraz tych wynikających z uwzględniania licznosci stanów separowalnych dla stanów cztero- i pięciokubitowych, uzyskano odpowiednio: $(\mathcal{B} = 5, q_F = 99.8\%, r = 6.9\%)$, $(\mathcal{B} = 4, q_F = 97.6\%, r = 31.4\%)$ oraz $(\mathcal{B} = 3.5, q_F = 92.5\%, r = 54.9\%)$. Należy jednak zauważyć, że w ostatnim przypadku poziom ufności nie spełnia wymaganego progu wiarygodności.

2.3 Podsumowanie rezultatów

W pracy [A] zaproponowano model pomiaru korelacji kwantowych przy niskiej statystyce pomiarowej. Wykorzystując ścisłą analizę statystyczną opartą na wnioskowaniu częstościowym oraz bayesowskim, przeformułowano standardowy formalizm świadków splątania tak, aby uwzględniał on skończoną liczbę kopii stanu, określony poziom wiarygodności oraz moc testu. Dla skrajnie niskiej statystyki, obejmującej jedynie 20 kopii stanu, wykazano możliwość przeprowadzenia wiarygodnej i efektywnej weryfikacji splątania kwantowego, która w sposób jawny uwzględnia ograniczenia wynikające ze skończonej liczby danych pomiarowych. Zaproponowana technika wyróżnia się prostotą implementacji i może być zastosowana do surowych danych eksperymentalnych.

Rozdział 3

Prawdopodobieństwo łamania nierówności Bella dla losowych kierunków pomiarowych oraz ograniczonej wydajności detekcji [B]

Pozostając w tematyce ograniczeń na obserwacje ściśle kwantowych korelacji motywowanych problemami eksperymentalnymi, jednym z największych wyzwań pozostaje ograniczona wydajność detekcji. Każdy poprawny test Bella powinien ją uwzględniać na etapie wyprowadzania nierówności oraz ograniczenia wynikającego z możliwych strategii modeli zmiennych ukrytych. Scenariusze Bella z losowymi pomiarami nie zawierały dotychczas takiej analizy. Praca [B] zamyka tę lukę w literaturze. Wyprowadzono w niej dolne ograniczenie na prawdopodobieństwo łamania nierówności Bella z dwoma losowymi kierunkami pomiarowymi oraz dwoma wynikami na stanowisko dla stanu maksymalnie splątanego dwóch kubitów uwzględniające wydajność detekcji. Przeanalizowano także bardziej złożone scenariusze zawierające więcej ustawień i cząstek, oraz wykazano, że tzw. typowość nielokalności jest zachowana jeśli wydajność detekcji przekracza jej krytyczną wartość.

3.1 Nierówności Bella z losowymi pomiarami

Standardowe testy Bella zakładają perfekcyjną znajomość lokalnych układów odniesienia związanych z kierunkami pomiarowymi oraz ich wzajemną kalibrację. Dla dwóch obserwatorów oznacza to, że zarówno Alicja, jak i Bob, mają pewność co do wykonywanych pomiarów oraz stosunków wzajemnych ustawień. Założenie kalibracji układów odniesienia może być jednak zniesione i stoją za tym dwie główne motywacje. Pierwsza odnosi się do praktycznych aspektów kalibracji układów odniesienia. Jeśli dystans pomiędzy obserwatorami jest duży, wymaga to zasobów komunikacyjnych i jest tym bardziej uciążliwe gdy stacje pomiarowe nie znajdują się w spoczynku. Druga pozwala na zbadanie fundamentalnego powiązania pomiędzy splątaniem zawartym w stanie, a możliwymi kierunkami pomiarowymi, które ujawniają nielokalność Bella. Technicznym narzędziem pozwalającym na relaksację kalibracji układów odniesienia są losowe pomiary.

3.2 Prawdopodobieństwo łamania nierówności Bella z ograniczoną wydajnością detekcji

Scenariusz Bella zdefiniowany jest poprzez liczbę obserwatorów N , pomiarów m_k i wyników na stanowisko pomiarowe o_k . Rezultaty zawarte w rozprawie ograniczają się do stanów kubitowych i eksperymentów z dwoma wynikami pomiarowymi na stronę oraz równą liczbą pomiarów m . Losowy k -ty pomiar w dowolnym stanowisku definiujemy poprzez pomiar projekcyjny zadany przez wektor $|i\rangle$ obrócony macierzą unitarną losowaną zgodnie z miarą Haara, tj. $|h_k^i\rangle = U_{[k]}|i\rangle$. Wybór bazy początkowej $|i\rangle$ jest dowolny tak długo jak pozostaje on spójny. W naszych rozważaniach wybrano bazę obliczeniową składającą się z wektorów własnych macierzy Pauliego σ_z . Wielkością, na której skupiono się w pracy [B] jest prawdopodobieństwo łamania nierówności Bella zdefiniowane jako

$$\mathcal{P}_V = \int d\Omega f(\Omega), \quad (3.1)$$

gdzie całka obliczana jest po wszystkich parametrach wchodzących w skład danego scenariusza, a f jest binarną funkcją przyjmującą wartość 1 gdy istnieje nierówność Bella łamana przez zadany stan i ustawienia oraz 0 w pozostałym przypadku. Wielkość ta opisuje łamanie wszystkich możliwych nierówności Bella w zadanym scenariuszu. W większości praktycznych obliczeń całka ta jest estymowana poprzez próbkowanie przestrzeni możliwych parametrów. Problem wydajności detekcji może być wprowadzony do formalizmu losowych pomiarów poprzez odpowiednie definicje możliwych wyników pomiarów. Dla prostoty, założmy sytuację w której mamy do czynienia z Alicją i Bobem wykonującymi po dwa pomiary na stanie dwóch kubitów. Każdy z obserwatorów losuje parę kierunków pomiarowych i zapisuje ± 1 w sytuacji, gdy detektor zadziałał oraz 0 gdy nie miało to miejsca. Zdarzenia te zachodzą z prawdopodobieństwem zadanym przez wydajność detekcji $\eta_{A/B}$. Równoległą strategią przypisywania wyników pomiaru jest uznanie braku detekcji za wynik -1 . Jeśli jednak nie zostanie to wyraźnie podkreślone, przedstawione rezultaty odnosić się będą do sytuacji z trzema wynikami pomiaru, nie z grupowaniem wyników. Po uzbieraniu dostatecznej statystyki pomiarowej, porównują oni swoje dane i pytają: czy złamaliśmy jakąś nierówność Bella? Na pytanie to można odpowiedzieć szukając odpowiedniego łącznego rozkładu prawdopodobieństwa. Średnia liczba pozytywnych odpowiedzi na to pytanie odpisuje wielkość $\mathcal{P}_V(\eta_A, \eta_B)$ - tym razem zależna już od wydajności detekcji.

3.2.1 Dwukubitowy stan maksymalnie splatany i dwa ustawienia na stanowisko

Dla przypadku stanu maksymalnie splatanego dwóch kubitów i dwóch pomiarów na stronę wykazano następujące dolne ograniczenia na prawdopodobieństwo łamania nierówności Bella. W układach charakteryzujących się asymetryczną wydajnością detekcji $\eta_A = 1, \eta_B = \eta$ praw-

dopodobieństwo łamania nierówności Bella $\mathcal{P}_V$ jest ograniczone z dołu przez

$$\begin{aligned} \mathcal{P}_V^c(\eta) = & \frac{1}{\eta^2(\eta^2-1)} \left[\pi + 2(\eta^2 - 1)\kappa - \pi\eta^4 \right. \\ & + 4\sqrt{2} \left(\sqrt{\eta^2 + \kappa} + \sqrt{\kappa^2(\eta^2 + \kappa)} - 2\eta^2\sqrt{\eta^2 + \kappa} \right) \\ & \left. + 8(\eta^4 - 1) \operatorname{arccot} \left\{ \sqrt{2} \sqrt{\frac{1}{\frac{\kappa}{\eta^2} + 1}} + \sqrt{\frac{\eta^2 - \kappa}{\eta^2 + \kappa}} \right\} \right]; \quad \eta > \frac{1}{\sqrt{2}}, \end{aligned} \quad (3.2)$$

gdzie $\kappa = \sqrt{2\eta^2 - 1}$. W przypadku symetrycznym $\eta_A = \eta_B = \eta$, ograniczenie to ma postać

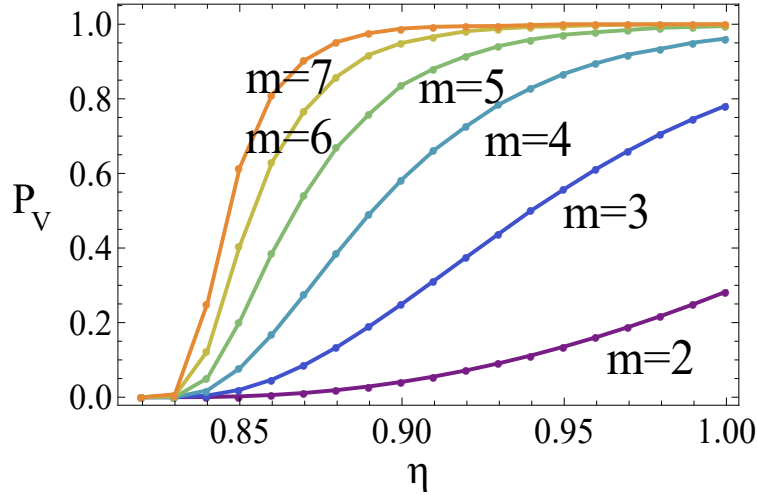
$$\begin{aligned} \mathcal{P}_V^c(\eta) = & \frac{2}{\eta^4} \left[2\gamma + \eta^2\pi[(\eta - 2)\eta + 2] \right. \\ & + \eta(\eta - 2) \left[2\sqrt{2}(\sqrt{\gamma\eta^2 + 2\gamma(1 - \eta) + \eta^4} \right. \\ & \left. \left. - \sqrt{-\gamma\eta^2 - 2\gamma(1 - \eta) + \eta^4}) + \gamma \right] + 2\gamma \right. \\ & \left. - 4\eta^2[(\eta - 2)\eta + 2] \arctan \left\{ \frac{4\sqrt{(\eta-1)^2((\eta-1)\eta+1)^2}}{\gamma((\eta-2)\eta+2)+\eta^4} \right\} \right]; \\ & \eta > \sqrt{5 + 4\sqrt{2}} - \sqrt{2} - 1, \end{aligned} \quad (3.3)$$

gdzie $\gamma = \sqrt{8(\eta - \eta^2) + 4\eta^3 + \eta^4 - 4}$. Powyższe wyniki zgadzają się z wyczerpującymi obliczeniami numerycznymi w ramach co najwyżej $\approx 3.5\%$ błędu względnego dla zadanych przedziałów η . Zatem ograniczenia te stanowią bardzo dobre przybliżenie badanej wielkości. Dolne ograniczenia na η stanowią górne ograniczenie na krytyczną wydajność detekcji η_{crit} , tj. taką wydajność poniżej której niemożliwym jest złamanie którejkolwiek nierówności Bella na danym stanie w ustalonym scenariuszu.

3.2.2 Wiele ustawień, typowość nielokalności Bella i kompensacja wydajności detekcji

W przypadku wielu ustawień m oraz symetrycznej wydajności detekcji η przeprowadzono obliczenia numeryczne oparte na programowaniu liniowym, wykorzystując technikę zaprezentowaną po raz pierwszy w [33]. Uzyskane rezultaty dla stanu maksymalnie splątanego przedstawiono na Rysunku 3.1.

Przy założeniu braku strat w liczbie wykrytych cząstek, prace [11, 12] wykazały, że nielokalność Bella jest zjawiskiem typowym. Oznacza to, że dla dostatecznie dużej liczby cząstek lub ustawień zaprzeczenie lokalnie realistycznego opisu eksperymentu staje się niemal pewne, tj. $\mathcal{P}_V \approx 100\%$. W pracy [B] pokazano, że zjawisko to utrzymuje się również w sytuacji, gdy wydajność detekcji jest ograniczona. Wyniki te obrazują także możliwość kompensacji ograniczonej efektywności η poprzez zwiększenie liczby ustawień pomiarowych (zob. Rys. 3.1).



Rysunek 3.1: Prawdopodobieństwo łamania nierówności Bella dla stanu maksymalnie splątanego dwóch kubitów, m ustawień na stanowisko pomiarowe oraz symetrycznej wydajności detekcji η . Rysunek zaczerpnięty z pracy [B].

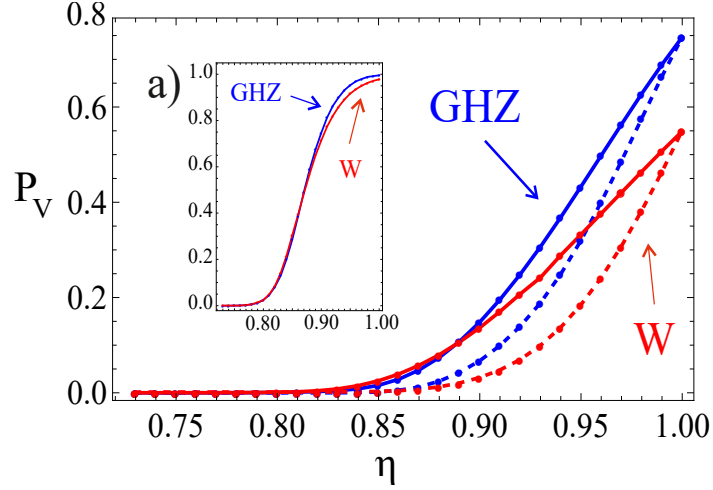
3.2.3 Stany trójkubitowe

Istnieją dwie klasy czystych stanów splątanych trzech kubitów, które nie są równoważne względem stochastycznych lokalnych operacji i klasycznej komunikacji (SLOCC) – stan GHZ oraz stan W. Ze względu na fakt, że formalizm losowych pomiarów nie rozróżnia stanów lokalnie równoważnych, obie te klasy stanowią naturalny obiekt analizy w omawianym kontekście. Ponownie wykonano dla nich wyczerpujące obliczenia numeryczne oparte na programowaniu liniowym. W tym przypadku zbadano dodatkowo strategie grupowania wyników. Uzyskane rezultaty przedstawiono na Rysunku 3.2.

Analiza danych numerycznych wykazała, że dla $m = 2$ przybliżona wartość η_{crit} jest niższa od najlepszego znanego wcześniej rezultatu $\eta_{crit} = 0.75$ [34]. Korzystając z odpowiednich kątów pomiarowych, wyprowadzono nową nierówność Bella dla obróconego stanu GHZ, charakteryzującą się niższą krytyczną wydajnością detekcji $\eta_{crit} = (\sqrt{10} - 1)/3 \approx 0.7208$. Dla trzech ustawień odtworzono nierówność CHSH z pomiarem projekcyjnym na jednym ze stanowisk, uzyskując $\eta_{crit} = 2/3$.

3.3 Podsumowanie rezultatów

W pracy [B] zbadano wpływ ograniczonej wydajności detekcji na prawdopodobieństwo łamania nierówności Bella dla losowych kierunków pomiarowych. Wyprowadzono dolne ograniczenie tej wielkości w przypadku stanu maksymalnie splątanego dwóch kubitów oraz dwóch ustawień na stanowisko. Następnie rozszerzono analizę na większą liczbę ustawień i cząstek, wykorzystując metody programowania liniowego. Wykazano, że typowość nielokalności – rozumiana jako niemal pewne łamanie nierówności Bella przy dostatecznie dużej liczbie ustawień pomiarowych – pozostaje zachowana tak długo, jak długo efektywność detekcji nie spada poniżej progu krytycznego. Oznacza to, że spadek prawdopodobieństwa łamania nierówności



Rysunek 3.2: Prawdopodobieństwo łamania nierówności Bella dla stanów GHZ (kolor niebieski) oraz W (kolor czerwony) dla $m = 2$ (rysunek główny) i $m = 3$ (rysunek a) ustawień. Wyniki zakładają symetryczną wydajność detekcji oraz dwa modele przetwarzania zdarzeń, w których nie nastąpiła detekcja cząstki. Linie ciągłe odpowiadają modelowi, w którym brak detekcji utożsamiany jest z wynikiem 0, natomiast linie przerywane przedstawiają przypadek, gdy brakowi detekcji przypisywany jest wynik -1 . Rysunek zaczerpnięty z pracy [B].

Bella może być kompensowany przez zwiększenie liczby ustawień pomiarowych na stronę. Na podstawie obserwowanych spadków prawdopodobieństwa łamania w przypadku trójkubitowego stanu GHZ wyprowadzono nową nierówność Bella dla dwóch ustawień, charakteryzującą się krytyczną wydajnością detekcji niższą niż najlepszy znany wcześniej rezultat.

Rozdział 4

Relacja pomiędzy splątaniem a superpozycją w kwantowych układach odniesienia [C]

Układ odniesienia rozumiany jest najczęściej jako pewien abstrakcyjny punkt w przestrzeni, wyposażony w zbiór kierunków, który nie posiada żadnych kwantowych cech. W ramach mechaniki kwantowej możliwa jest bardziej ogólna konstrukcja, w której układ odniesienia utożsamiony jest z wybraną fizyczną cząstką kwantową. Formalizm ten określa się mianem kwantowych układów odniesienia, natomiast operacje umożliwiające ich zmianę nazywane są transformacjami kwantowych układów odniesienia [13].

Okazuje się, że w takim opisie ściśle kwantowe efekty, takie jak superpozycja czy splątanie, zależą od tego, w jakim kwantowym układzie odniesienia są one opisywane. W pracy [C] scharakteryzowano dowolne transformacje kwantowych układów odniesienia w kontekście tych zasobów oraz wyprowadzono relacje zachodzące pomiędzy nimi dla czystych stanów dwuczęściowych. Głównym rezultatem pracy jest twierdzenie wykazujące, że suma ilości superpozycji na podukładzie dwuczęściowego stanu czystego oraz jego splątania – mierzonych za pomocą miar opartych na entropii von Neumanna – jest niezmiennikiem rozważanych transformacji.

4.1 Kwantowe układy odniesienia

Rozważmy układ trzech cząstek: A , B i C . Niech kwantowy układ odniesienia będzie stowarzyszony z cząstką C . Dla zadanej zwartej grupy G oraz przestrzeni Hilberta $L^2(G) \otimes L^2(G)$, operacja opisująca transformację kwantowego układu odniesienia $C \rightarrow A$ dana jest wyrażeniem [35]

$$S^{(C) \rightarrow (A)} = \int_G dg |g^{-1}\rangle_{CA} \langle g| \otimes U^\dagger B(g), \quad (4.1)$$

gdzie $|g\rangle g \in G$ stanowi bazę lokalnej przestrzeni Hilberta, a $U(g)$ jest lewą regularną reprezentacją grupy G . Dla grup dyskretnych należy przyjąć $\mathbb{C}(G)$ zamiast $L^2(G)$, a całkę zastąpić sumą. W najprostszym przypadku grupy dwuelementowej $\mathbb{Z}_2$, stowarzyszonej z pojedynczym kubitem, mamy $\mathbb{C}(\mathbb{Z}_2) \cong \mathbb{C}^2$, a transformacja przyjmuje postać

$$S^{(C) \rightarrow (A)} = |0\rangle_{CA} \langle 0| \otimes \mathbb{1}_B + |1\rangle_{CA} \langle 1| \otimes \sigma_B^x. \quad (4.2)$$

Operacja ta ma zatem postać kwantowej kontrolowanej bramki NOT. W pracy [C] wykazano, że dowolne transformacje kwantowych układów odniesienia są operacjami niekoherentnymi zachowującymi ilość superpozycji globalnego stanu.

4.2 Ilość splątania i superpozycji podukładu jako niezmiennik transformacji

Jednymi z miar splątania i superpozycji (nazywaną w kontekście teorii zasobów koherencją) badanych w pracy [C] są miary oparte o względną entropię von Neumanna. Dla dwuczęściowych stanów czystych AB entropia splątania zdefiniowana jest jako [36, 37]

$$\mathcal{E}_e[|\psi\rangle_{AB}] = \mathcal{S}[\rho_A], \quad (4.3)$$

gdzie $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ to stan zredukowany podukładu A , a $\mathcal{S}[\rho] = -\text{Tr}[\rho \log \rho]$ to entropia von Neumanna. Względna entropia koherencji stanu ρ przyjmuje postać [38]

$$\mathcal{C}_e[\rho] = \mathcal{S}[\rho_d] - \mathcal{S}[\rho], \quad (4.4)$$

gdzie ρ_d to część diagonalna stanu ρ .

Zgodnie z Twierdzeniem 1 z pracy [C], dla dowolnego stanu czystego $|\psi_{AB}^{(C)}\rangle$ opisywanego z perspektywy kwantowego układu odniesienia stowarzyszonego z cząstką C , oraz stanu $|\psi_{BC}^{(A)}\rangle = S^{(C)\rightarrow(A)}|\psi_{AB}^{(C)}\rangle$ otrzymanego po dowolnej transformacji $S^{(C)\rightarrow(A)}$, zachodzi

$$\mathcal{E}_e[|\psi^{(C)}\rangle_{AB}] + \mathcal{C}_e[\rho_A^{(C)}] = \mathcal{E}_e[|\psi^{(A)}\rangle_{BC}] + \mathcal{C}_e[\rho_C^{(A)}]. \quad (4.5)$$

Powyższe równanie pokazuje, że suma miar splątania czystego stanu dwuczęściowego oraz koherencji jego podukładu stanowi niezmiennik transformacji kwantowych układów odniesienia. Równoważną postać tego wyniku można zapisać jako

$$\Delta\mathcal{E}_e = -\Delta\mathcal{C}_e, \quad (4.6)$$

gdzie symbol Δ oznacza zmianę danej wielkości przed i po transformacji. Dla miar opartych o względną entropię zachodzi więc $\Delta\mathcal{E}_e \Delta\mathcal{C}_e \leq 0$. Zgodnie z Twierdzeniem 2 z pracy [C], zależność ta jest uniwersalna dla dowolnych miar splątania i koherencji, jeżeli istnieje taki kwantowy układ odniesienia, dla którego czysty stan dwuczęściowy przed transformacją jest produktowy lub jego stan zredukowany jest niekoherentny. Zastosowanie powyższego rezultatu do innych kwantifikatorów omawianych zasobów zostało szczegółowo omówione w pracy [C].

4.3 Względność splątania a Nielokalność Bella

Splątanie kwantowe powiązane jest ściśle z nielokalnością Bella, omawianą w poprzednich rozdziałach. Ponieważ zmiana kwantowego układu odniesienia może prowadzić do całkowitej utraty splątania przy jednoczesnym wzroście koherencji podukładu, możliwa jest sytuacja, w której w jednym układzie odniesienia stan wykazuje nielokalność Bella, a w innym już

nie. Na pierwszy rzut oka mogłoby się więc wydawać, że łamanie nierówności Bella wyróżnia pewną klasę układów odniesienia i nie pozwala traktować ich jako równoważnych. Jednakże transformacje kwantowych układów odniesienia są operacjami unitarnymi działającymi na cały stan dwóch cząstek, a ich faktoryzacja nie jest możliwa. W konsekwencji pomiar lokalnej wielkości na stanie splątanym $|\psi^{(C)}\rangle_{AB}$ odpowiada pomiarowi niefaktoryzowalnej wielkości na produktowym stanie $|\psi^{(A)}\rangle_{BC}$ z perspektywy A . Mamy bowiem $\text{Tr}(O\rho_{AB}^{(C)}) = \text{Tr}(OS^{(A)\rightarrow(C)}\rho_{BC}^{(A)}[S^{(A)\rightarrow(C)}]^\dagger) = \text{Tr}([S^{(A)\rightarrow(C)}]^\dagger OS^{(A)\rightarrow(C)}\rho_{BC}^{(A)}) = \text{Tr}(O'\rho_{BC}^{(A)})$. Dla dwóch kubitów, przykładowo, operator $\sigma_x \otimes (\sigma_x + \sigma_z)/\sqrt{2}$ przechodzi w $(\sigma_x \otimes \mathbb{1} - \sigma_y \otimes \sigma_y)$. Niefaktoryzowalność prawdopodobieństw, która w jednym układzie odniesienia stoi w sprzeczności z modelami lokalnych zmiennych ukrytych, w innym układzie wynika z nielokalności obserwabli. Zjawisko łamania nierówności Bella zachodzi więc w każdym układzie odniesienia, jednak jego źródło i interpretacja mają różny charakter. Należy przy tym podkreślić, że powyższe rozważania dotyczą perspektyw cząstek kwantowych, a nie rzeczywistej zmiany aparatury pomiarowej.

4.4 Podsumowanie rezultatów

W pracy [C] wyprowadzono ilościowe związki pomiędzy splątaniem czystych stanów dwuczęściowych a koherencją ich podukładów, zachodzące przy transformacjach kwantowych układów odniesienia. Dla wybranych kwantyfikatorów splątania i koherencji wykazano, że zmiana splątania przy przejściu do nowego kwantowego układu odniesienia jest w pełni kompensowana przez zmianę ilości superpozycji na podukładzie badanego stanu. Ponadto udowodniono, że dla dowolnych miar omawianych zasobów spadek miary splątania zawsze wiąże się ze wzrostem koherencji, jeśli jeden z początkowych zasobów jest równy zero. Prawdziwe jest też odwrotne stwierdzenie. Wykazano również, że dowolna operacja zmiany kwantowego układu odniesienia zachowuje koherencję globalnego stanu, a uzyskane rezultaty mają istotne konsekwencje dla interpretacji nielokalności Bella z perspektywy układów kwantowych.

Rozdział 5

Poligamia łamania nierówności Bella w układach wielu cząstek [D]

Korelacje kwantowe prowadzące do łamania nierówności Bella podlegają wielu ograniczeniom dotyczącym ich struktury oraz możliwych wartości. Jednym z najważniejszych rezultatów stojących u podstaw kwantowej kryptografii niezależnej od urządzeń jest monogamia nielokalności [14, 39, 40, 41]. Zgodnie z nią w układzie trzech cząstek ($N = 3$) niemożliwe jest złamanie więcej niż jednej dwuczęściowej nierówności Bella opartej na dwóch ustawieniach pomiarowych na stanowisko. W scenariuszu wielocząstkowym złamanie więcej niż jednej dwuustawieniowej nierówności Bella pomiędzy co najmniej trzema obserwatorami staje się możliwe [42, 43, 44, 45, 46, 47, 48]. Nie zaproponowano jednak dotąd żadnej nierówności oraz stanu, które pozwalałyby na łamanie *wszystkich* możliwych nierówności, tj. nierówności na każdym możliwym zbiorze obserwatorów o ustalonej liczności.

W pracy [D] przedstawiono systematyczną metodę identyfikacji stanów kwantowych N kubitów oraz generowania dwuustawieniowych nierówności Bella dla $N - 1$ obserwatorów, które pozwalają osiągnąć to zadanie dla $N > 3$. Innymi słowy wykazano, że jeżeli liczba kubitów w układzie jest większa niż trzy, to możliwe jest złamanie wszystkich dwuustawieniowych, $(N - 1)$ -cząstkowych nierówności Bella jednocześnie, wykazując tym samym poligamiczną naturę wielocząstkowej nielokalności. Uzyskane wyniki zostały potwierdzone eksperymentalnie dla $N = 6$ przy pomocy wielofotonowych stanów splątanych w bazie polaryzacyjnej.

5.1 Monogamia nielokalności Bella

Rozważmy nierówność CHSH, dla której obowiązuje ścisła monogamia

$$\text{CHSH}_{AB} \equiv A_1 B_1 + A_2 B_2 + A_2 B_1 - A_1 B_2 < 2, \quad (5.1)$$

gdzie $A_{1/2}$ i $B_{1/2}$ to odpowiednio ustawienia pomiarowe obserwatorów A i B . Okazuje się, że w układzie trzech obserwatorów: A , B i C , którzy chcą złamać powyższą nierówność parami zachodzi [14, 39, 40, 41]

$$\langle \text{CHSH}_{AB} \rangle^2 + \langle \text{CHSH}_{AC} \rangle^2 \leq 8, \quad (5.2)$$

gdzie poprzez $\langle \text{CHSH}_{AB/AC} \rangle$ oznaczono wartość wyrażenia CHSH mierzonego pomiędzy A i B oraz A i C na dowolnym stanie trzech kubitów. Nierówność (5.2) stanowi równanie dysku

o promieniu $2\sqrt{2}$, co oznacza, że niemożliwym jest złamanie obu z rozważanych nierówności jednocześnie, tj. gdy $\langle \text{CHSH}_{AB} \rangle > 2$, to $\langle \text{CHSH}_{AC} \rangle < 2$. Powyższy rezultat pozostaje prawdziwy dla dowolnego lokalnego wymiaru układu kwantowego d [39, 40, 41]. Ze względu na pełną charakteryzację nielokalności przez zbiór nierówności CHSH w scenariuszach o dwóch ustawieniach i dwóch wynikach pomiarowych, wynik ten ma duże znaczenie fundamentalne, a monogamia łamania nierówności Bella jest jedną z najważniejszych zasad fundamentów mechaniki kwantowej.

5.2 Poligamia nielokalności Bella

W pracy [D] wykazano, że w przypadku nierówności wielokubitowych możliwe jest złamanie dwuustawieniowych nierówności Bella pomiędzy *wszystkimi* zbiorami $N-1$ obserwatorów jednocześnie dla $N > 3$. Ze względu na naturalną sprzeczność tego wyniku z monogamią nierówności Bella zachodzącą dla przypadku nierówności dwuczęściowych, zjawisko to nazwano poligamią łamania nierówności Bella, lub poligamią nielokalności. Podobne zachowania zaobserwowano w przeszłości dla nierówności wieloustawieniowych [49] oraz dla wybranych, choć nie wszystkich, podzbiorów układu N -kubitowego [42, 43, 44, 45, 46, 47, 48]. Wyniki zaprezentowane w artykule [D] dostarczają systematycznej metodologii pozwalającej na identyfikację odpowiednich stanów kwantowych oraz sposobu generowania poligamicznych nierówności Bella. Stanowią one także pierwszy dowód poligamicznego charakteru nielokalności Bella dla układów o dowolnej liczbie cząstek $N > 3$.

5.2.1 Nierówności Mermin–Ardehaliego–Belinskiego–Klyshki

Jednym z uogólnień nierówności CHSH na układy wielocząstkowe są nierówności Mermin–Ardehaliego–Belinskiego–Klyshki (MABK) w postaci $\langle M_N \rangle \leq 2^{(N-1)/2}$ [50, 51, 52]. Naturalnym pierwszym krokiem w scenariuszu wielocząstkowym jest więc zbadanie ich własności pod kątem współdzielenia nielokalności. Dla ustawień prowadzących do maksymalnego łamania nierówności MABK przez $N-1$ kubitowy stan GHZ operator Bella ma postać [52, 53, 54]

$$M_{N-1} = 2^{N-1}(|D_{N-1}^0\rangle\langle D_{N-1}^{N-1}| + |D_{N-1}^{N-1}\rangle\langle D_{N-1}^0|), \quad (5.3)$$

gdzie $|D_{N-1}^e\rangle$ oznaczają stany Dickego $(N-1)$ kubitów. Ważną obserwacją jest to, że dla takich samych ustawień na każdą ze stron badana nierówność prowadzi do operatora permutacyjnie niezmienniczego. Oznacza, to że jego wartości oczekiwane na dowolnych stanach zredukowanych N -kubitowego stanu permutacyjnie niezmienniczego są równe. Jeśli istnieje zatem taki globalny stan symetryczny, dla którego wybrany stan zredukowany prowadzi do złamania lokalnie realistycznego ograniczenia dla $N-1$ kubitów, wszystkie podzbiory $N-1$ kubitów prowadzą do złamania nierówności MABK. W rezultacie dzięki symetrii nierówności, pomiarów i stanu problem wykazania istnienia poligamii Bella znacznie się upraszcza.

Dowolny permutacyjnie niezmienniczy stan czysty N kubitów można zapisać w bazie stanów Dickego poprzez

$$|\psi_N\rangle = \sum_{i=0}^N c_i |D_N^i\rangle \quad (5.4)$$

Wartość oczekiwana M_{N-1} , rozszerzonego trywialnie o macierz identycznościową przy dowolnym kubicie, na powyższym stanie wynosi

$$\mathcal{M}_{N-1} \equiv \langle \psi_N | M_{N-1} \otimes \mathbb{1} | \psi \rangle = \frac{2^{N-1}}{\sqrt{N}} (c_0 c_{N-1} + c_1 c_N) \quad (5.5)$$

Optymalizując to wyrażenie po amplitudach c_i otrzymamy $\mathcal{M}_{N-1}^{\max} = 2^{N-2}/\sqrt{N}$. Przykładowy stan osiągający tę wartość dany jest poprzez

$$|\phi\rangle = \frac{1}{\sqrt{2}} (|W_N\rangle + |1\rangle^{\otimes N}), \quad (5.6)$$

gdzie $|W_N\rangle \equiv |D_N^1\rangle$ to stan W dla N kubitów. Okazuje się, że $\mathcal{M}_{N-1}^{\max}$ jest większe od lokalnego ograniczenia na M_{N-1} począwszy od $N = 5$. Zatem wszystkie N nierówności MABK mierzone na $(N-1)$ -kubitowych stanach zredukowanych (5.6) łamane są jednocześnie dla każdego $N > 4$, tym samym wykazując poligamię nielokalności Bella.

5.2.2 Program liniowy dla nierówności poligamicznych

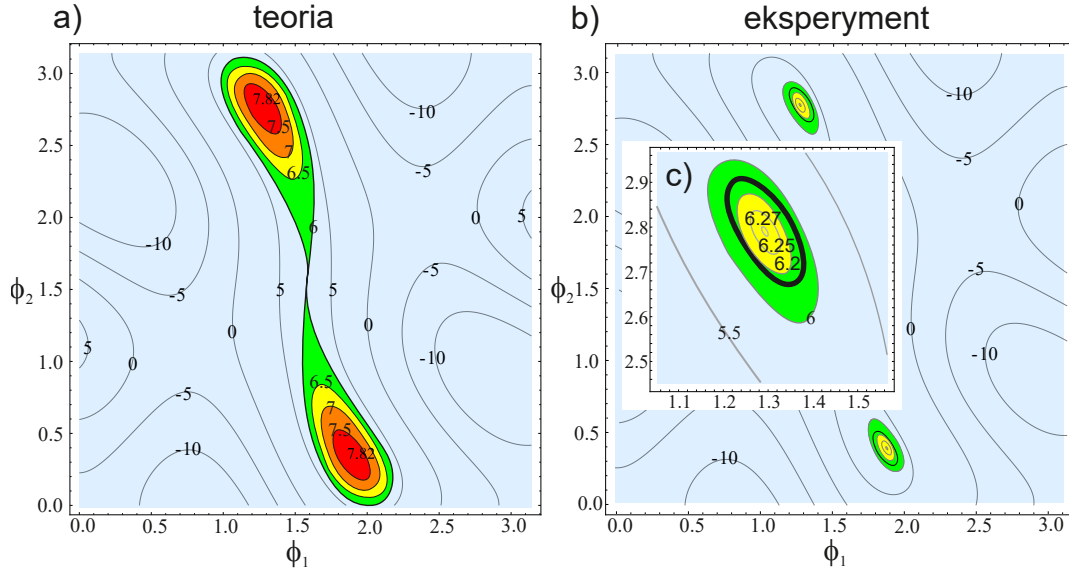
Wyniki przedstawione w poprzedniej sekcji wykazują istnienie poligamii łamania nierówności Bella dla $N > 4$. Wiedząc, że zjawisko to jest niemożliwe dla przypadku trzech cząstek, otwartym problemem pozostaje więc pytanie dotyczące istnienia takiego zachowania dla $N = 4$. Ponownie ze względu na własności symetrii permutacyjnej założono takie same ustawienia pomiarowe na każdą ze stron oraz stany w postaci (5.4). Wykorzystując autorską metodę opartą o programowanie liniowe znaleziono $\langle I_{ABC} \rangle \leq 6$, gdzie

$$\begin{aligned} I_{ABC} = & 2 \text{sym}[A_1] - \text{sym}[A_1 B_1] - \text{sym}[A_1 B_2] \\ & + \text{sym}[A_2 B_2] + 2A_1 B_1 C_1 + \text{sym}[A_2 B_1 C_1] \\ & - 2 \text{sym}[A_2 B_2 C_1] - A_2 B_2 C_2. \end{aligned} \quad (5.7)$$

Poprzez $\text{sym}[\cdot]$ oznaczono symetryzację wyrażenia po wszystkich permutacjach dolnych indeksów $\text{sym}[A_k B_l C_m] = \sum_{\pi(k,l,m)} A_k B_l C_m$, zakładając $A_0 = B_0 = C_0 = 1$. Otrzymana nierówność należy do zbioru Śliwy [55]. Zmiana oznaczeń obserwatorów z ABC na dowolny trójelementowy podzbiór $ABCD$ definiuje pozostałe nierówności wchodzące w skład przedstawianych rozważań. Dla stanu $|\psi\rangle = \cos \theta |D_4^1\rangle + \sin \theta |1111\rangle$ z $\theta = 0.144$ każda z czterech nierówności jest łamana o 0.154, co rozwiązuje problem istnienia poligamii Bella dla $N = 4$. Inne interesujące przykłady nierówności poligamicznych otrzymanych za pomocą wykorzystanej metody opisano szczegółowo w pracy [D]. Przykładowa implementacja programu dostępna jest w repozytorium Zenodo (<https://zenodo.org/records/13893953>).

5.2.3 Realizacja eksperymentalna

Zjawisko poligamii łamania wielocząstkowych nierówności Bella przetestowano eksperymentalnie wykorzystując wielofotonowe stany splecione w polaryzacyjnych stopniach swobody. W celu zmniejszenia wymagań eksperymentalnych dotyczących krytycznej widzialności stanu kwantowego względem nierówności (5.7) oraz nierówności MABK wykorzystano ponownie



Rysunek 5.1: Wartość kwantowa wyrażenia I_{ABCDE} definiującego poligamiczną nierówność Bella dla $N = 6$ uśredniona po wszystkich zredukowanych stanach pięciu kubitów. Panel a) przedstawia przewidziane teoretycznie wartości dla stanu $|D_6^3\rangle$ w funkcji kątów pomiarowych ϕ_1 i ϕ_2 dla obserwacji $\cos \phi_i \sigma_x + \sin \phi_i \sigma_z$. Panel b) zawiera wyniki oparte o dane eksperymentalne wraz z regionem wyszczególnionym poprzez czarną krzywą (zbliżenie c)), w którym każda z sześciu nierówności jest łamana. Rysunek zaczerpnięty z pracy [D].

metodę opartą o programowanie liniowe. Dla dostępnego w eksperymencie stanu $|D_6^3\rangle$ znaleziono nierówność $\langle I_{ABCDE} \rangle \leq 6$ o krytycznej widzialności 76.71%. Szczegółowy opis eksperymentu dostępny jest w pracach [56, 57]. Wartość wyrażenia I_{ABCDE} uśrednionego po wszystkich zbiorach pięciofotonowych w funkcji kątów pomiarowych, oparta o dane eksperymentalne, została porównana z przewidywaniami teoretycznymi na Rysunku 5.1. Ogólny charakter wyników jest zgodny z przewidywaniami. Ze względu na problemy związane ze skończoną statystyką oraz niedoskonałości eksperymentalne średnia wartość wyrażenia Bella nie jest jednak równoważna wartości każdego z wyrażeń. Obszar ustawień, dla których każda z nierówności łamana jest jednocześnie, zakreślony został czarną krzywą na panelu C. Dla optymalnych ustawień pomiarowych każda z nierówności łamana jest o co najmniej jedno odchylenie standardowe. Uzyskane w oparciu o dane eksperymentalne łamania są zatem istotne, a poligamia nierówności Bella została potwierdzona doświadczeniem.

5.2.4 Podsumowanie rezultatów

W pracy [D] wykazano możliwość równoczesnego łamania nierówności Bella na wszystkich $(N-1)$ -kubitowych podukładach stanu N cząstkowego dla $N > 3$. W układach zawierających więcej niż cztery cząstki zjawisko to okazało się możliwe dla nierówności MABK. W przypadku $N = 4$ problem został rozwiązany poprzez wykorzystanie autorskiej metody opartej o programowanie liniowe, pozwalającej na znajdowanie nierówności poligamicznych w oparciu o symetrię permutacyjną. Przewidywania teoretyczne zostały potwierdzone w eksperymencie optycznym dla $N = 6$. Pomiary na splątanym w polaryzacyjnych stopniach swobody sześciok-

fotonowym stanie Dickego wykazały możliwość jednoczesnego łamania sześciu nierówności Bella na raz. Uzyskane wyniki wykazują poligamiczną naturę nielokalności Bella w układach wielu kubitów oraz wskazują na o wiele mniejsze niż dotychczas uważano ograniczenia na możliwość współdzielenia tych korelacji.

Rozdział 6

Szybkość wytwarzania silnie splątanych stanów kwantowych z oddziaływaniami dwuciałowymi [E]

Wytwarzanie stanów kwantowych o ściśle kwantowych własnościach stanowi jedno z kluczowych zagadnień teorii informacji kwantowej oraz nieodzowny element eksperymentów kwantowych. Najczęściej stosowanym podejściem do realizacji tego zadania jest sekwencyjne stosowanie bramek kwantowych działających co najwyżej na parach kubitów. W takim modelu wystarczające jest rozważanie oddziaływań o co najwyżej strukturze dwuciałowej. Minimalny czas wymagany do otrzymania zadanego stanu kwantowego z ustalonego stanu początkowego określany jest mianem kwantowego ograniczenia prędkości (*ang.* quantum speed limits) [16, 17, 58]. Dla optymalnych operacji dwukubitowych ograniczenia te mogą być nasycane na poziomie pojedynczych bramek, a ich realizacja jest możliwa przy wykorzystaniu powszechnie występujących oddziaływań fizycznych. Odmienny charakter ma sytuacja w przypadku układów fizycznych, w których ewolucja zachodzi w sposób samoistny. Hamiltonian opisujący dynamikę układu wielu cząstek nie prowadzi w ogólności do obrazu skończonej sekwencji, kolejno realizowanych oddziaływań. Co więcej, Hamiltoniany umożliwiające najszybsze osiągnięcie zadanego stanu splątanego, począwszy od stanu produktowego, zawierają zazwyczaj oddziaływania wyższych rzędów, tj. oddziaływania wielociałowe [59, 60]. W pracy [E] przeanalizowano szybkość wytwarzania N -kubitowych stanów GHZ, W, Dickego oraz AME dla $N \leq 7$ w przypadku ewolucji generowanych przez Hamiltoniany o strukturze co najwyżej dwuciałowej. Zastosowanie heurystycznych metod optymalizacji nieliniowej, uzupełnionych rachunkami analitycznymi, pozwoliło wykazać istotny wpływ struktury oddziaływań na możliwość realizacji rozważanego zadania. W szczególności zaobserwowano, że narzucone ograniczenia mogą prowadzić do szybszego, w porównaniu ze standardowym obwodem kwantowym o tej samej całkowitej energii, generowania stanu GHZ. Ponadto zaobserwowano interesujące zjawisko, w którym wytworzenie stanu o większej liczbie cząstek nie wymaga większej ilości czasu. Ze względu na powszechność oddziaływań dwuciałowych w układach fizycznych analizowany problem umożliwia głębsze zrozumienie fundamentalnych ograniczeń dotyczących wytwarzania splątania kwantowego w układach wielocząstkowych. Jednocześnie pozwala on na identyfikację klas Hamiltonianów posiadających własności szczególnie istotne z punktu widzenia efektywnego generowania silnie splątanych stanów kwantowych.

6.1 Oddziaływania dwuciałowe i kwantowe ograniczenia szybkości generowania stanów kwantowych

Dowolny Hamiltonian opisujący układ N kubitów z oddziaływaniami dwuciałowymi można zapisać w postaci

$$H = \sum_{i,j \in \mathcal{I}_g} \sum_{\mu_i, \mu_j = x, y, z} h_{\mu_i \mu_j} \sigma_{\mu_i}^{(i)} \sigma_{\mu_j}^{(j)} + \sum_{i, \mu_i} b_{\mu_i} \sigma_{\mu_i}^{(i)}, \quad (6.1)$$

gdzie współczynniki $h_{\mu_i \mu_j}$ opisują siłę oddziaływań pomiędzy parami kubitów, natomiast b_{μ_i} odpowiadają lokalnym członom jednociłowym, zwykle interpretowanym jako zewnętrzne pole magnetyczne lub stany energetyczne atomów dwupoziomowych. Struktura pierwszej sumy w (6.1) określona jest przez graf oddziaływań $\mathcal{I}_g$, opisujący geometrię sprzężeń w układzie. W pracy [E] przeanalizowano grafy oddziaływań o ustalonym zasięgu geometrycznym $\mathcal{I}$. Przykładowo, $\mathcal{I} = 1$ odpowiada oddziaływaniom najbliższych sąsiadów, $\mathcal{I} = 2$ kolejnym najbliższym sąsiadom, natomiast $\mathcal{I} = N - 1$ opisuje oddziaływania typu „każdy z każdym”. Rozważana klasa Hamiltonianów obejmuje m.in. modele Heisenberga, dalekozasięgowe modele Isinga, oddziaływania występujące w układach nadprzewodzących kubitów czy oddziaływanie Kaplana-Shekhtmana-Entina-Wohlmana-Aharoniego. W celu zwiększenia efektywności obliczeń, Hamiltonian w każdym przypadku dostosowywano do symetrii badanego stanu.

Zaniedbując strukturę oddziaływań, minimalny czas potrzebny do przekształcenia stanu początkowego $|\psi(0)\rangle$ w stan $|\psi(t)\rangle$ ograniczony jest przez tzw. kwantowe ograniczenia prędkości. Jednym z fundamentalnych rezultatów w tym zakresie jest nierówność Mandelstama–Tamma [16]

$$t_{\min} = \arccos(|\langle \psi(0) | \psi(t) \rangle|) \frac{\hbar}{\Delta H}, \quad (6.2)$$

gdzie ΔH oznacza odchylenie standardowe Hamiltonianu w stanie początkowym, a dalej przyjęto $\hbar = 1$. Ponieważ ewolucja czasowa $U = \exp(-iHt)$ jest liniowa względem energii i czasu, możliwe jest przeskalowanie energii układu w celu skrócenia czasu ewolucji. W relacji (6.2) efekt ten uwidacznia się poprzez czynnik ΔH . W obliczeniach przeprowadzonych w pracy [E] uwzględniono to poprzez odpowiednią normalizację Hamiltonianu,

$$H \rightarrow \frac{H - \mathbb{1}E_{\min}}{E_{\max} - E_{\min}}, \quad (6.3)$$

gdzie $E_{\min}$ oraz $E_{\max}$ są odpowiednio najmniejszą i największą wartością własną Hamiltonianu H . Taka procedura umożliwia analizę wpływu struktury oddziaływań niezależnie od całkowitej szerokości widma energetycznego.

6.2 Ograniczenia na szybkość wytwarzania splątania przy oddziaływaniach dwuciałowych

Analiza minimalnej liczby poziomów energetycznych Hamiltonianu (6.1) oraz odpowiadającej im wariancji energii w stanie początkowym pozwala na analityczne oszacowanie minimalnego czasu ewolucji $t_{\min}$ dla trzech kubitów z wykorzystaniem relacji Mandelstama–Tamma. W

pracy [E] pokazano, że przybliżony minimalny czas potrzebny do wytworzenia trójkubitowych stanów GHZ oraz W ze stanu $|000\rangle$ wynosi

$$t_{\min}^{\text{GHZ}} = 2\pi, \quad t_{\min}^W = \pi, \quad (6.4)$$

jeśli dozwolone są jedynie oddziaływania dwuciałowe. W przypadku Hamiltonianów zawierających oddziaływania trójciałowe otrzymano natomiast

$$t_{\min}^{\text{GHZ}} = \frac{\pi}{2}, \quad t_{\min}^W = \pi. \quad (6.5)$$

Oznacza to, że dla stanu W możliwa jest saturacja ograniczenia Mandelstama–Tamma, natomiast w przypadku stanu GHZ czas generacji przy użyciu oddziaływań dwuciałowych jest czterokrotnie dłuższy. Porównanie tych estymacji z wynikami numerycznymi przedstawiono w kolejnej sekcji rozdziału.

6.2.1 Optymalizacja wierności stanu

W przypadku większej liczby kubitów zastosowano heurystyczne metody nieliniowej optymalizacji w celu maksymalizacji wierności stanu $|\psi(t)\rangle = e^{-iHt}|0\rangle^{\otimes N}$ względem zadanego stanu docelowego $|\psi\rangle$. Mianowicie rozważano

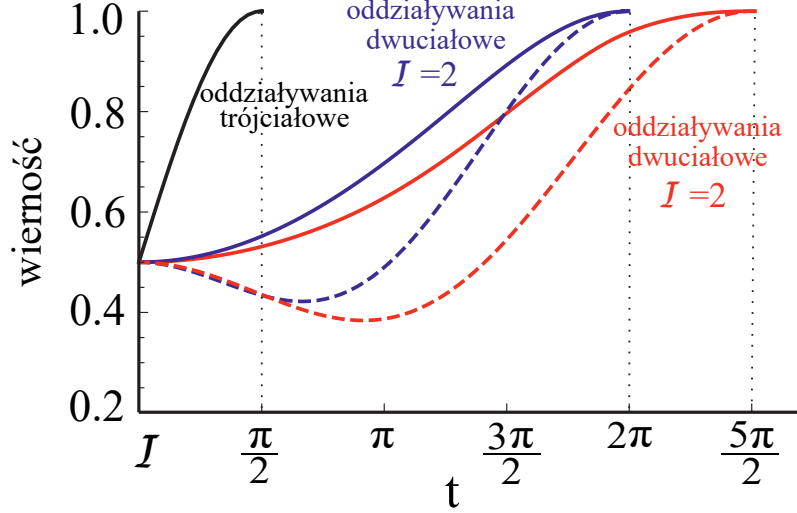
$$\mathcal{F}(t) = \max_{h_{\mu\nu}, b_\mu} |\langle\psi|e^{-iHt}|0\dots 0\rangle|^2. \quad (6.6)$$

Wybór stanu początkowego motywowany jest standardową konwencją stosowaną w algorytmach kwantowych. Obliczenia numeryczne przeprowadzono w środowisku Python z wykorzystaniem lokalnych metod BFGS dostępnych w bibliotece SciPy oraz algorytmów genetycznych z pakietu PyGAD. Równolegle wykonano obliczenia stosując Wolfram Mathematica 13.0 i dostępne w nim metody *Differential Evolution* oraz *Random Search* dla 10^4 iteracji.

Maksymalizację wierności przeprowadzano dla kolejnych punktów czasowych t z krokiem Δt rzędu 10^{-1} – 10^{-4} , zależnie od dynamiki zmian $\mathcal{F}(t)$. Przy zastosowaniu metod analitycznych warunków $\mathcal{F}(t') = 1$ wyznacza minimalny czas t' potrzebny do uzyskania stanu docelowego. Uwzględniając ograniczoną precyzję metod numerycznych, za spełnienie tego warunku przyjęto $\mathcal{F}(t') \geq 1 - 10^{-6}$, co pozwalało utożsamić t' z $t_{\min}$.

Dla $N = 3$ wyniki numeryczne, zarówno dla ogólnego Hamiltonianu dwuciałowego, jak i dla Hamiltonianu dostosowanego do symetrii stanu, okazały się zgodne z wcześniejszymi oszacowaniami analitycznymi. Minimalna wierność pomiędzy ewoluowanym w czasie stanem $|000\rangle$, a stanem GHZ dla oddziaływań najbliższych sąsiadów ($\mathcal{I} = 1$), oddziaływań typu każdy z każdym ($\mathcal{I} = 2$) oraz oddziaływań trójciałowych przedstawiona została na Rysunku 6.1. Dla większych N konfiguracja oddziaływania $\mathcal{I} = N - 1$ także okazała się być najbardziej optymalną pod względem poszukiwanego $t_{\min}$. Uzyskane wyniki dla zaadoptowanych do symetrii stanu Hamiltonianów oraz powyższego $\mathcal{I}$ podsumowano w Tabeli 6.1. Podany dla pięciokubitowego stanu AME minimalny czas ewolucji wyznaczony został z mniejszą, wynoszącą 1%, dokładnością. W przypadku stanów GHZ zaobserwowano interesujące zjawisko, w którym wytworzenie stanu o większej liczbie cząstek niekoniecznie wymaga większej ilości czasu. Ekstrapolacja uzyskanych rezultatów wykazała

$$t_{\min}^{\text{GHZ},2} = \pi \lceil N/2 \rceil^2 / 2. \quad (6.7)$$



Rysunek 6.1: Zależność czasowa minimalnej wierności pomiędzy trójkubitowym stanem GHZ a stanem $|000\rangle$ ewoluującym za pomocą oddziaływań dwukubitowych (kolory czerwony i niebieski) oraz trójkubitowych (kolor czarny), uzyskana metodą optymalizacji numerycznej. Punkty końcowe poszczególnych krzywych wyznaczają odpowiadające im kwantowe ograniczenia prędkości. Krzywymi przerywanymi przedstawiono wierność stanu, którego dynamika opisywana jest przez optymalny Hamiltonian. Dla oddziaływań najbliższych sąsiadów ($\mathcal{I} = 1$, kolor czerwony) minimalny czas potrzebny na realizację postawionego problemu jest większy niż w przypadku oddziaływań typu każdy z każdym ($\mathcal{I} = 2$, kolor niebieski), który pozostaje w zgodzie z analitycznie oszacowanym czasem $t_{\min}$. W przypadku oddziaływań trójkubitowych relacja Mandelstama–Tamma jest nasycana. Rysunek zaczerpnięty z pracy [E].

Co istotne, standardowy obwód kwantowy oparty o bramki Hadamarda i CNOT, przy tej samej energii całkowitej, spełnia nierówność $N^2\pi \geq t_{\min}^{\text{GHZ},2}$. Zatem uzyskana optymalna ewolucja prowadzi do szybszej, w porównaniu ze standardowym obwodem kwantowym, generacji wielocząstkowego stanu GHZ.

Wykonane obliczenia numeryczne pozwoliły również na wyznaczenie jawnych postaci Hamiltonianów prowadzących do osiągnięcia stanu docelowego w minimalnym czasie $t_{\min}$. Dla nieparzystej liczby kubitów przykładowy Hamiltonian dwuciałowy osiągający $t_{\min}^{\text{GHZ},2}$ ma postać

$$H = \sum_{i=1}^N \sum_{j=i+1}^N (\sigma_x^{(i)} \sigma_x^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)}) + 2 \sum_{i=1}^N \sigma_y^{(i)}. \quad (6.8)$$

Hamiltonian ten, przedstawiony w nieunormowanej postaci, odpowiada modelowi XXY z jednorodnym polem magnetycznym skierowanym wzdłuż osi y oraz topologią oddziaływań typu „każdy z każdym”. Dla $N = 3$ redukuje się on do modelu najbliższych sąsiadów z periodycznymi warunkami brzegowymi.

Stan	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	MT
$ GHZ\rangle$	$2\pi \approx 6.28$	$2\pi \approx 6.28$	$\frac{9\pi}{2} \approx 14.14$	$\frac{9\pi}{2} \approx 14.14$	$8\pi \approx 25.13$	$\frac{\pi}{2} \approx 1.57$
$ W\rangle$	$\pi \approx 3.14$	$\frac{\sqrt{11}\pi}{\sqrt{2}} \approx 7.37$	$\frac{9\pi}{\sqrt{5}} \approx 12.64$	≈ 18.76	≈ 25.60	$\pi \approx 3.14$
$ D_4^2\rangle$		$\frac{15}{2}\pi \approx 23.56$				
$ AME\rangle$	$2\pi \approx 6.28$	—	≈ 10.72		—	≈ 2.42

Tabela 6.1: Podsumowanie wyników dotyczących minimalnych czasów ewolucji prowadzących do wytworzenia rozważanych stanów kwantowych przy użyciu dwuciałowych Hamiltonianów zaadaptowanych do symetrii stanu oraz topologii oddziaływań typu „każdy z każdym”. Dla porównania w ostatniej kolumnie zestawiono odpowiadające im kwantowe ograniczenia prędkości wynikające z relacji Mandelstama–Tamma. W przypadku stanów AME podano odpowiednie ograniczenie dla $N = 5$.

6.3 Podsumowanie rezultatów

W pracy [E] przeanalizowano możliwie najszybsze czasy wytwarzania N -kubitowych stanów GHZ, W, Dickego oraz AME dla $N \leq 7$ z wykorzystaniem Hamiltonianów dwuciałowych zaadaptowanych do symetrii rozważanych stanów. Wykorzystując relacje Mandelstama–Tama uzyskano przybliżone t_{min} dla stanów GHZ i W trzech kubitów poprzez estymacje maksymalnego odchylenia standardowego umożliwiającego ewolucję do pożądanego stanu. Dla $N \geq 3$ przeprowadzono wyczerpujące obliczenia numeryczne polegające na maksymalizacji wierności pomiędzy stanem ewoluowanym a stanem docelowym względem wszystkich parametrów Hamiltonianu. W przypadku $N = 3$ otrzymane wyniki numeryczne okazały się być zgodne z oszacowaniami analitycznymi. Dla wszystkich analizowanych stanów, z wyjątkiem trójkubitowego stanu W, uzyskane t_{min} okazały się być znacznie dłuższe niż, osiągnęte przez Hamiltoniany zawierające oddziaływania wielociałowe, t_{min} wynikające z ograniczenia Mandelstama–Tama. Praca [E] podaje także jawne postacie optymalnych Hamiltonianów dwuciałowych realizujących uzyskane minimalne czasy ewolucji. Szczególnie interesującą obserwacją jest fakt, że w przypadku stanów GHZ minimalny czas ewolucji generowanej przez oddziaływania dwuciałowe nie musi rosnąć wraz ze wzrostem liczby cząstek. Przeprowadzona analiza ujawnia wyraźny, ilościowy związek pomiędzy szybkością wytwarzania nieklasycznych korelacji a charakterem oddziaływań odpowiedzialnych za dynamikę układu.

Rozdział 7

Weryfikacja oddziaływań wielociałowych poprzez kwantową informację Fishera [F]

Większość oddziaływań analizowanych w mechanice kwantowej, a w szczególności w teorii informacji kwantowej, ma charakter fundamentalnie dwuciałowy. Poza czystą ciekawością badawczą okazuje się, że możliwie najszybsza ewolucja pomiędzy dwoma stanami kwantowymi [59, 60], generowanie silnie splątanych stanów podstawowych [18] czy bezpośrednia implementacja wybranych kodów kwantowej korekcji błędów [61], wymagają Hamiltonianów zawierających hipotetyczne oddziaływania wielociałowe. Pierwsze z wymienionych zagadnień zostało omówione w poprzednim rozdziale. Potencjalne zastosowania oddziaływań wielociałowych, a także potrzeba pogłębionego zrozumienia ich roli w układach wielu ciał, doprowadziły do rozwoju zaawansowanych technik ich inżynierii oraz symulacji [62, 63, 64, 19]. Równocześnie, trudności związane z ich eksperymentalną realizacją oraz identyfikacją układów wykazujących istotne oddziaływania wyższych rzędów wskazują na konieczność opracowania metod ich wykrywania i weryfikacji [65]. W pracy [F] zaproponowano metodę weryfikacji oddziaływań k -ciałowych opartą na analizie kwantowej informacji Fishera mierzonej na stanie produktowym. Przekroczenie maksymalnej wartości tej wielkości, dozwolonej dla ustalonego rzędu sprzężeń k , pozwala wnioskować o obecności oddziaływań wyższych niż k -ciałowe. Zaproponowana technika znajduje zastosowanie w modelach typu Isinga oraz w sytuacjach, w których struktura oddziaływań jedno- i dwuciałowych jest znana.

7.1 Oddziaływania wyższych rzędów i kwantowa informacja Fishera

Hamiltoniany opisujące oddziaływania k -ciałowe w układach wielu kubitów można konstruować poprzez uogólnienie wyrażenia (6.1) o iloczyny tensorowe co najwyżej k macierzy Pauliego. W ogólności operator

$$H = \sum_j h_j^{(k)} \quad (7.1)$$

nazywamy k -lokalnym lub k -ciałowym (tj. zawierającym oddziaływania k -tego rzędu) wtedy i tylko wtedy, gdy każdy z operatorów $h_j^{(k)}$ działa nietrywialnie na co najwyżej k cząstek.

W dalszej części pracy przez H_k oznaczono Hamiltoniany zawierające wyłącznie oddziaływania k -ciałowe, natomiast poprzez $\mathcal{H}^{(K)}$ Hamiltoniany obejmujące oddziaływania co najwyżej rzędu K .

Kwantowa informacja Fishera stanowi jeden z kluczowych wyznaczników efektów o ściśle kwantowym charakterze. Pozwala ona m.in. na wyznaczanie fundamentalnych ograniczeń dokładności pomiarowej, które przekraczają możliwości klasyczne [66]. Dla operatora hermitowskiego A oraz stanu kwantowego ρ definiuje się ją jako

$$F[\rho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2, \quad (7.2)$$

gdzie $|k\rangle$ oraz λ_k oznaczają odpowiednio wektory i wartości własne macierzy gęstości ρ . Suma przebiega wyłącznie po parach indeksów spełniających warunek $\lambda_k + \lambda_l \neq 0$. Dla dowolnego stanu mieszanego ρ istnieje zawsze stan czysty $|\psi\rangle$ taki, że

$$F[\rho, A] \leq 4(\Delta A)_{|\psi\rangle}^2 \equiv F[|\psi\rangle, A], \quad (7.3)$$

co oznacza, że maksymalizacja kwantowej informacji Fishera po wszystkich stanach może zostać ograniczona do optymalizacji wariancji operatora A po stanach czystych. W standardowym podejściu metrologicznym, przy kodowaniu unitarnym, rozważane Hamiltoniany są najczęściej jednociłowe i prowadzą do tzw. skalowania Heisenberga, dla którego $F \sim N^2$ [67, 68]. Kwantowa informacja Fishera była jednak badana również w kontekście układów wielociłowych, gdzie zaobserwowano jakościowe zmiany w jej zachowaniu, zob. [69, 70, 71, 72, 73].

7.2 Wielociłowe oddziaływania typu Isinga

W pracy [F] w pierwszej kolejności rozważono szczególny przypadek układów wielu kubitów, który formalnie przypomina standardowe podejście metrologiczne. Wielociłowy Hamiltonian typu Isinga o ustalonym rzędzie oddziaływań zdefiniowano jako

$$H_k = \mathcal{N} \sum_{(i_1, \dots, i_k) \in G_k} \sigma_z^{(i_1)} \sigma_z^{(i_2)} \dots \sigma_z^{(i_k)}, \quad (7.4)$$

gdzie $\mathcal{N}$ jest stałą normalizacyjną, natomiast G_k oznacza graf oddziaływań typu „każdy z każdym”. Sumowanie przebiega po wszystkich k -elementowych podzbiorach cząstek, co zapewnia permutacyjną niezmienniczość operatora. Dla przypadku $k = 1$, przy odpowiednim wyborze normalizacji, otrzymujemy standardowy Hamiltonian metrologiczny

$$H_1 = \frac{1}{2} \sum_{i=1}^N \sigma_z^{(i)}. \quad (7.5)$$

Istotnym przykładem jest również przypadek $k = 2$ oraz $\mathcal{N} = J$, prowadzący do postaci

$$H_2 = J \sum_{i < j}^N \sigma_z^{(i)} \sigma_z^{(j)}. \quad (7.6)$$

Powyższy Hamiltonian opisuje dalekozasięgowy model Isinga na pełnym grafie oddziaływań i stanowi szczególny przypadek modelu Lipkina-Meshkova-Glicka.

Ogólny Hamiltonian typu Isinga zawierający co najwyżej oddziaływania K -ciałowe można zapisać jako

$$\mathcal{H}^{(K)} = \mathcal{N} \sum_{k \leq K} \alpha_k H_k, \quad (7.7)$$

gdzie α_k są rzeczywistymi współczynnikami określającymi wagi poszczególnych członów. W pracy [F] skupiono się głównie na przypadku $\alpha_k = 1$ dla wszystkich k . Stała normalizacyjna $\mathcal{N}$ została dobrana w taki sposób, aby nie naruszyć ani klasycznego, ani heisenbergowskiego skalowania, tj. $\|\mathcal{H}^{(K)}\| = \max_{|\phi\rangle} \|\mathcal{H}^{(K)} |\phi\rangle\| = \frac{N}{2}$. Taki wybór sprawia, że Hamiltonian „wygląda” tak, jakby nie zawierał oddziaływań wielociałowych. W konsekwencji otrzymujemy górne ograniczenie na wariancję oraz kwantową informację Fishera, które dla dowolnego k oraz dowolnego stanu kwantowego $|\psi\rangle$, splątanego bądź nie, nie mogą przekroczyć wartości N^2 . Warunek ten zrealizowano poprzez wybór $\mathcal{N} = N/(2 \max_i |E_i|)$, gdzie E_i oznaczają wartości własne operatora $\mathcal{H}^{(K)}$. Warto podkreślić, że wszystkie prezentowane wyniki pozostają prawdziwe również dla Hamiltonianów równoważnych względem lokalnych operacji unitarnych.

7.3 Procedura weryfikacji oddziaływań wielociałowych

Proponowana metoda weryfikacji oddziaływań wielociałowych opiera się na obserwacji dotyczącej możliwych wartości kwantowej informacji Fishera na stanach produktowych. Dla Hamiltonianu o ustalonym rzędzie oddziaływań k oraz liczbie kubitów N zachodzi może

$$B_k \equiv \max_{|\psi_{\text{prod}}\rangle} F[|\psi_{\text{prod}}\rangle, H_k] < B_{k+1}. \quad (7.8)$$

W konsekwencji, mając dostęp do oddziaływań wielociałowych, pomiar kwantowej informacji Fishera na stanach separowalnych ρ_{sep} umożliwia ich weryfikację w sytuacji, gdy $F[\rho_{\text{sep}}, H] > B_k$. Dla Hamiltonianów typu Isinga wartości B_k są ściśle uporządkowane względem rzędu oddziaływania k .

Pierwszym interesującym przypadkiem badanego problemu jest możliwość wykrywania oddziaływań wykraczających poza reżim dwuciałowy. Dla Hamiltonianu typu Isinga o $k = 2$ bezpośrednie obliczenia analityczne oraz numeryczne dla $N \leq 13$ prowadzą do

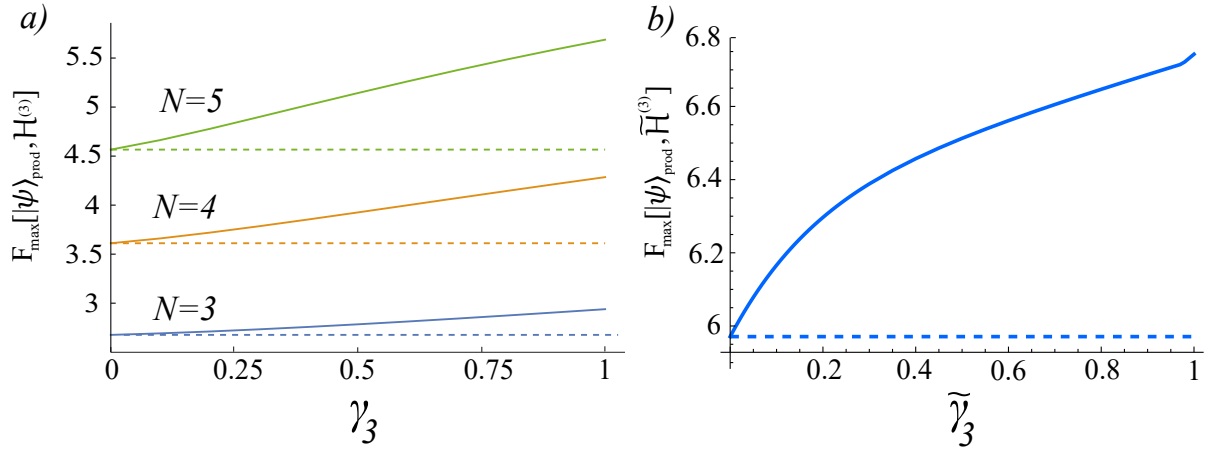
$$B_2 = \frac{2N(N-1)}{2(N-2)+1}. \quad (7.9)$$

Dla układów zawierających oddziaływania jedno- i dwuciałowe $\mathcal{H}^{(2)} = H_1 + H_2$ otrzymano ograniczenie

$$\mathcal{B}_{1+2} = \max_p \left(-\frac{16Np(p-1)}{(N+1)^2} \left[(N-2)^2 + 2(N-1)p((2N-3)p - (2N-5)) \right] \right), \quad (7.10)$$

co pozwala na weryfikację obecności oddziaływań co najmniej trójciałowych poprzez warunek

$$F[\rho, \mathcal{H}^{(K)}] > \mathcal{B}_{1+2} \Rightarrow K \geq 3. \quad (7.11)$$



Rysunek 7.1: Zależność maksymalnej wartości kwantowej informacji Fishera na stanie produktowym od parametru sprzężenia oddziaływań trójciałowych dodanych do odpowiednich Hamiltonianów jedno- i dwuciałowych. Panel a) przedstawia przypadek $N = 3, 4, 5$ kubitów, w którym Hamiltonian bazowy dany jest dwuciałowym Hamiltonianem typu Isinga z podłużnym polem magnetycznym, natomiast wkład oddziaływań wyższego rzędu opisany jest przez człon $\gamma_3 H_3$. Panel b) ilustruje analogiczną zależność dla trójkubitowego modelu XY z parametrami δ oraz b dobranymi tak, aby maksymalizować wartość $F[\rho_{\text{sep}}, H_{XY}]$. W tym przypadku oddziaływanie trójciałowe ma postać $\tilde{\gamma}_3 \sigma_x^1 \sigma_x^2 \sigma_x^3$. Rysunek zaczerpnięty z pracy [F].

Maksymalizacja B_{1+2} może zostać przeprowadzona analitycznie, gdyż rozważana funkcja jest wielomianem czwartego stopnia w parametrze p . Ze względu na obszerną postać rozwiązania jawnego nie zostało ono przytoczone wprost. Przykład zastosowania proponowanej metody przedstawiono dla dwuciałowego Hamiltonianu typu Isinga z podłużnym polem magnetycznym oraz kontrolowanym członem trójciałowym $\gamma_3 H_3$, oznaczonego jako $\mathcal{H}^{(3)}$. Przekroczenie ograniczenia $\mathcal{B}_{1+2}$ jednoznacznie sygnalizowało obecność niezerowego sprzężenia γ_3 (zob. Rys. 7.1a dla $N = 3, 4, 5$).

7.3.1 Przypadek znanej struktury oddziaływań dwuciałowych

Przedstawiona konstrukcja może zostać zastosowana do dowolnego Hamiltonianu o znanej strukturze oddziaływań jedno- i dwuciałowych. Taki scenariusz umożliwia identyfikację ukrytych oddziaływań wyższych rzędów, które mogłyby zostać następnie wzmocnione za pomocą technik symulacji kwantowych. W pracy [F] przeanalizowano w tym kontekście model XY na pełnym grafie oddziaływań z poprzecznym polem magnetycznym

$$\mathcal{H}_{XY} = \mathcal{N} \left\{ J \sum_{i < j} [(1 + \delta) \sigma_x^i \sigma_x^j + (1 - \delta) \sigma_y^i \sigma_y^j] + b \sum_i \sigma_z^i \right\},$$

gdzie $\mathcal{N}$ jest stałą normalizacyjną, J stałą oddziaływania, δ parametrem anizotropii, natomiast b określa siłę zewnętrznego pola magnetycznego. Optymalizacja kwantowej informacji Fishera po stanach produktowych oraz parametrach Hamiltonianu $\mathcal{H}_{XY}$ dla $N = 3$ kubitów, przy zachowaniu ustalonej normy, prowadzi do wartości granicznej $B_{XY} \approx 5.97$.

Dodanie członu trójciałowego $\sigma_x^1 \sigma_x^2 \sigma_x^3$ umożliwia przekroczenie tej wartości, ujawniając tym samym obecność oddziaływań wyższego rzędu. Zależność maksymalnej kwantowej informacji Fishera na stanie produktowym od parametru sprzężenia $\tilde{\gamma}_3$ dla Hamiltonianu XY z dodatkowym członem $\tilde{\gamma}_3 \sigma_x^1 \sigma_x^2 \sigma_x^3$ przedstawiono na Rysunku 7.1b. Dla losowych sprzężeń trójciałowych $\gamma_{ijk} \in [-0.5, 0.5]$ oraz dodanego członu $\sum_{i,j,k=1}^3 \gamma_{ijk} \sigma_i^1 \sigma_j^2 \sigma_k^3$ estymowana, na podstawie 10^5 próbek, częstość łamania ograniczenia B_{XY} dla optymalnego, z góry ustalonego stanu produktowego wynosi 1.4%. Przez stan optymalny rozumie się tutaj trójkubitowy stan produktowy osiągający maksymalną wartość kwantowej informacji Fishera dla Hamiltonianu $\mathcal{H}_{XY}$. Nie przeprowadzano optymalizacji po stanach produktowych dla pełnego Hamiltonianu, aby uwzględnić możliwie największy brak informacji na temat struktury oddziaływań wyższego rzędu. Choć wykrycie generycznych oddziaływań o nieustalonej strukturze przy wykorzystaniu przedstawionej metody jest zasadniczo możliwe, prawdopodobieństwo uzyskania jednoznacznie pozytywnej konkluzji testu pozostaje niewielkie.

7.4 Podsumowanie rezultatów

W pracy [F] zaproponowano nową metodę weryfikacji obecności oddziaływań wielociałowych w układach N kubitów, opartą na analizie kwantowej informacji Fishera. Wykorzystując bezpośrednie obliczenia analityczne i numeryczne wykazano uporządkowanie maksymalnej kwantowej informacji Fishera na stanach produktowych względem rzędu oddziaływań k dla Hamiltonianów typu Isinga o ustalonym k i liczbie kubitów N . Wyznaczono dla nich również jedno- i dwuciałowe ograniczenia, których przekroczenie jednoznacznie wskazuje na obecność oddziaływań co najmniej trójciałowych. Proponowana technika nie ogranicza się wyłącznie do modeli typu Isinga. Weryfikacja oddziaływań wyższych rzędów jest możliwa także w sytuacji, w której znana jest struktura oddziaływań jedno- i dwuciałowych. Zastosowanie tej procedury zilustrowano na przykładzie trójkubitowego modelu XY. Dodanie do niego kontrolowanego członu trójciałowego lub Hamiltonianu z losowymi sprzężeniami trójciałowymi prowadziło do przekroczenia odpowiedniego ograniczenia na kwantową informację Fishera, ponownie wskazując na obecność oddziaływań wyższego rzędu.

Rozdział 8

Dyskusja i podsumowanie uzyskanych wyników

Celem niniejszej rozprawy było zbadanie fizycznych ograniczeń związanych z wytwarzaniem oraz obserwacją korelacji o charakterze ściśle kwantowym. Przedstawione wyniki obejmują zarówno zagadnienia o charakterze fundamentalnym, jak i problemy istotne z punktu widzenia praktycznych zastosowań w technologiach kwantowych. Do pierwszej z tych kategorii należą prace poświęcone analizie ograniczeń wynikających z wyboru kwantowego układu odniesienia, możliwości współdzielenia nielokalności Bella w układach wielocząstkowych oraz struktury oddziaływań dwu- i wielociałowych. Druga grupa zagadnień dotyczy realistycznych uwarunkowań eksperymentalnych, takich jak silnie ograniczona statystyka pomiarowa, skończona wydajność detekcji oraz brak współdzielonego układu odniesienia. Uzyskane rezultaty jednoznacznie wskazują na istotny wpływ tych czynników na możliwości generowania i obserwacji nieklasycznych korelacji. Badane ograniczenia mogą dostarczać potencjalnych korzyści lub osłabiać, lecz nie całkowicie eliminować, analizowane nieklasyczne efekty. W tych przypadkach zaproponowano także strategie rozwiązywania potencjalnie występujących problemów.

Do głównych rezultatów rozprawy należą: (1) wprowadzenie modelu pomiaru korelacji kwantowych opartego na skończonej statystyce, (2) przeprowadzenie ścisłej analizy statystycznej indykatorów splątania wyznaczanych na skrajnie małych zbiorach danych, (3) wyprowadzenie analitycznych ograniczeń na prawdopodobieństwo łamania nierówności Bella dla dwukubitowych stanów maksymalnie splątanych w eksperymentach z dwoma ustawieniami na stanowisko pomiarowe, (4) przeprowadzenie obliczeń numerycznych dotyczących prawdopodobieństwa łamania nierówności Bella w zależności od wydajności detekcji dla stanów dwukubitowych i liczby ustawień $m \leq 7$ oraz stanów trójkubitowych dla dwóch i trzech ustawień (5) wykazanie istnienia typowości nielokalności dla wydajności detekcji powyżej progu krytycznego, (6) zaproponowanie dwuustawieniowej nierówności Bella, o niższej niż dotychczas najlepszej wydajności krytycznej, dla stanu GHZ trzech kubitów, (7) charakteryzacja transformacji kwantowych układów odniesienia jako operacji niekoherentnych zachowujących globalną ilość superpozycji, (8) wykazanie ilościowej relacji pomiędzy splątaniem a superpozycją podukładu dla dwuczęściowych stanów czystych powiązanych operacją zmiany kwantowego układu odniesienia, (9) wykazanie poligamii nielokalności Bella dla dwuustawieniowych nierówności $(N - 1)$ -kubitowych mierzonych w układzie N obserwatorów dla

każdego $N > 3$, (10) wprowadzenie metodologii pozwalającej na identyfikację stanów oraz generowanie nierówności Bella o własnościach poligamicznych, (11) wyznaczenie minimalnych czasów potrzebnych do wytworzenia wybranych silnie splątanych stanów kwantowych przez ewolucje generowane zaadoptowanymi do symetrii Hamiltonianami o strukturze dwuciałowej oraz podanie jawnej postaci optymalnych generatorów, (12) zaproponowanie metody weryfikacji oddziaływań wielociałowych opartej na analizie kwantowej informacji Fishera dla Hamiltonianów typu Isinga oraz sytuacji, w której struktura oddziaływań dwuciałowych jest znana.

Przedstawione w niniejszej rozprawie wyniki pogłębiają zrozumienie fizycznych ograniczeń związanych z wytwarzaniem i obserwacją korelacji kwantowych oraz dostarczają narzędzi umożliwiających ich badanie – również w realistycznych warunkach eksperymentalnych. Uzyskane rezultaty mają znaczenie zarówno fundamentalne, jak i aplikacyjne, wskazując na potencjalne zastosowania w obecnych oraz przyszłych eksperymentach i technologiach kwantowych, w szczególności w kontekście certyfikacji zasobów kwantowych, testów nielokalności oraz charakteryzacji złożonych oddziaływań w układach wielu cząstek.

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ARTICLE OPEN



Valid and efficient entanglement verification with finite copies of a quantum state

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Detecting entanglement in multipartite quantum states is an inherently probabilistic process, typically with a few measured samples. The level of confidence in entanglement detection quantifies the scheme's validity via the probability that the signal comes from a separable state, offering a meaningful figure of merit for big datasets. Yet, with limited samples, avoiding experimental data misinterpretations requires considering not only the probabilities concerning separable states but also the probability that the signal came from an entangled state, i.e. the detection scheme's efficiency. We demonstrate this explicitly and apply a general method to optimize both the validity and the efficiency in small data sets providing examples using at most 20 state copies. The method is based on an analytical model of finite statistics effects on correlation functions which takes into account both a Frequentist as well as a Bayesian approach and is applicable to arbitrary entanglement witnesses.

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INTRODUCTION

Quantum entanglement has long been recognized as an important prerequisite of modern quantum technologies. Its detection is accordingly a well-studied topic with a plethora of different methods available. The field has evolved towards strategies directly applicable to experimental data which inevitably is limited to a finite number of detection events. If this number is large, various forms of entanglement witnesses^{1–3} provide reliable entanglement verification⁴. Interestingly, also the analysis of smaller data sets allows to detect entanglement⁵, most recently with quantum state verification methods^{6–8} and tailored game-like protocols in which particles are measured one by one^{9–12}. These methods are of high practical relevance in all cases where only a limited amount or only partial data is accessible, e.g. when an inefficient source has to be characterized quickly or one is interested in entanglement detection in large-scale quantum systems.

The finiteness of data sets used to derive a conclusion about whether the state is entangled or not inevitably leads to a probabilistic nature of this conclusion. Common certification schemes aim to ensure a small error if the outcome of the method speaks in favour of entanglement, i.e. a small error of a false positive. Yet, in the case of low statistics, this might come at the price of a large probability of rejecting an actually entangled state, i.e. a false negative. This tradeoff is illustrated in Fig. 1, which shows that for smaller statistics one is required to go to more and more strict criteria to reliably distinguish between results compatible with entangled states and results compatible with separable states. In consequence, due to the increased strictness, it becomes also more and more improbable that an entangled state will pass the test. It thus becomes crucial to properly apply the apparatus of statistical hypothesis testing^{13–15} to the scenario of entanglement certification in order to avoid significant

misinterpretations of conclusions based on small sample sizes and to properly optimize the usage of these scarce resources.

Here, we analyse the consequences of limited datasets for the application of entanglement witnesses based on correlation functions. As mentioned above, there are competing interests in choosing a certification strategy. The first is achieving a certain level of validity, i.e. trust in a positive result of the entanglement certification. The second interest is to ensure that the test can correctly assess as many input states as possible, which we call its efficiency. In general, different certification strategies will exhibit different efficiencies even if they achieve the same validity and vice versa. Thus, both quantifiers should be taken into account when choosing an optimal test strategy. We develop an algorithm for the optimization of experimental entanglement tests which simultaneously ensures high validity with optimal efficiency while being independent of the system's size.

Although the general framework of statistical analysis which is employed here is well-known to statisticians, it has not yet been comprehensively applied in the field of entanglement certification, where it becomes especially relevant in the context of limited datasets. The present work contributes to closing this interdisciplinary gap. The results are illustrated with various examples where different entanglement witnesses are applied to general classes of quantum states. While our examples all consider systems of qubits (the most commonly encountered systems in quantum information), our general analysis is by no means limited to two-dimensional systems and can be applied to any witness which is a function of generalized correlation tensor elements.

RESULTS

Entanglement certification as hypothesis testing

There are two main approaches to statistical hypothesis testing, the Frequentist and the Bayesian one (see e.g. ref. ¹³ for a

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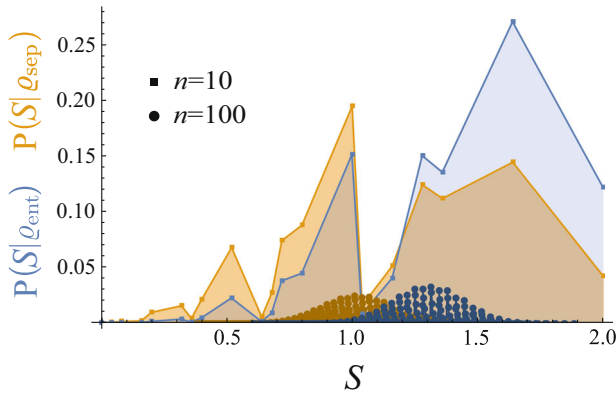


Fig. 1 Illustration of finite statistics effects on entanglement verification. The plot shows the probability distributions to obtain a specific value of a certain nonlinear entanglement witness S discussed in the “Methods” subsection “Certifying entanglement”. The witness uses only two correlation measurements, each estimated either with $n = 10$ (squares) or $n = 100$ (circles) copies of the states. The probability distribution given an entangled state is marked in blue. The probability in orange is computed for a separable state that saturates the value of the witness in the case of infinite data sets, i.e. a separable state that tends to yield large values of S . For larger statistics (see circles), the two distributions have small overlap making it easy to verify that an entangled state was prepared. For smaller statistics (see squares) the overlap is very significant and care has to be taken even if big values of the witness are observed. More detailed discussion on this issue is provided in the “Results” subsection “Limited significance of entanglement certification”. Here we develop universal tools, applicable to any entanglement witness, that extend it to the domain of finite statistics and can be employed to efficiently use every single copy of a state. For examples see the subsection “Optimizing entanglement certification strategies” in the “Results” section.

Frequentist and refs. 14,15 for a Bayesian perspective). These approaches employ different quantifiers to capture different figures of merit. In general, both are useful to statistically evaluate an entanglement certification scheme, but dependent on the context, such as e.g. the availability of prior information or the practical goal of certification, one of these frameworks might be more suited than the other. In the following, we will first express entanglement detection as a hypothesis test and subsequently show how it is statistically analysed from the Frequentist or Bayesian viewpoints.

Entanglement certification can be reduced to a so-called simple hypothesis test, where the parameter space contains just two elements corresponding to the two disjoint hypotheses ‘the state is separable’ and ‘the state is entangled’. We denote the two parameters as ‘sep’ and ‘ent’, respectively. The outcome space $\mathcal{O}$ is given by the set of all possible outcomes generated by a particular entanglement detection function under consideration. Depending on the chosen approach to statistical inference, the theory of hypothesis testing now allows to construct the optimal test to choose between the two hypotheses for any given outcome $Q \in \mathcal{O}$.

In non-randomized tests the outcome space is divided into two disjoint sets $\mathcal{O}_{\text{acc}}$ (acceptance set) and $\mathcal{O}_{\text{rej}} = \mathcal{O} \setminus \mathcal{O}_{\text{acc}}$ (rejection set) such that entanglement is certified whenever $Q \in \mathcal{O}_{\text{acc}}$. Note, that sometimes it can be optimal to perform a (partially) randomized test, where there exists a subset of outcomes for which the decision between the hypotheses is made randomly. For simplicity, we present our analysis in terms of non-randomized tests and note that randomized test can be treated equivalently. Furthermore, the usefulness of a randomized test is illustrated in the generalized scenario part of the Optimizing Entanglement Certification Strategies section.

In typical entanglement certification schemes the outcomes Q approximate theoretical quantities Q^∞ which are functions of the states that would be observed when the number of measurements tends to infinity. From this point on we use the superscript ∞ to indicate such ideal values. The inference of entanglement is usually justified by the fact that values in the acceptance set $\mathcal{O}_{\text{acc}}$ cannot be attained by any separable state, i.e.

$$P(Q^\infty \in \mathcal{O}_{\text{acc}} | \text{sep}) = 0. \quad (1)$$

If the experiment is performed with sufficient statistical data the experimental result Q approximates the theoretical Q^∞ very well and thus a result in $\mathcal{O}_{\text{acc}}$ entanglement can be certified essentially with certainty. Crucially, however, if the amount of empirical data is sufficiently small, the statistical fluctuations of the outcomes lead to a manifestly statistical nature of the conclusion and a proper analysis of the validity of the statistical inference becomes necessary.

Frequentist approach

In the Frequentist approach, the validity of the test is given by the level of confidence $q_F = 1 - \alpha$, that one or the other hypothesis is correct, with α being the level of significance. In our simple certification scenario, the significance α is the upper bound on the probability q_F with which the experiment will wrongly certify entanglement if the state is separable. As this is a statement about the average performance of the test, it does not allow any probabilistic assertions about the correctness of a particular single run of the procedure. For example, if we perform certification of entanglement with the values typically used for hypothesis testing, i.e. $q_F = 95\%$ ($\alpha = 5\%$), but are given separable states only, we would certify the state to be entangled on average in roughly 5% of the runs.

In a certification scenario, the level of significance only depends on the likelihood of type I errors (false positives). Thus, to achieve a certain validity q_F , the acceptance set has to be chosen such that the likelihood of yielding a value in the acceptance set given a separable state is at most $1 - q_F$, i.e.,

$$\sum_{Q \in \mathcal{O}_{\text{rej}}} P(Q | \text{sep}) \geq q_F. \quad (2)$$

The likelihood function for the opposite hypothesis $P(Q | \text{ent})$ is irrelevant as far as the level of significance is concerned.

Although the distinction between types of errors in Frequentist inference is well known, particularly in the field of entanglement certification with small data sets, often the focus lies exclusively on the level of significance^{5,9–11,16,17}. We want to argue, that in most practical applications, one is not only interested in a valid method but also in its ability to positively certify at least some entangled states. This tradeoff becomes particularly crucial in the case of very low statistics. Thus, hypothesis tests should also be optimized based on a second quantifier, the power of the test, which characterizes exactly this ability to provide a positive certification corresponding to minimizing e_2 , the error of type II (probability of false negatives). The power r is given by

$$r = 1 - e_2 = \sum_{Q \in \mathcal{O}_{\text{acc}}} P(Q | \text{ent}). \quad (3)$$

The concept of statistical power is essential for finding optimal tests and properly assessing the usefulness of statistical conclusions. See Fig. 2 for an illustration. Thus, in our terminology, the maximization of power corresponds to a maximization of the efficiency of the test.

Bayesian approach

In a Bayesian approach to certification, probabilities for both hypotheses can be calculated given a certain outcome, Q , i.e. $P(\text{ent} | Q)$ and $P(\text{sep} | Q) = 1 - P(\text{ent} | Q)$. Validity in this context is

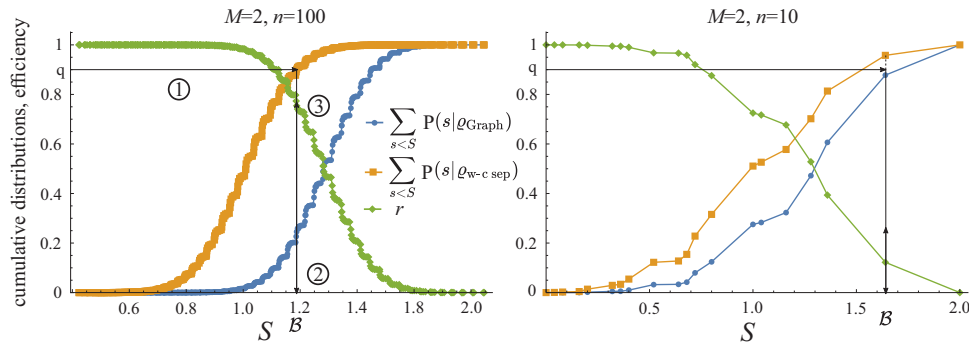


Fig. 2 Validity and power in the Frequentist approach applied to the nonlinear witness S . Given a required minimal, e.g., $q_F = 0.9$, an appropriate outcome space over the values of S (for definition see “Entanglement certification” in the Methods section) is chosen by specifying a bound B such that the separable correlations have at least q_F mass of its probability distribution below B , i.e., the cumulative probability distribution of the worst case separable correlations (orange data) reaches q_F (arrows marked with encircled 1 and 2). For this bound, one then evaluates the cumulative probability of the entangled state (blue data). The power, r , is defined as one minus this probability and is plotted in green. The power at the determined bound is marked by an arrow with encircled 3. The higher the power, the better the entanglement detection works, i.e., the higher the probability to actually detect the entangled state as such (see Fig. 1). Reducing the statistics from $n = 100$ (left panel) to $n = 10$ (right panel) lowers the power of the detection from about 75% to about only 12% if the same validity $q_F = 0.9$ (level of significance 10%) is required.

readily identified with the Bayesian level of acceptance q_B , which is the minimal value of the posterior probability for which the hypothesis ent is accepted. Entanglement is thus certified whenever $P(\text{ent}|Q) \geq q_B$. This means that the acceptance set $\mathcal{O}_{\text{acc}}$ has to be chosen such that

$$P(\text{ent}|Q) \geq q_B, \quad \text{for all } Q \in \mathcal{O}_{\text{acc}}, \quad (4)$$

i.e., whenever a value of $Q \in \mathcal{O}_{\text{acc}}$ is observed, the certification of entanglement is credible at least with degree q_B . Note, that as given by Bayes theorem, the posterior $P(\text{ent}|Q)$ is a function of both $P(Q|\text{sep})$ and $P(Q|\text{ent})$, as well as of the quotient of priors $P(\text{ent})/P(\text{sep})$.

To arrive at a formulation of Bayesian hypothesis testing in terms of validity and efficiency, one has to consider the connection between the level of acceptance and so-called loss functions. For a given outcome Q and weights β and $(1-\beta)$ for false positive and false negative respectively, the (posterior) loss function L_Q is given by

$$L_Q = \begin{cases} \beta P(\text{sep}|Q) & \text{if } Q \in \mathcal{O}_{\text{acc}} \\ (1-\beta) P(\text{ent}|Q) & \text{if } Q \in \mathcal{O}_{\text{rej}}, \end{cases} \quad (5)$$

i.e. the respective probability of error multiplied by the corresponding weight. It can be shown that for each Q this loss is optimized if the rejection and acceptance sets are chosen simply based on an acceptance level of $q_B = \beta^{14}$. Thus for optimal tests, we can simply identify the acceptance level q_B with the (relative) weight β given to the error of false positive.

A priori, i.e., without the knowledge of the outcome Q , an expected loss L can be defined as

$$L = \sum_Q L_Q P(Q) = q_B P(Q \in \mathcal{O}_{\text{acc}}|\text{sep})P(\text{sep}) + (1-q_B)P(Q \in \mathcal{O}_{\text{rej}}|\text{ent})P(\text{ent}), \quad (6)$$

which corresponds to a weighted sum of the probabilities for the errors of both types. This is also equivalent to the average loss if the procedure is applied multiple times. In analogy to the power for Frequentist tests, the average loss L allows to compare different Bayesian test strategies with equal acceptance levels. From a Bayesian perspective maximal efficiency of a test thus corresponds to minimal loss L .

Finally, let us also briefly mention why the relevant priors are not discussed in the usual detection of quantum entanglement, via witnesses, in the limit of many measurement results⁴. In such a case, when the number of state copies is large: $n \gg 1$, it is possible to choose acceptance sets $\mathcal{O}_{\text{acc}}$ for which it is practically

impossible for a separable state to exceed the bound of the witness, i.e., $P(Q \in \mathcal{O}_{\text{acc}}|\text{sep}) \approx 0$, as discussed above. Therefore, Bayes rule simplifies and the posterior becomes $P(\text{ent}|Q) \approx 1$ independently of the priors. In consequence, for any outcome in the acceptance set entanglement is certified with validities $q_F \approx q_B \approx 1$ both from the Frequentist and the Bayesian perspective and the distinction between the two approaches becomes superfluous.

Effects of finite statistics

The standard experimental scheme for entanglement certification in qubit systems is based on the measurement of correlation functions or simply correlations $T_{\mu_1 \dots \mu_N}$ defined by

$$T_{\mu_1 \dots \mu_N} = \langle R_{\mu_1} \dots R_{\mu_N} \rangle = \text{Tr}(\rho \sigma_{\mu_1} \otimes \dots \otimes \sigma_{\mu_N}), \quad (7)$$

where σ_{μ_k} is the μ_k th local Pauli matrix of the k th party with $\mu_k \in \{0, x, y, z\}$ and σ_0 being the identity. The correlation is thus the average of the product of local measurement outcomes $R_{\mu_k} = \pm 1$ when suitable local Pauli measurements are executed. Note that definition (7) does not only include the full correlations, for which all parties perform measurements but also marginal correlations, i.e. values of T to which only a subset of parties contributes, which formally correspond to setting the remaining indices $\mu_k = 0$. To simplify the notation in the following we write the correlations as T_j , where the index $j \in [1, 4^N]$ indicates the set of chosen measurement settings over all parties.

Consider a broad class of procedures for entanglement certification, in which a quantity Q^∞ , e.g., an entanglement witness, is calculated as a function of the correlations T_j . Note that by choosing Q^∞ tailored to witness various types of entanglement, e.g. genuine multipartite entanglement, one can employ the further discussed method for testing the statements about, e.g. genuine multipartite entanglement or biseparability. For the usual application of the conditions imposed by Q^∞ , the actual correlation functions are estimated with a finite number of experimental trials. We denote the actually measured correlations resulting from finite statistics by t_j .

Thus, instead of the ideal quantum mechanically predicted value Q^∞ based on T_j obtained in the limit of infinite statistics, experimentally we have only access to the modified value Q , calculated from t_j . We therefore begin with modelling t_j .

Let $\{n_j\}$ denote the number of state copies measured for each setting j , which can vary from setting to setting. For the j th

measurement setting, we have

$$\tau_j = \frac{n_j^+ - n_j^-}{n_j} \quad (8)$$

where $n_j^\pm$ are the numbers of trials in which the product of local results equals $R_{\mu_1} \cdots R_{\mu_N} = \pm 1$, with $n_j^+ + n_j^- = n_j$.

For large n_j we obtain $\tau_j \approx T_j$, but for small n_j the measured τ_j take potentially different values which are furthermore restricted to the discrete set $\tau_j \in \{-1, -1 + \frac{2}{n_j}, \dots, 1 - \frac{2}{n_j}, 1\}$. Given an N -qubit state, expressed by its correlations T_j , it is possible to explicitly calculate the ideal probability distribution $P(\tau_j)$ over these discrete values by employing the binomial distribution as demonstrated in Methods (Witnesses in finite statistics). The mean and variance of this distribution are given as

$$\langle \tau_j \rangle = T_j, \quad \text{Var}(\tau_j) = \sigma_j^2 / n_j, \quad (9)$$

where σ_j^2 is a function of the ideal quantum prediction equal to $\sigma_j^2 = 1 - T_j^2$, see also¹⁸. Note that the variance depends on the correlation value itself, e.g. a perfect correlation has no variance. This already hints that the method is most useful for states with high values of T_j^2 .

Considering a particular entanglement certification procedure with the outcome $Q = f(\tau_{j_1}, \dots, \tau_{j_M})$ (M being the number of correlation measurements), it becomes possible to calculate explicitly the probability distribution over the outcomes

$$P(Q) = \sum_{\substack{\tau_{j_1}, \dots, \tau_{j_M} \\ f(\tau_{j_1}, \dots, \tau_{j_M}) = Q}} \prod_{k=1}^M P(\tau_{j_k}). \quad (10)$$

In the “Methods”, subsection “Witnesses in finite statistics” section, we illustrate this procedure for both linear^{1–3} and non-linear entanglement witnesses^{17,19–24}. The linear witness $W = \sum_j a_j \sigma_j$ with expectation value $\text{Tr}(W\rho) = \sum_j a_j T_j$ denoted as E_W^∞ , extension to finite statistics has the following properties:

$$\langle E_W \rangle = E_W^\infty, \quad \text{and} \quad \text{Var}(E_W) = \sum_j a_j^2 \text{Var}(\tau_j). \quad (11)$$

For nonlinear witnesses, further work is required as one needs to calculate the effect of finite statistics on nonlinear functions of τ_j . We describe a general procedure in the “Methods” section and here briefly illustrate the case when the witness depends on the square of correlations. Its mean and variance read

$$\langle \tau_j^2 \rangle = T_j^2 + \text{Var}(\tau_j), \quad \text{Var}(\tau_j^2) = \frac{2(n_j - 1)(1 - T_j^2)[(2n_j - 3)T_j^2 + 1]}{n_j^3}. \quad (12)$$

Note that the mean value of the square has a consistent shift by exactly the variance of the estimation from the finite data. This is a consequence of taking the square—the negative values in the distribution are transformed into positive values leading to a systematic deviation. This fact is important in the experimental analysis of finite data sets and has already been addressed in ref.²⁵. As a final illustration assume the quadratic witness $S^\infty = \sum_j T_j^2$ that sums up M squared correlations (see the subsection “Certifying entanglement” in the “Methods” section), each measured with the same number of copies n , i.e. $n_j = n$. We obtain

$$\langle S \rangle = \frac{(n-1)S_M^\infty + M}{n}, \quad \text{with} \quad \text{Var}(S) = \sum_{i=1}^M \text{Var}(\tau_i^2), \quad (13)$$

where now apart from the statistical uncertainty also a shift of the mean value is present.

Limited significance of entanglement certification

The modelling of probability distributions for the outcomes of single correlation measurements now allows us to calculate explicitly the probability distribution over outcomes of the

entanglement witnesses Q . This calculation is always based on particular state spaces corresponding to the hypotheses ent and sep. Initially, for simplicity, we consider a simple scenario, where the test is supposed to distinguish between having a particular entangled state of the form

$$\rho = \frac{3}{4} |\text{GHZ}\rangle \langle \text{GHZ}| + \frac{1}{4} \mathbb{1}/2^N, \quad (14)$$

or an arbitrary separable state. Here, $|\text{GHZ}\rangle$ is an N -qubit GHZ state, for any N , and $\mathbb{1}/2^N$ corresponds to the so-called white noise with no correlations at all. Note that in realistic scenarios the expected distribution of input states might be more complicated as we also demonstrate below. The probability distributions corresponding to the sep hypothesis are modelled based on a worst-case estimation of correlations compatible with separability as discussed in detail in the “Methods” section (see the subsections “Estimating probability distributions” and “Correlations compatible with separability”).

Now consider the linear witness $W = \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \cdots \otimes \sigma_x - \sigma_x \otimes \cdots \otimes \sigma_x + 1$ with only $n = 10$ clicks per single correlation measurement, i.e., 20 state copies in total. Due to this very small number of data points, separable states can now give rise to outcomes in the acceptance set, i.e., $P(E_W < 0 | \text{sep}) > 0$, where we follow the convention that separable states admit $E_W^\infty \geq 0$ in the case of infinite statistics. In fact, in this example, the probability that the witness is negative on a suitable separable state is as large as 41.5%. The straightforward solution seems to be to make the boundary more strict, i.e. require sufficiently negative values to certify entanglement. However, even with the strictest nontrivial bound of $-4/5$ we find that the violation by a separable state is still equal to 2.5%, i.e. $P(E_W \leq -4/5 | \text{sep}) = 2.5\%$. At the same time, the state ρ gives rise to values $E_W < -4/5$ with probability of 26.7%. While it is clearly more probable for the entangled state to violate the bound than for any separable state, still, given the finite values of both probabilities, the question remains what the appropriate probabilistic conclusion about the quality of such certification should be.

The nonlinear criterion $S = \langle \sigma_y \otimes \sigma_y \otimes \sigma_x \otimes \cdots \otimes \sigma_x \rangle^2 + \langle \sigma_x \otimes \cdots \otimes \sigma_x \rangle^2$ has been illustrated in Fig. 1. Again we consider the sum of only two squared correlations, each estimated in 10 experimental trials. Our analytical tools return the plotted $P(S | \text{ent})$ based on the above-mentioned entangled state ρ . Figure 1 clearly illustrates that finite statistics leads to the violation of the bound of 1 by a separable state. Again the immediate solution seems to be making the bound more strict. We find that $P(S = 2 | \text{sep}) = 4.2\%$, but at the same time $P(S = 2 | \rho) = 6.9\%$. With this bound, the procedure would wrongly certify entanglement (false positive) in at most only 4.2% of the cases, but at the same time, it would fail to correctly certify it (false negative) in roughly 93% of the cases where the state was entangled. Furthermore, for any single experiment yielding value $S = 2$, we are clearly not justified to conclude that the state is entangled, as a separable state might have given rise to the same value of S with a comparable probability.

This type of explicit calculation can be applied to any witness based on correlation values to evaluate whether it can be applied efficiently in the case of finite statistics. In particular, it becomes necessary to put $P(Q \in \mathcal{O}_{\text{acc}} | \text{sep})$ in relation to the probability $P(Q \in \mathcal{O}_{\text{acc}} | \text{ent})$ via a proper statistical analysis, either Frequentist or Bayesian.

Resource optimization

Based on the possibility of quantitatively evaluating any entanglement certification procedure in terms of validity and efficiency, we now tackle the question of how to best use a small number n of available copies of a multi-qubit quantum state. The experimental choice to be made is about the amount of settings M

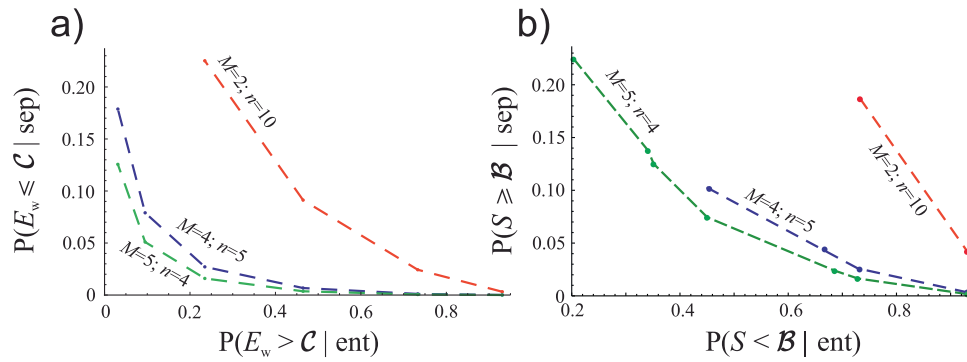


Fig. 3 Optimal use of 20 copies. Plot for ρ in Eq. (15) with $p = 3/4$, e.g. a GHZ state mixed with white noise. Panel (a) is for the linear witness, panel (b) for the quadratic witness. The discreteness of the outcome space for just 20 copies is clearly visible in the plot. To maximize validity and efficiency both probabilities need to be simultaneously minimized. This is best realized by dividing the 20 copies into 5 correlation measurements, each correlation estimated in 4 experimental runs.

and the distribution of the copies among those $\{n_j\}$, where $\sum_j n_j = \eta$. Furthermore, a suitable outcome space $\mathcal{O}_{\text{acc}}$ has to be chosen. By explicitly calculating the distributions $P(Q|\text{sep})$ and $P(Q|\text{ent})$, and assuming a certain prior the validity and efficiency of the procedure can be characterized, no matter if a Frequentist or Bayesian approach is chosen. In practice, a reasonable scenario could be to require a certain minimal validity and then choose a combination of experimental parameters with a suitable acceptance space, such as to maximize efficiency. This yields the following optimization problem:

Given a certain minimal validity $q_{\min}$ and a total number η of state copies, what set of M observables, each estimated on $\{n_j\}$ copies of the state, and what acceptance set $\mathcal{O}_{\text{acc}}$ are to be chosen such that an observation of $Q \in \mathcal{O}_{\text{acc}}$ certifies entanglement with a validity $q \geq q_{\min}$, while keeping the efficiency as high as possible.

Note that when Q is a function of marginal correlations corresponding to commuting operators it can be possible to measure them simultaneously from a single global measurement setting. This is the approach chosen in ref. ⁹, where a single state copy is used to obtain values for several marginal correlations providing the outcome for a linear witness. Thus, in general, given a function Q also of marginal correlations it is of high interest to group these into as few measurement settings as possible. Since our example witnesses S and W are solely based on full correlations this particular layer of optimization is not relevant for them.

Optimizing entanglement certification strategies

Scenario: First of all, let us emphasise that the proper statistical analysis is especially crucial in the case of a very limited data set composed of at most a few tens of data points. In the examples below we assume that no more than 20 copies of multi-particle state from a possibly entangled source have been successfully measured. The small amount of data makes it practically impossible to distinguish entangled states which are close to separable ones. Thus, the best candidates are states which are strongly non-classically correlated. Generalizing Eq. (14), here, we therefore consider mixed states of the form

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\rho_n, \quad (15)$$

where a dominant admixture comes from a pure state $|\psi\rangle$, assumed to admit perfect correlations, and the remaining part, ρ_n , describes experimental noise. Since our theoretical analysis just depends on the number of correlations (measurement settings) it is thus independent of the number of involved particles. See ref. ²⁶ for fidelity estimation with the same feature.

Accordingly, we will not specify the state $|\psi\rangle$ exactly as the only parameters that enter the witnesses are correlations. We would like to emphasise this point, the methods work for any GHZ states, graph states, cluster states, absolutely maximally entangled states etc. Although in practice the perfect correlations of $|\psi\rangle$ will be reduced by various types of noise, which we consider via the addition of ρ_n , we show in the “Methods” subsection “Effects of different types of noise” that the probability to observe a certain measurement outcome depends almost entirely on the parameter p , and not a type of noise as long as it is uniform and unbiased. Clearly, it is possible to introduce a specific noise that results in different success probability characteristics. In such a case one can still apply our methods, but here we focus on random noises as they are the usual models of experimental imperfections. Thus, we restrict our examples to the case of white noise, i.e. $\rho_n = \mathbb{1}/2^N$. In effect, the model (15) encompasses a plethora of experimental situations that are of interest in quantum information.

Consider a certification scenario where we expect that the tested state is either from the family (15) or a separable state modelled according to the worst-case correlations compatible with separability (see Estimating Probability Distributions and Correlations Compatible with Separability in the Methods), with $P(\text{sep}) = P(\text{ent}) = 1/2$. We assume 20 available state copies and require a target validity of $q_F = q_B = 97.5\%$. This will help us to compare and highlight the differences in reasoning that occur from Frequentist and Bayesian perspectives on statistical inference. According to our optimization scheme, we want to determine the set of measurements, i.e., their amount M and in general also the measurement directions, together with an optimal distribution of clicks per setting $\{n_j\}$. For simplicity here, we restrict the presentation to cases where we assume that all n_j 's are the same and equal to n , and hence we constrain the product $Mn = 20$. Let us first set $p = 3/4$ and specify the Bayesian loss function as Eq. (31).

Linear witness. To illustrate how to optimally distribute the 20 copies for the linear witness W , as an example we choose a procedure where the acceptance set is chosen via an upper bound C . We extend the linear witness to M different settings as $E_W^\infty = T_1 - T_2 - \dots - T_M + 1$. It turns out that the worst case separable state, i.e. the state which maximizes the probabilities in the acceptance set, for a sufficiently low bound C has the correlations $T_1 = -1/M$ and $T_i = 1/M$ for $i = 2, \dots, M$. In Fig. 3a we plot $P(E_W \leq C|\text{sep})$ and $P(E_W > C|\text{ent})$, which are directly related to validity and efficiency. As the simultaneous optimization of the two quantifiers is based on keeping both probabilities as small as possible, clearly the optimal procedure is realized for $M = 5$ and $n = 4$.

From the Frequentist perspective to achieve the minimally required validity of 97.5%, a bound of $C_F = -2.5$ has to be chosen, which leads to an efficiency of $r \approx 76.5\%$. The highest validity achievable with 20 state copies is 99.996% with a bound of $C_F = -4$. However, it comes at the cost of efficiency of just $r \approx 6.9\%$.

The Bayesian framework requires $P(\text{ent}|E_W) \geq 97.5\%$ for all values in the acceptance set. This is satisfied if the bound C_B is chosen as -3 . The associated efficiency, in this case, is given by the loss function of $L \approx 0.007$.

From the above results, one can see a clear difference in the choice of the acceptance sets for each reasoning. Namely, the Frequentist bound is less strict than the Bayesian counterpart, implying instances in which experimenters would disagree on the conclusions about the entanglement certification. For comparison, from the Frequentist perspective, the adjusted bound of $C_B = -3$ that is optimal for the Bayesian approach results in the level of confidence given by 99.64% and the corresponding power of $r \approx 53.5\%$. As expected, for the lower bound confidence is increased and power reduced, however, we emphasise that in general the Bayesian approach does not always yield stricter acceptance sets.

Quadratic witness. Just as for the case of the linear witness, we introduce a bound B to define the acceptance set also in the example of the quadratic witness. Figure 3b presents the two relevant quantities $P(S \geq B|\text{sep})$ and $P(S < B|\text{ent})$. Again, the best use of copies is given by $n = 4$ and $M = 5$.

Based on the minimally required validity of 97.5%, we calculate the bound B for the Frequentist approach to be $B_F = 4$ and in the Bayesian approach we get $B_B = 5$. This leads to an efficiency of $r \approx 31.4\%$ and $L \approx 0.013$ respectively. Recall that the chosen priors for the Bayesian inference are $P(\text{ent}) = P(\text{sep}) = 0.5$. In general, the examined state is entangled iff $p > 1/(2^{N-1} + 1)^{27}$. Thus, given our state model, a more realistic scenario would take into account the cardinality of the separable states within the family (15) in the prior probabilities. For example, with a uniform distribution over p and $N = 4$, the natural prior is $P(\text{ent}) = 8/9$ and $P(\text{sep}) = 1/9$. This leads to the change of the acceptance set with $B_B = 4$ and efficiency $L \approx 0.018$. With increasing N the efficiency in the Bayesian approach is changing. As the last example, for $N = 5$ the prior is $P(\text{ent}) = 16/17$ and $P(\text{sep}) = 1/17$, the Bayesian bound is given by $B_B = 3.5$ and the corresponding efficiency reads $L \approx 0.015$.

Again, for equal priors Frequentist reasoning yields a greater acceptance set. However, here one can explicitly see that for more physically motivated priors this is no longer true and C_B can be smaller than C_F . For all of the discussed cases, confidence and power are ($B = 5$, $q_F = 99.8\%$, $r = 6.9\%$), ($B = 4$, $q_F = 97.6\%$, $r = 31.4\%$) and ($B = 3.5$, $q_F = 92.5\%$, $r = 54.9\%$) respectively. Note that the last level of confidence does not meet the required validity condition.

Generalized scenario: Finally, in a more general example, we relax the condition of an equal amount of copies per setting and apply our optimization algorithm to determine the optimal amount of settings and distribution of state copies over these settings, such as to achieve optimal efficiency for certification with the quadratic witness S . It turns out that different certification strategies are found to be optimal, depending on whether a Frequentist or Bayesian approach is chosen. We calculate the distribution $P(p|\text{ent})$ based on the model of a family of noisy three-qubit GHZ states (14), with a mean $\bar{p} = 0.8$ and a standard deviation $\Delta p = 0.1$, such that for the entangled states

$$P(p) = \begin{cases} \mathcal{N}e^{-\frac{(p-\bar{p})^2}{2(\Delta p)^2}} & p \in [p_{\min}, 1] \\ 0 & \text{else,} \end{cases} \quad (16)$$

with proper normalization $\mathcal{N}$ and the lower bound $p_{\min} = 1/5$ given by the requirement that the states have to be entangled, see the quadratic witness example in the previous subsection. We set the maximal number of settings to $M = 3$, allow at most 13 state copies and require a minimal validity of 70%. Furthermore, we again choose not to model the separable state space and instead estimate $P(S|\text{sep})$ via the worst-case optimization used above. In this example for the prior, we assume that it is twice as likely to encounter an entangled state from our family than any separable state.

If the optimization is based on a Frequentist understanding of validity, i.e., achieving at least 70% confidence, it is optimal to use 12 of the 13 possible measurement runs and distribute them equally among the three settings as $n_1 = 4$, $n_2 = 4$ and $n_3 = 4$ (denoted as [444] for brevity). Figure 4a illustrates the resulting probability distribution $P(S|\text{ent})$ and the worst case distribution $P(S|\text{sep})$ obtained via the discussed optimization, where the best acceptance set turns out to be $\mathcal{O}_{\text{acc}} = \{0, 1, 2.25, 3\}$. The probability $P(S \in \mathcal{O}_{\text{acc}}|\text{sep})$ is only 29.8% and thus the confidence remains above 70% while at the same time the power $r = P(S \in \mathcal{O}_{\text{acc}}|\text{ent})$ is maximized to 66.4%. Estimating also the Bayesian loss for this acceptance interval yields a value of 0.137. For comparison, Fig. 4b illustrates the estimated posterior distribution. From the Bayesian perspective, the acceptance set has to be chosen as $\{2.25, 3\}$ which leads to a minimal level of acceptance of 76.8% and a loss of 0.149. The corresponding power would be 65.6%.

This example illustrates two seemingly peculiar consequences of the need to use a worst-case estimation. The first concerns the benefit of using a randomized test. To maximize the power of the test, the outcomes 0 and 1 are included in the acceptance set. As seen in Fig. 4 these are not valuable points since they have a significantly higher likelihood given a separable state than an entangled state. In principle, it would be much more beneficial to add a point with a better likelihood ratio, but as the optimization shows in this case the confidence would drop below 70%. This problem is resolved by the Neymark–Pearson lemma¹³, which proves that the optimal strategy consists of simply choosing the points with the best likelihood ratios (after corresponding optimization) and make a probabilistic conclusion whenever results with the lowest ratio are obtained. In this manner, the confidence level of 70% could be reached exactly while maximizing the power of the test even further. Second, the loss of the interval $\{0, 1, 2.25, 3\}$ is estimated to be smaller than the loss of the interval $\{2.25, 3.0\}$, while still the latter is chosen as the Bayesian optimum. This discrepancy is a consequence of the fact that different worst-case estimates are chosen to find the acceptance sets and to calculate the loss, as explained further in the “Methods” subsection “Estimating the probability distributions”. These loss values are only used to compare the Bayesian performance given different distributions of measurement settings.

Figure 4d shows the posterior distribution for the optimal case according to the Bayesian approach, i.e. achieving at least 70% level of acceptance with as little expected loss as possible. It turns out that in this case, it is better to use only 11 of the 13 possible measurements and distribute them as [533] among the three settings. The requirement of $q_B \geq 70\%$ leads to the choice of acceptance set $\mathcal{O}_{\text{acc}} = \{2.36, 3\}$ leading to a loss of 0.146, which is smaller than the value of 0.149 obtained with the distribution [444]. The corresponding power would be 50.4%. Conversely, as illustrated in Fig. 4c, with [444] only a power of 53.5% with a level of confidence of 71.1% (and a loss of 0.1603) can be reached from the point of view of the Frequentist approach making that choice suboptimal as compared with the best choice presented above. Note, that also here a randomized test would yield better results.

It might seem strange that in both cases a smaller than maximal number of copies is deemed optimal, but this is mainly a consequence of the strong discreteness of the outcome space. Indeed the maximal number of 13 measurements might provide

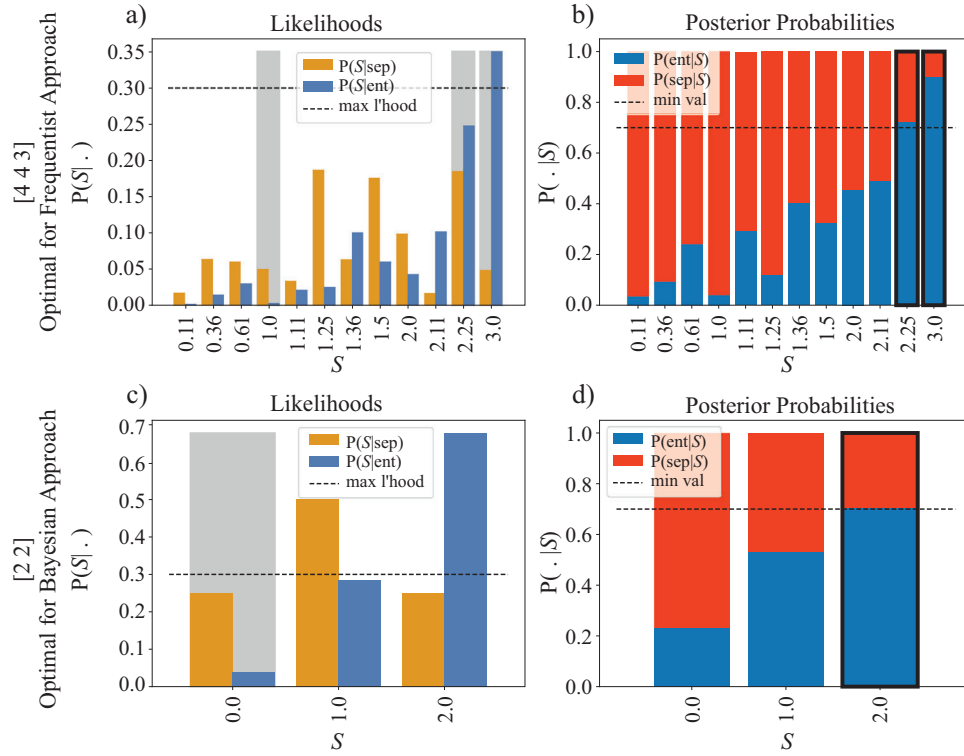


Fig. 4 Optimal choices of distributions of states for Bayesian and Frequentist figures of merit. Panel **a** shows the estimated distributions $P(S|ent)$ (blue bars) and $P(S|sep)$ (orange bars) for the choice [444] which is optimal from the Frequentist point of view. Note that the shown $P(S|sep)$ is actually the worst-case estimate $G_{acc}(Q)$, optimized for one particular acceptance set. The acceptance set $\mathcal{O}_{acc} = \{0, 1, 2.25, 3\}$ (marked by the grey background) is chosen by maximizing $P(S \in \mathcal{O}_{acc}|ent)$, i.e., includes as much as possible of blue bar length while not allowing $P(S \in \mathcal{O}_{acc}|sep)$ (estimated by $G_{acc}(S \in \mathcal{O}_{acc}|sep)$) to exceed $1 - 70\% = 30\%$ (dashed black line), i.e., not getting too much orange bar length into the acceptance set. A power of 66.4% is reached. Panel **b** shows the posterior probability $P(ent|S)$ for that choice of experiment. The acceptance set (marked by the black boxes) is given by all values of S for which $P(ent|S) = 1 - P(sep|S) \geq 70\%$ (dashed horizontal line) leading to $\mathcal{O}_{acc} = \{2.25, 3\}$. Note, that the posterior is based on the point-wise worst-case optimization G_S which is different from the one employed in panel **a**, see Estimating Probability Distributions in the Methods. Panels **c** and **d** show the respective distributions for the choice [533] optimal from the Bayesian point of view. With the acceptance set $\mathcal{O}_{acc} = \{2.36, 3\}$ for the Bayesian analysis, it leads to a smaller estimated loss. On the other hand, the optimal Frequentist power of 53.5% with the interval $\{0.58, 2.36, 3\}$ is significantly lower than for the first testing strategy. Note, for example in panel **(c)** it might appear that choosing, say the point 1.22 instead of 0.58 might be more beneficial for the Frequentist. However, it needs to be taken into account that the shown distribution is obtained from a worst-case optimization for $\mathcal{O}_{acc} = \{0.58, 2.36, 3\}$ and that with appropriate parameters $G_{acc}(S \in \{1.22, 2.36, 3\}|sep) \geq 30\%$ can be achieved.

maximal information about the state. However, the ability to harness this information within the given bounds on the levels of confidence and acceptance is restricted by the need to choose the acceptance and rejection sets from a small number of outcomes. It can thus happen that for a smaller number of measurements, the probabilities distribute more favourably such that the relevant quantifiers can reach better values.

Finally, it is important to keep in mind that although we subsume the level of confidence and level of acceptance by the term validity, they quantify different figures of merit which correspond to different practical interests. While the Frequentist is only interested in the average correctness of the conclusion of the certification method, the Bayesian wants to quantify the correctness of the conclusion on a case-by-case basis (and becomes dependent on the prior). Note, for example, that with the distribution [444] although it is much more probable for a separable state to yield a value 1 than for an entangled state (see Fig. 4a, given that result, the Frequentist would still certify entanglement as this decision increases the overall power without violating the significance level. On the other hand, if, as done here, a worst-case optimization of correlations compatible with separability is used, the Frequentist approach requires prior assumptions only to estimate the power (by modelling the space of entangled states) and can provide an assertion of the confidence which is completely independent of all priors.

DISCUSSION

In entanglement certification, it is usually of interest to achieve a sufficiently high validity, i.e. be sufficiently convinced that the certification does not falsely conclude entanglement. Here we showed explicitly that for small data sets it is necessary to consider not only $P(Q|sep)$, i.e. the probability of a separable state to yield particular values of the witness Q , but also $P(Q|ent)$ of entangled states. Such proper design of an experiment is necessary if we are to make robust conclusions in the presence of finite statistics and allows us to define and evaluate the validity and efficiency of the method. Only with these two quantifiers, it becomes possible to optimize the usage of the available experimental resources.

We have introduced a universal procedure that allows the extension of any entanglement witness to the domain of finite statistics. It is based on an analytical calculation of probability distributions over possible outcomes when estimating correlation functions experimentally with finite resources. Based on these probability distributions we recalled the measures of validity and efficiency, applicable to any entanglement detection scheme, from the points of view of both Frequentist and Bayesian interpretation. Illustrative examples were provided based on linear as well as nonlinear witnesses and broad families of states.

The methods introduced here are directly applicable to raw data and should be especially helpful for a resource-efficient estimation of the performance of a known apparatus subject to

variations of external parameters or in multipartite experiments with rare detection events, e.g., multi-photon setups based on coincidence clicks. For example, our scheme could be employed to quickly certify the quality of a large quantum processor before a time-consuming computation task.

METHODS

Certifying entanglement

Linear witnesses W admit the following expansion in terms of the Pauli basis

$$W = \sum_j a_j \sigma_j. \quad (17)$$

The expectation value $E_W \equiv \langle W \rangle$ given a particular state ρ , is then a linear function of the correlations T_j given by

$$E_W = \text{Tr}(W\rho) = \sum_j a_j T_j. \quad (18)$$

Measuring each correlation and computing the expectation value of the witness function provides information about the separability of the state. Constructions of such functions are tailored to a specific entangled state for which $E_W < 0$, whereas the expectation value on separable states is never negative. For example, a linear witness detecting entanglement of a three-qubit GHZ state could be chosen as

$$W = \sigma_y \otimes \sigma_y \otimes \sigma_x - \sigma_x \otimes \sigma_x \otimes \sigma_x + 1, \quad (19)$$

where the constant ensures that the witness is non-negative on separable states and negative on the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$, written in the standard σ_z basis.

Entanglement in state ρ can also be revealed via nonlinear witnesses. As an example we consider the scheme introduced in ref. ¹⁹. Consider the so-called correlation length S defined by

$$S \equiv \sum_{j=1}^{3^N} T_j^2, \quad (20)$$

which is the sum of all *full* correlations squared. Accordingly, the sum goes only up to 3^N (the marginal correlations are not included). It has been proven^{17,20–24} that a value of $S > 1$ certifies entanglement. Since S sums up only non-negative terms, entanglement is verified as soon as a subset of M correlations violates the bound, i.e., $S_M > 1$. We omit the subscript whenever the number of measured settings is clear.

Witnesses in finite statistics

Since the product of qubit measurement results is either equal to $+1$ or -1 , the probability that in n trials we obtain n_+ products equal to $+1$ is given by the binomial distribution:

$$p_i(n_+) = \binom{n}{n_+} \left(\frac{1+T_i}{2} \right)^{n_+} \left(\frac{1-T_i}{2} \right)^{n-n_+}, \quad (21)$$

where $(1 \pm T_i)/2$ is the probability to obtain the product equal to ± 1 in a single trial, estimated from the ideal quantum mechanical prediction. All random variables are assumed to be independent and identically distributed. Using Eq. (8) the correlation T_i is also binomially distributed as

$$P(\tau_i) = p_i\left(\frac{n}{2}(1+T_i)\right). \quad (22)$$

Note that due to the finiteness of n also τ_i can take on a finite set of values $\{-1, -1 + \frac{2}{n}, \dots, 1 - \frac{2}{n}, 1\}$.

In the case of nonlinear witnesses, these correlations have to be further processed. We illustrate this process using the quadratic witness S since it can be easily generalized to any power. In order to write the probability distribution of different values of S we first

note that

$$P(\tau_i^2) = P(\tau_i) + P(-\tau_i) \quad \text{for } \tau_i \neq 0, \quad (23)$$

because both values $\pm \tau_i$ give the same square. In the special case of $\tau_i = 0$, the probability $P(\tau_i^2)$ is just given by $P(\tau_i = 0)$. Since S is defined as the sum of squares, a particular value of S can be realized in many ways, e.g. the value $S = 1$ can be achieved if any one $\tau_i^2 = 1$ and all the other squared correlations are zero, or when all $\tau_i^2 = 1/M$, etc. The same reasoning applies to the linear case. Assuming that all the squared correlations τ_i^2 are independently distributed, the values of S satisfy:

$$P(S) = \sum_{\tau_1^2 + \dots + \tau_M^2 = S} P(\tau_1^2) \dots P(\tau_M^2). \quad (24)$$

These distributions were used in the “Results” section to derive mean values and variances of the introduced quantities.

Estimating the probability distributions

For a simple hypothesis test, determining validity and efficiency requires the calculation of the two likelihoods $P(Q|\text{sep})$ and $P(Q|\text{ent})$. In practice, however, the parameter space is more complicated and one has to consider the full subspaces $\mathcal{H}_{\text{sep}}$ and $\mathcal{H}_{\text{ent}}$ of separable and entangled states. These spaces would be modelled via distributions $P(\rho_j|\text{sep})$ and $P(\rho_j|\text{ent})$ of separable and entangled states ρ_j and each of those states would give rise to certain likelihood of outcomes $P(Q|\rho_j)$. Although one option would be to expand the discussion towards so-called composite hypothesis tests, the simple scenario can be recovered by reducing these state spaces to single parameters (‘sep’ and ‘ent’). This is done by considering the average distributions of outcomes given a separable or entangled state with

$$P(Q|\text{sep}) = \sum_j P(Q|\rho_j)P(\rho_j|\text{sep}), \quad (25)$$

$$P(Q|\text{ent}) = \sum_j P(Q|\rho_j)P(\rho_j|\text{ent}). \quad (26)$$

Unfortunately, evaluating such expressions is notoriously daunting²⁸.

In some cases, it is possible to avoid the modelling of the state space and instead consider a worst-case distribution of values Q . For $P(Q|\text{sep})$ this distribution can either be calculated point-wise for each Q , denoted as G_Q , or for the whole acceptance interval $\mathcal{O}_{\text{acc}}$, denoted as G_{acc} . These distributions have to satisfy

$$\forall \rho \in \mathcal{H}_{\text{sep}} \quad \forall Q \in \mathcal{O}_{\text{acc}} \quad G_Q(Q) \geq P(Q|\rho) \quad \text{and} \quad (27)$$

$$\forall \rho \in \mathcal{H}_{\text{sep}} \quad G_{\text{acc}}(Q \in \mathcal{O}_{\text{acc}}) \geq P(Q \in \mathcal{O}_{\text{acc}}|\rho), \quad (28)$$

where G_Q is stronger in the sense that it satisfies the condition for G_{acc} as well, i.e. $G_{\text{acc}}(Q \in \mathcal{O}_{\text{acc}}) \leq \sum_{Q \in \mathcal{O}_{\text{acc}}} G_Q(Q)$. The strategy to find the upper bounds is based on finding a set of M correlations T_j corresponding to $\bar{\rho}$ such that $P(Q|\bar{\rho})$ satisfies the criterion in question. Note that $\bar{\rho}$ might not even be a proper density matrix which is why we refer to such correlations as compatible with separability. Furthermore, in the case of G_Q the value at each Q can correspond to a different $\bar{\rho}$.

G_{acc} can be used straightforwardly to choose Frequentist acceptance sets by modifying condition (2) to

$$\sum_{Q \in \mathcal{O}_{\text{acc}}} P(Q|\text{sep}) \leq \sum_{Q \in \mathcal{O}_{\text{acc}}} G_{\text{acc}}(Q|\text{sep}) \leq 1 - q_f. \quad (29)$$

If the criterion is satisfied for G_{acc} it will also be satisfied for the actual distribution $P(Q|\text{sep})$. Thus, the estimation can only lead to a stricter choice of the acceptance set and ensures that the minimal level of significance is attained.

In the Bayesian case for each outcome Q the minimal validity q_B is assured as in Eq. (4), by providing a lower bound on $P(\text{ent}|Q)$

using G_Q as

$$P(\text{ent}|Q) = \frac{P(Q|\text{ent})P(\text{ent})}{P(Q|\text{sep})P(\text{sep}) + P(Q|\text{ent})P(\text{ent})} \geq \frac{P(Q|\text{ent})P(\text{ent})}{G_Q(Q)P(\text{sep}) + P(Q|\text{ent})P(\text{ent})} \geq q_B. \quad (30)$$

To estimate the expected loss, however, it is sufficient to employ the weaker worst-case estimate G_{acc} since the scenario is considered a priori, i.e. before a particular outcome Q is known. It is thus sufficient to only maximize the probability in the whole interval $\mathcal{O}_{\text{acc}}$, independently of how $G_{\text{acc}}(Q)$ is related to $P(Q|\text{sep})$ point-wise. The loss (6) is thus estimated as

$$\begin{aligned} L &= q_B P(Q \in \mathcal{O}_{\text{acc}}|\text{sep})P(\text{sep}) + (1 - q_B)P(Q \in \mathcal{O}_{\text{rej}}|\text{ent})P(\text{ent}) \\ &\leq q_B G_{\text{acc}}(Q \in \mathcal{O}_{\text{acc}})P(\text{sep}) + (1 - q_B)P(Q \in \mathcal{O}_{\text{rej}}|\text{ent})P(\text{ent}). \end{aligned} \quad (31)$$

Since different worst-case estimates are employed, i.e. G_Q to find the acceptance interval and G_{acc} to estimate the loss, it is no longer the case that the optimal choice of acceptance intervals minimizes the loss. However, the worst-case estimate of the loss with G_{acc} is still useful to compare scenarios with different numbers of copies and distributions of copies over settings, for each of which G_Q has been used as a rule to find the acceptance sets.

In analogy to $P(Q|\text{sep})$ in principle, both the Frequentist and the Bayesian approach could proceed given a lower bound on $P(Q|\text{ent})$. However, this is completely unfeasible for several reasons. For example, for the non-linear witness S : (i) entangled states can easily yield very small values of S due to statistical fluctuations, (ii) some entangled states have small (or even vanishing) S^∞ , (iii) there are bad choices of measurement settings, e.g. ones that give S^∞ close to 1. Thus, $P(S|\text{ent})$ is obtained by modelling a certain state distribution $P(\rho_j|\text{ent})$, which fits the currently employed apparatus for state generation, as illustrated in Results (Optimizing Entanglement Certification Strategies). Similar problems appear for linear witnesses. Note, that the probability distributions $P(\rho_j|\text{sep})$ and $P(\rho_j|\text{ent})$ do not describe a statistical uncertainty of the form encountered in mixed states. Instead, they capture prior expectations as to what separable or entangled states are to be submitted to the certification procedure. In other words, due to nonlinearity of the function $P(Q|\rho_j)$ it does not hold that $P(Q|\text{sep}) = P(Q|\bar{\rho}_{\text{sep}})$ with a mean separable state $\bar{\rho}_{\text{sep}} = \sum_j \rho_j P(\rho_j|\text{sep})$ and similarly for $P(Q|\text{ent})$.

Correlations compatible with separability

We have written an optimization algorithm in order to find the set of correlation functions that maximize the probabilities in the acceptance sets while being compatible with separability, i.e. the correlations do not violate the witness in the limit of infinite sample size. It verifies that for equal distribution of copies over the settings, i.e. each $n_i = n$ for each $i = 1, \dots, M$, and sufficiently high bound on the linear witness $\mathcal{C}$ ($\mathcal{B}$ for the non-linear witness) all of the correlations can be taken as $T_i = 1/M$ ($T_i^2 = 1/M$). The algorithm can be also applied to a model probability distribution over Q for separable states and arbitrary distributions of copies as well as more complicated disjoint acceptance sets (see the “Results” subsection “Optimizing entanglement certification strategies”).

The basic algorithm works as follows. We compute $P(Q)$ and the cumulative probability function. They are parameterized with to-be-determined correlations $\{T_1, T_2, \dots, T_M\}$ as well as the bound defining the acceptance set. The heart of the computation is the numerical maximization of $\sum_{Q \in \mathcal{O}_{\text{acc}}} P(Q)(P(Q)\forall_{Q \in \mathcal{O}_{\text{acc}}})$, for the fixed value of bound, with respect to $\{T_1, T_2, \dots, T_M\}$. We constrain the set of possible correlations to those that yield $W^\infty \geq 0$ for the linear and $S^\infty \leq 1$ for the non-linear witness, with each $T_i \in [-1, 1]$. Two widely available out-of-the-box methods were used for finding the

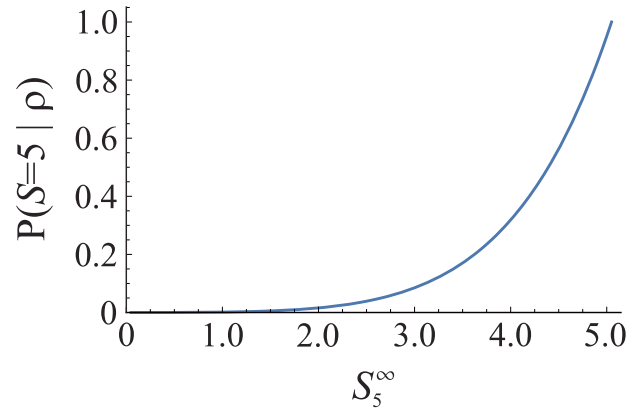


Fig. 5 The efficiency of entanglement detection for different noises ρ_n and their admixtures p in Eq. (15). The plot shows the probability of measuring $S = 5$ given p (vertical axis) as a function of S_5^∞ (horizontal axis). The plotted curve assumes white noise and is given analytically as $[(1 + 6p^2 + p^4)/8]^5$. The data for the other types of noise, i.e., pure and mixed noises according to Eqs. (33) and (34), overlap with the presented one. Thus, different types of unbiased noises are not relevant to the considered criteria.

maximum: simulated annealing and Nelder-Mead algorithm^{33,34}. Both gave similar results with the Nelder-Mead method being more precise in the considered problem. The output of the algorithm is a list of $\{T_1, T_2, \dots, T_M\}$ that maximizes $\sum_{Q \in \mathcal{O}_{\text{acc}}} P(Q)(P(Q)\forall_{Q \in \mathcal{O}_{\text{acc}}})$ for the given bound, or any other acceptance set, under the stated constraints.

Effects of different types of noise

We analyse here the effects due to different forms of noise in Eq. (15) on the efficiency of the entanglement certification. In particular, we consider the following unbiased noises

$$\rho_{\text{wn}} = \mathbb{1}/2^N, \quad \text{white noise} \quad (32)$$

$$\rho_{\text{rp}} = |\psi_r\rangle\langle\psi_r|, \quad \text{Haar random pure state noise} \quad (33)$$

$$\rho_{\text{wn}} = \rho_r, \quad \text{random mixed states}, \quad (34)$$

where Haar randomness refers to the sampling of states according to the rotationally invariant measure. Data for the random states was computed numerically by scanning p in the range $[0.5, 1]$ with steps of 0.01, sampling 1000 random four-qubit states for each p , choosing the five biggest correlations, computing S_5^∞ and finally the probability of $S = 5$ in finite statistics. In this way, practically all the values of S_5^∞ were scanned. These calculations have shown that the probability of success does not depend on the kind of noise and it is described solely by the value of p . The probability of success $P(S = 5|p)$ for a continuous choice of p is presented in Fig. 5.

DATA AVAILABILITY

The data supporting figures are available at <https://doi.org/10.48550/arXiv.2208.01983>.

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AUTHOR CONTRIBUTIONS

P.C., W.L. and L.K. started the project. P.C. and L.K. formulated the finite statistics model for entanglement witnesses. J.D. analysed and performed a careful statistical analysis, further developed by J.D., P.C., L.K. and W.K. P.C., J.D., L.K. and J.M. prepared detailed examples and perfected the resource optimization strategy. W.L., T.P. and H.W. supervised the project. All authors contributed to analysing the results, as well as writing the manuscript.

COMPETING INTERESTS

The authors declare no competing interests.

ADDITIONAL INFORMATION

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PAPER

How likely are you to observe non-locality with imperfect detection efficiency and random measurement settings?

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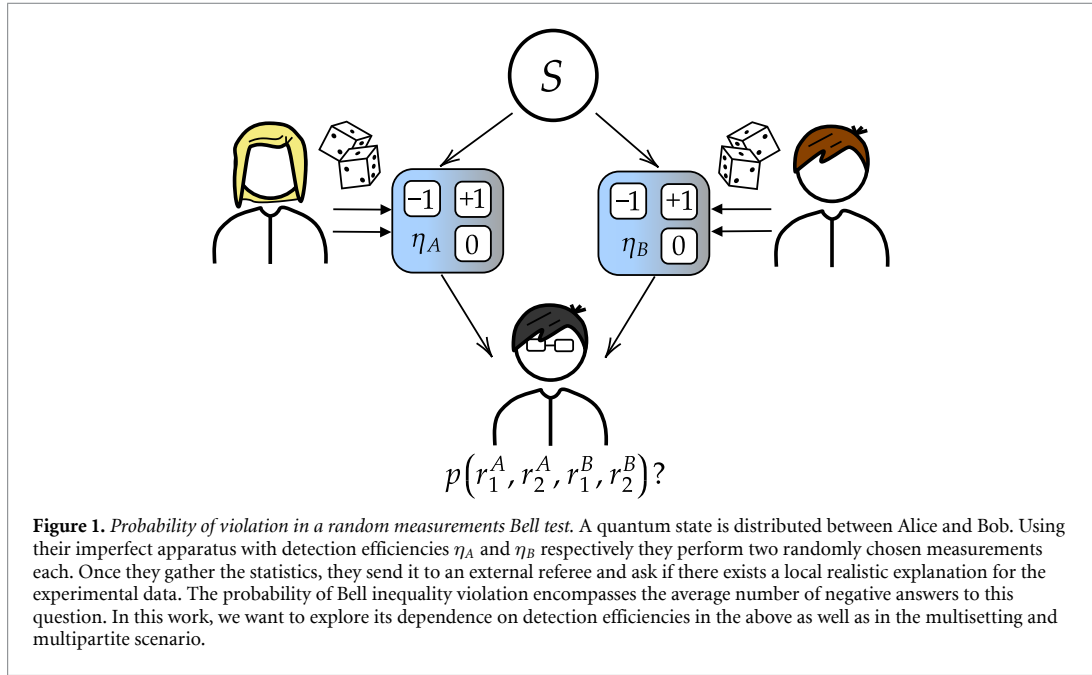
E-mail: pawel.cieslinski@phdstud.ug.edu.pl**Keywords:** Bell non-locality, quantum information, quantum foundations, random measurements, detection efficiency, probability of local realism violation

Abstract

Imperfect detection efficiency remains one of the major obstacles in achieving loophole-free Bell tests over long distances. At the same time, the challenge of establishing a common reference frame for measurements becomes more pronounced as the separation between parties increases. In this work, we tackle both of these issues by examining the impact of limited detection efficiency on the probability of Bell inequality violation with Haar random measurement settings. We derive analytical lower bounds on the violation probability for a two-qubit maximally entangled state, which is tight for correlation inequalities and perfect detection efficiencies. We further investigate it numerically for more qubits and settings using two detection efficiency models and an original method based on the linear programming. Beyond that, we show that the so-called typicality of Bell inequality violation, i.e. almost certain violation of local realism with sufficiently many particles or random measurement directions, holds even if the detection efficiency is limited. Our findings reveal that increasing the number of measurement settings can compensate for efficiency losses above the critical threshold. Additionally, we determine critical detection efficiencies for both two-party and three-party scenarios. For two parties, we recover previously established results, while for three parties, we derive a symmetric critical efficiency of $\eta_{\text{crit}} = 2/3$ for the W and GHZ states within the binning model. In cases involving the no-detection outcome, we present a modified inequality using a pair of orthogonal observables for each party with $\eta_{\text{crit}} \approx 0.7208$, which is notably less than 0.75 for the GHZ state. These results offer deeper insights into the limitations and possibilities for certifying non-locality in the presence of limited detection efficiency, shedding light on its robustness in practical Bell tests.

1. Introduction

Bell non-locality [1] is a well-established resource in quantum cryptography [2–6], randomness generation [7–10] and communication [11–14]. One of the main problems in taking full advantage of its capabilities is the imperfect detection efficiency [15–21], which becomes especially daunting in the case of long-distance Bell experiments. On the other hand, a large separation between distinct parties complicates calibration and establishing a common reference frame for the measurement settings. Both of these problems are the key issues in performing quantum information protocols using satellites in space, see e.g. [22–24]. Randomised measurements have proven themselves useful in many areas of quantum information [25, 26], among which is the possibility of certifying non-locality without the need for a shared reference frame [25]. Thus, incorporating detection efficiency into the random measurements Bell test can address both of the mentioned problems.



In this paper, we explore the dependence of the detection efficiency on the probability of local realism violation when the measurement directions are chosen at random. For simplicity, imagine a two-party scenario where Alice and Bob perform two Haar randomly chosen measurements (i.e. measurement directions chosen uniformly and randomly from a Bloch sphere) on a shared two-qubit state with imperfect detectors, see figure 1. Whenever the detectors click, they record ± 1 and 0 otherwise. Later we will also consider a different model for the imperfect detection efficiency based on the outcome binning strategy (also referred to as assignment strategy), however, for now, if not stated differently, we treat the no-detection event as 0. Then, they gather their statistics and send it to an external referee who checks whether there exists a local realistic model, i.e. a joint probability distribution compatible with the obtained information. How often will he be unable to do so? This is exactly the question of our interest. Formally, one can define the probability of Bell inequality violation as [27, 28]

$$\mathcal{P}_V = \int d\Omega f(\Omega), \quad (1)$$

where the integral goes over all the parameters within a Bell scenario, and f is a binary function that outputs 1 whenever the given settings allow one to violate some Bell inequality and 0 otherwise. This distribution was already studied in the past in several ways and contexts, see e.g. [27–37]. In practical calculations, the above integral is estimated by a large sampling ($\approx 10^9$) over the parameter space. Such a sample size ensures high significance of the numerical calculations. Moreover, this quantity can be treated as an equivalent description of non-locality for a full set of tight Bell inequalities in a given scenario [38].

Here, we derive an analytical detection efficiency dependent lower bound on the probability of Bell inequality violation for a two-qubit maximally entangled state, which is exact for correlation inequalities and perfect efficiencies. We further explore this dependence numerically for more parties and settings using an original method based on linear programming first used in [33]. Our findings extend the previous results on the random measurements Bell tests by including the inevitable limitations of experimental devices and contain quantitative predictions for direct practical use. Since our approach does not refer to a specific inequality but encompasses an entire Bell scenario, we were able to determine the symmetric critical efficiencies η_{crit} for the three-qubit GHZ as well as the W state and improve the best-known results for the two- and three-setting inequalities. In the case of perfect detectors, this problem was already studied in references [27, 29, 33, 34]. One of the conclusions made in [33, 34] was that with a growing number of particles or measurement settings, violation of local realism is almost certain—the so-called typicality of Bell violation. Our findings extend this conclusion to the case of imperfect detectors and furthermore show that any drop in efficiency above the critical one can be compensated with additional settings. This shows that the misalignment of the measurement directions and a drop in detection efficiency can be overcome with a higher number of measurements.

2. Lower bound on the probability of violation for a two-qubit maximally entangled state

Consider a probability-based CHSH inequality

$$I = \sum_{x,y=0}^1 P(a \oplus b = xy|x, y) \leq 3, \quad (2)$$

violated maximally by $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ with suitably chosen measurement directions. By including detection efficiencies η_A and η_B as well as the optimal local strategies, it can be expressed as

$$\begin{aligned} I(\eta_A, \eta_B) &= \eta_A(1 - \eta_B)M_A + \eta_B(1 - \eta_A)M_B \\ &\quad + \eta_A\eta_B Q + (1 - \eta_A)(1 - \eta_B)X \\ &= 2\eta_A(1 - \eta_B) + 2\eta_B(1 - \eta_A) \\ &\quad + \eta_A\eta_B Q + 3(1 - \eta_A)(1 - \eta_B) \leq 3 \end{aligned} \quad (3)$$

where M_i and X denote the CHSH value when either one or both detectors did not click respectively, and Q corresponds to I [21]. The main premise of this work is to determine the probability of observing $I(\eta_A, \eta_B) > L = 3$ when the measurements are chosen randomly according to the Haar measure [25]. Examining a single CHSH variant is sufficient to determine the probability of interest as the two-setting two-output scenario is fully characterised by a set of four CHSH inequalities with each of them having an equal contribution to $\mathcal{P}_V$ [29]. Note that, whenever $\bar{Q} = \eta_A\eta_B Q + 3(1 - \eta_A)(1 - \eta_B) > L$, the inequality has to be violated. Thus, determining the probability of $\bar{Q}$ violation denoted as $\mathcal{P}_V^c$ would yield a lower bound on $\mathcal{P}_V$ as a function of the detection efficiencies. Note that such an approach will be exact for perfect η and correlation-based inequalities. We also expect that it should approximate the exact probability up to a small percentage error in some range of large η . Following this idea, we derive the considered bound $\mathcal{P}_V^c$ by extending the proof presented in [29] to our scenario. All details are included in appendix A and lead to the following:

Proposition 1. *For the asymmetric detection efficiency $\eta_A = 1, \eta_B = \eta$ the probability of violation $\mathcal{P}_V$ is lower bounded by*

$$\begin{aligned} \mathcal{P}_V^c(\eta) &= \frac{1}{\eta^2(\eta^2 - 1)} \left[\pi + 2(\eta^2 - 1)\kappa - \pi\eta^4 \right. \\ &\quad + 4\sqrt{2} \left(\sqrt{\eta^2 + \kappa} + \sqrt{\kappa^2(\eta^2 + \kappa)} - 2\eta^2\sqrt{\eta^2 + \kappa} \right) \\ &\quad \left. + 8(\eta^4 - 1) \operatorname{arccot} \left\{ \sqrt{2} \sqrt{\frac{1}{\frac{\kappa}{\eta^2} + 1} + \sqrt{\frac{\eta^2 - \kappa}{\eta^2 + \kappa}}} \right\} \right]; \quad \eta > \frac{1}{\sqrt{2}}, \end{aligned} \quad (4)$$

where $\kappa = \sqrt{2\eta^2 - 1}$.

Proposition 2. *For the symmetric detection efficiency $\eta_A = \eta_B = \eta$ the probability of violation $\mathcal{P}_V$ is lower bounded by*

$$\begin{aligned} \mathcal{P}_V^c(\eta) &= \frac{2}{\eta^4} \left[2\gamma + \eta^2\pi[(\eta - 2)\eta + 2] \right. \\ &\quad + \eta(\eta - 2) \left[2\sqrt{2} \left(\sqrt{\gamma\eta^2 + 2\gamma(1 - \eta) + \eta^4} \right. \right. \\ &\quad \left. \left. - \sqrt{-\gamma\eta^2 - 2\gamma(1 - \eta) + \eta^4} \right) + \gamma \right] + 2\gamma \\ &\quad \left. - 4\eta^2[(\eta - 2)\eta + 2] \arctan \left\{ \frac{4\sqrt{(\eta - 1)^2((\eta - 1)\eta + 1)^2}}{\gamma((\eta - 2)\eta + 2) + \eta^4} \right\} \right]; \\ &\quad \eta > \sqrt{5 + 4\sqrt{2}} - \sqrt{2} - 1, \end{aligned} \quad (5)$$

where $\gamma = \sqrt{8(\eta - \eta^2) + 4\eta^3 + \eta^4 - 4}$.

Constraints on the allowed values of η reflect the upper bounds on the critical efficiencies. The above bounds agree well with the linear programming results and can recover $\mathcal{P}_V$ with good accuracy in an entire range of η . For a direct comparison with numerics, see section 3.

3. Numerical results

An alternative approach to this problem, in general, is to search for the joint probability distributions of the form

$$P_{LR}(r_1^1, r_2^1, \dots, r_m^1, r_1^2, \dots, r_m^N) = P_{LR}(\vec{r}^1, \dots, \vec{r}^N), \quad (6)$$

describing the statistics of measurement outcomes $r_k^\mathcal{O}$ obtained for the party $\mathcal{O}$ measuring the setting k . For example, in the two-party, two-setting scenario for qubits, we have $P_{LR}(r_1^1, r_2^1, r_1^2, r_2^2) = P_{LR}(r_1^A, r_2^A, r_1^B, r_2^B)$, where A and B stand for Alice and Bob, respectively, and $r_k^\mathcal{O} = \pm 1$. For the sake of generality, we will introduce all of the concepts using number labelling of the parties. Later, however, we will constrain ourselves to at most three parties and use alphabetical labels for clarity. In general, quantum mechanics allow one to measure probability distributions that do not adhere to the above. Let $M_k^\mathcal{O} = \sum_{r^i = \pm 1} r_k^i |r_k^i\rangle\langle r_k^i|$ be the k th observable measured by $\mathcal{O}$. For the given state ρ quantum mechanical probabilities yield

$$P_Q(r_{k_1}^1, \dots, r_{k_N}^N | M_{k_1}^1, \dots, M_{k_N}^N) = \text{Tr}[\rho |r_{k_1}^1\rangle\langle r_{k_1}^1| \otimes \dots \otimes |r_{k_N}^N\rangle\langle r_{k_N}^N|]. \quad (7)$$

If the local realistic model for this distribution exists, then the above probabilities can be explained by the following marginal distributions

$$P_Q(r_{k_1}^1, \dots, r_{k_N}^N | M_{k_1}^1, \dots, M_{k_N}^N) \stackrel{\text{LR}}{=} \sum_{r_{k'_1}^1, \dots, r_{k'_N}^N = 0}^{d-1} P_{LR}(\vec{r}^1, \dots, \vec{r}^N) = P_{LR}(r_{k_1}^1, \dots, r_{k_N}^N), \quad (8)$$

where $k'_i \neq k_i$ ($i = 1, \dots, N$). The above represents the probability of obtaining measurement outcomes $r_{k_1}^1, \dots, r_{k_N}^N$ when measuring observables $M_{k_1}^1, \dots, M_{k_N}^N$. Again, in the two-party two-setting scenario for qubits we have, e.g. $P_{LR}(r_1^1, r_1^2) = P(r_1^1, -1, r_1^2, -1) + P(r_1^1, -1, r_1^2, 1) + P(r_1^1, 1, r_1^2, -1) + P(r_1^1, 1, r_1^2, 1)$. To include the imperfect detection efficiencies we treat the local events where no particle was detected as an additional measurement outcome 0. As a result, more probabilities P_{LR} , with $r_k^\mathcal{O} = 0, \pm 1$, that need to be taken into account [39]. Finally, if the measurement statistics is incompatible with the joint probability distribution description, then no local realistic model exists, and thus, there is a Bell inequality that is violated with the given state and settings [40, 41]. It is worth noting that such inequality can be violated in a very weak manner [36]. This strongly points out the relevance of including the detection efficiency in such a Bell scenario. For completeness we will also consider a model in which the outcomes are grouped in just two categories. There, we will consider a single imperfect detector per party that records the $+1$ events. If no $+1$ was detected, we treat it as -1 . However, if not stated differently, we will work with the first scenario.

Before we proceed to the numerical results, we will specify the notion of random measurements in the studied context. Randomised settings are performed by applying a unitary transformation from the Haar ensemble U on the projection operators associated with the Pauli matrix σ_z , i.e. $U_{[k]}|i\rangle\langle i| = |h_k^i\rangle\langle h_k^i|$, $i = 0, 1$. The explicit form of $U_{[k]}$ as a function of measurement angles can be found in [33]. Using such parametrisation, Haar sampling reduces to drawing angles from a uniform distribution. Generating $m^\mathcal{O}$ unitaries will produce $m^\mathcal{O}$ random settings, enumerated by k , for party $\mathcal{O}$. For each sample of tensor product of Haar unitaries $U_{[k_1^1]} \otimes \dots \otimes U_{[k_N^N]}$ and a given state, we generate the measurement statistics and look for a local realistic model as presented above using the linear programming method first used in [33]. If the joint probability distribution does not exist for the produced statistics we put $f = 1$ ($f = 0$ otherwise), and by having a large enough sample, estimate $\mathcal{P}_V$ (1) which is the quantity of our interest. For all of our calculations, the number of random unitaries varies between $10^3 - 10^{10}$ depending on the number of studied settings and parties.

3.1. Two-qubit maximally entangled states

Results for a two-qubit maximally entangled state and two settings per party obtained through the discussed techniques are shown in figure 2. The general behaviour of the probability of violation agrees with a basic intuition as it decreases monotonously with each of the efficiencies η . However, here the important information lies not only in the general behaviour of the probability of interest but in our ability to make quantitative predictions on $\mathcal{P}_V$. The range of the studied efficiencies goes from the approximate μ_{crit} to 1. Note that it covers μ which allow for the device independent quantum key distribution, see e.g. [42] that

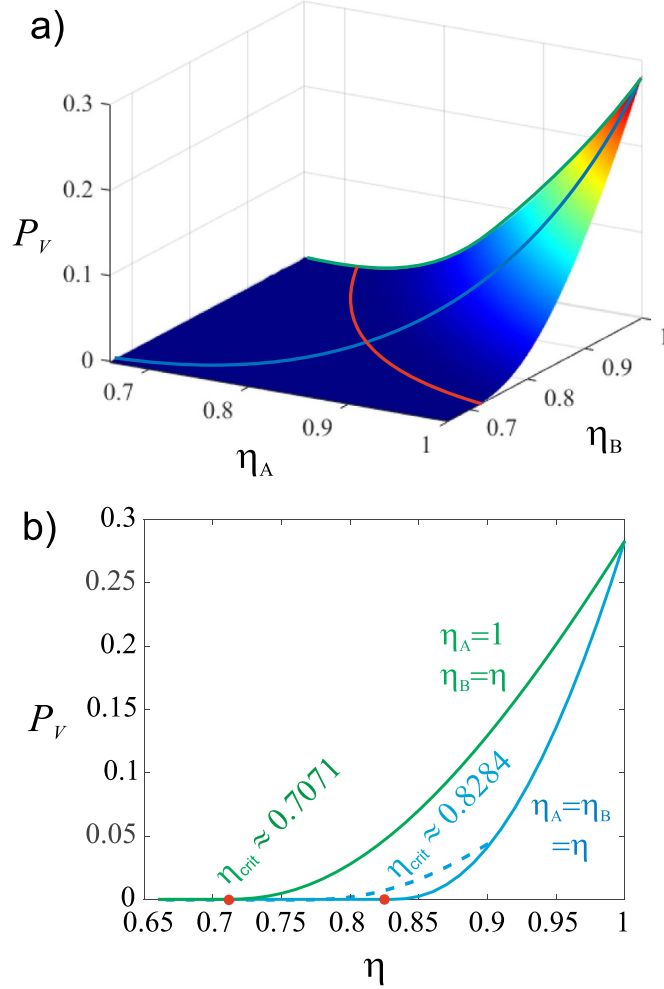


Figure 2. Probability of Bell inequality violation $\mathcal{P}_V$ as a function of detection efficiencies η_A and η_B for a two-qubit maximally entangled state and two settings per party. Each point on the plot was computed with a sample of $10^6 - 10^{10}$ linear programming evaluations and a step of $\Delta\eta = 10^{-3}$. The solid red line on the (a) panel represents the critical efficiency η_{crit} and separates the region of $\mathcal{P}_V$ being ≈ 0 up to statistical significance and numerical error. The additional coloured lines correspond to the two significant cases of symmetric ($\eta_A = \eta_B = \eta$) and asymmetric ($\eta_A = 1, \eta_B = \eta$) detection efficiency scenarios marked in blue and green respectively. For clarity, they are plotted separately on panel b). There, an additional case of $\mu_A = 0.9$ and $\mu_B = \mu$ was plotted (blue dashed curve) to illustrate a more nuanced asymmetric scenario. As expected, the maximal $\mathcal{P}_V$ occurs for the perfect detection case and reproduces the known result derived in [29]. As the efficiencies drop down, the probability of violation decreases monotonously, and here we provide the exact predictions for its value at any given (η_A, η_B) . The above numerical results obtained through the linear programming are in good agreement with the derived lower bounds (5) and (6). For more details, see the main text.

tolerates efficiencies above ≈ 0.685 , and its experimental demonstration [43]. For the perfect detection case, we recover $\mathcal{P}_V \approx 28.32\%$ [29, 33]. On the other hand, we can localise the critical efficiencies yielding $\mathcal{P}_V \approx 0$, up to statistical significance and numerical error, for any pair of (η_A, η_B) denoted by the red line. Their values in the symmetric (blue line) and asymmetric case (green line) coincide with the known results [44, 45]. Note that in many important situations where the entangled states are distributed between long-range separated parties, the detection efficiency is drastically reduced due to the transmission losses. In a direct scenario, loophole free violation of any standard Bell inequality is impossible. However, modern quantum repeaters or memories promise the solution to this problem and can effectively pass most of the detection efficiency to the measuring devices [23]. Assuming that, take an example of space-to-ground links where the asymmetric detection efficiency scenario can be better described when $\mu_A < 1$ [46, 47], e.g. $\mu_A = 0.9$ and $\mu_B = \mu$. In this case, the probability of violation drops as compared to the standard asymmetric scenario and is plotted explicitly with a dashed line in figure 2.

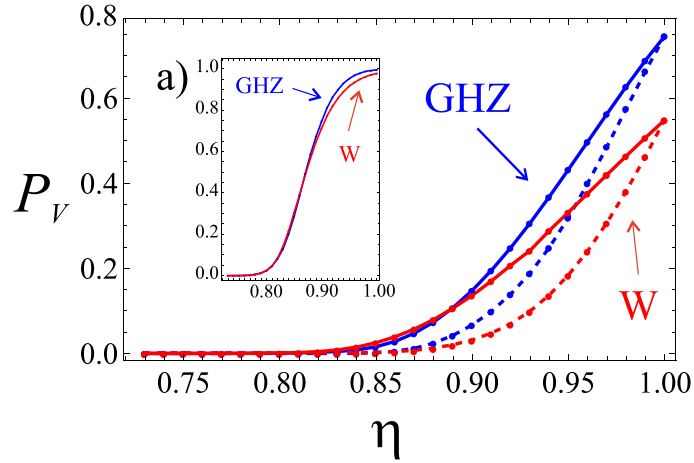


Figure 3. Probability of violation for the three-qubit GHZ (blue) and W (red) state with symmetric detection efficiencies $\eta_A = \eta_B = \eta_C = \eta$ and an extra no-detection outcome (solid line) and binning (dashed line for) models. For small efficiencies, in the three-outcome model, the probability of violation is higher for the W state in both $m = 2$ and $m = 3$, see inset a), cases. Then, both curves cross. For the $m = 2$ case, it happens at $\eta = 0.89$ and for $m = 3$ it can be localised between $\eta = 0.86$ and $\eta = 0.87$. Afterwards, the GHZ state probability grows faster and reaches its maximal value at $\eta = 1$, which differs from the W state maxima significantly for $m = 2$. The highest P_V is observed for the GHZ state and $m = 3$ scenario, where it reaches almost 100%. Note that for the extra no-detection outcome model and both states, the probability of Bell inequality violation for $m = 3$ as compared to $m = 2$ is greater for all η . This shows that introducing more settings can increase the probability of violation in the imperfect detection randomised scenarios. In general, both states give similar predictions for the overall behaviour of P_V , and this also holds for the binning model. Nevertheless, in a quantitative sense, the binning model performs worse and the violation probability is always smaller.

We further observe that in the asymmetric case the lower bound $\mathcal{P}_V^c$ (5) agree with P_V up to $\approx 3.5\%$ relative error within the whole range of η , i.e. $\eta_{\text{asym}} \in [\eta_{\text{crit}}^c, 1]$, where $\eta_{\text{crit}}^c = 1/\sqrt{2}$. In the symmetric case (6), the same holds up to $\approx 2.2\%$ error.

3.1.1. Binning results for $N = 2$

The problem of $\mathcal{P}_V$ computation can also be approached with the binning model for imperfect detection efficiency. In such a scenario, there only exist two distinct outcomes associated with a two-element positive operator valued measure (POVM)

$$\begin{aligned} P_+^O &= \eta_O |h_k^0\rangle\langle h_k^0| \\ P_-^O &= \mathbb{1} - P_+^O, \end{aligned} \quad (9)$$

where again $|h_k^0\rangle\langle h_k^0|$ is a random projective measurement associated with the +1 outcome. Results for the symmetric and asymmetric scenarios obtained within that framework are practically indistinguishable from the ones presented in the previous section. This is due to the fact that within the limited detection efficiency two-setting two-party scenario, the three-outcome approach does not provide any extra inequalities [48]. However, this will no longer be the case for the three-qubit states in the subsequent section.

3.2. Three-qubits W and GHZ states

For three qubits, we focus on two entangled states which are inequivalent under local unitaries and can yield different probabilities in the randomised scenario, i.e. the GHZ and W states defined as

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad (10)$$

$$|\text{W}\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle). \quad (11)$$

Here, we will examine two- and three-setting Bell inequalities in the same context. By performing analogous calculations, we evaluate the probability of violation in the case of symmetric efficiencies $\eta_A = \eta_B = \eta_C = \eta$ and a step of 10^{-2} , see figure 3. We observe that for small efficiencies, the probability of violation is higher in the case of the W state for both $m = 2$ and $m = 3$ measurement settings. Then, both curves cross. It is

interesting from the non-local resources point of view due to the mentioned inequivalence of the studied states. From a practical perspective, if we know our efficiencies, it turns out that it might be as beneficial to prepare the W state as the GHZ one depending on their generation feasibility. After this point, the GHZ state probability exceeds the one for the W states and reaches its maximum value for either choice of m . Interestingly, the probability of Bell inequality violation for $m = 3$ is greater for all η as compared with the two-setting case and the maximum $\mathcal{P}_V$ for both states is close to 1 (see inset *a*) in figure 3). This points to the fact that introducing more settings increases the probability of violation, thus expanding the previous results [33, 34] to the imperfect detection scenarios. This observation will be explored in more detail in the subsequent section.

For the three-qubit case, we can also approximate the critical detection efficiencies η_{crit} without referring to a specific inequality. However, due to its statistical nature, it has an intrinsic statistical and numerical uncertainty. Up to our knowledge, the best previous results for η_{crit} and the studied states are $\eta_{\text{crit}}(\text{GHZ}_3) = 0.75$ [16] for $m = 2$ and can be reduced only by introducing many ($m = 17$) additional settings [17]. For the W state they are given as $\eta_{\text{crit}}(\text{W}_3) = 0.83747$ for $m = 2$ and $\eta_{\text{crit}}(\text{W}_3) = 0.6$ for $m = 3$ [18].

From our data, we observe that the approximated η_{crit} are smaller than the previously known best results. In order to determine them in a reliable manner, one would have to find the corresponding inequalities and prove that the violation vanishes for a given η . Unfortunately, this is a very daunting task in such a model. Nevertheless, we have found an inequality that improves the Mermin inequality efficiency threshold in such an approach. See appendix B for details. By taking into account the non-signalling constraints and rewriting the original problem in the Collins–Gisin format, we arrived at the inequality with $\eta_{\text{crit}} = (\sqrt{10} - 1)/3 \approx 0.7208$. The quantum value was evaluated on the rotated GHZ state, but since it is equivalent to the original state under local unitaries they are indistinguishable in the random measurement setting, and our reasoning is consistent.

Despite finding an inequality that allowed us to lower the η_{crit} , we found that this more complex approach is not necessary. By turning off one of the detectors, here the one measuring the -1 events, for each of the observers and moving onto the binning model, we observe that even smaller η_{crit} can be found. Now, we will follow this path.

3.2.1. Binning results for $N = 3$ and critical detection efficiencies

Again, we shift our focus to the binning model. As above, we will limit ourselves to the symmetric detection efficiency scenario parametrised by a single η . The numerical results for the $\mathcal{P}_V$ are presented in figure 3. Here we see strong quantitative differences, especially for the W state. The probabilities in both approaches to the detection efficiency are, as expected, equal for $\eta = 1$, but then drop down faster than before. The general behaviour stays the same.

The advantage of such a formulation of the detection efficiency model is that the observables are standard qubit POVMs, and it becomes feasible to find the exact inequalities violated for small η and derive the critical detection efficiencies. From the numerical calculations, we find the measurement settings for which only few joint probability distributions were found and $\mathcal{P}_V \approx 0$. Then we use the linear programming method introduced in [49] to find the corresponding inequalities. For both of the studied states and $m = 2$ we arrive at $\langle I_C \rangle \leq 1$ with

$$I_C = -A_1 + \frac{1}{2}(\mathbb{1} + A_1)[B_1C_1 + B_2C_1 + B_1C_2 - B_2C_2] \quad (12)$$

where X_i for $i = 1, 2$ and $X = A, B, C$ are the observables for each party. Note that there is only one non-trivial setting on Alice's side. The term $1/2(\mathbb{1} + A_1)$ can be interpreted as a projection and is known to improve the detection efficiency, see e.g. [50]. Note that the remaining part of I_C is the CHSH expression, which immediately points to the previous work on a similar subject, e.g. [44, 51–53]. Moreover, this inequality is tight as it is a facet of the Bell–Pitowsky polytope [54]. Now, solving I_C for η and maximizing the violation we get $\eta_{\text{crit}} = 2/3$. This is exactly the minimal detection efficiency for two-qubit states obtained in [44]. Moreover, after a suitable projection, the state on which the CHSH expression is measured is the Eberhard state

$$|\psi_E\rangle = \frac{1}{\sqrt{1 + \alpha^2}}(|00\rangle + \alpha|11\rangle), \quad (13)$$

for some $\alpha \ll 1$. From the experimental perspective, the observed violations have a clear filtering interpretation. Performing analogous optimisation for $m = 3$ we again arrive at (12) and $\eta_{\text{crit}} = 2/3$.

All of the obtained symmetric critical detection efficiencies are summarised and presented in table 1.

Table 1. Summary of the symmetric critical detection efficiencies for the studied states. The first column of the table refers to the state under consideration, and is followed by the imperfect detection model and the number of settings m in the Bell scenario. The subscripts in the state's name refer to the number of parties. Note that beyond recovering the previously known results for two-qubits, we also present new critical detection efficiencies in the tripartite scenario. For more details, see the main text.

State	Detection model	m	η_{crit}
$ \psi^-\rangle$	Both	2	$2/(1 + \sqrt{2}) \approx 0.8284$
$ \text{GHZ}_3\rangle$	Three-outcome	2	$(\sqrt{10} - 1)/3 \approx 0.7208$
$ \text{GHZ}_3\rangle$	Binning	2,3	$2/3$
$ \text{W}_3\rangle$	Binning	2,3	$2/3$

4. Typicality of local realism violation and efficiency compensation

In reference [33, 34] the authors showed that with perfect detectors, non-locality is typical, meaning that by extending the number of particles or randomly chosen settings the local realism violation becomes almost certain, i.e. $\mathcal{P}_V \approx 100\%$. Here, we investigate this problem in a more practical scenario. We determined the probability of Bell inequality violation for the maximally entangled two-qubit state and the whole range of symmetric detection efficiencies with a step of 10^{-2} and different number of measurement settings for Alice and Bob. For clarity, we chose $m^A = m^B = m$ and performed the analysis for $m \leq 7$. The results are depicted in figure 4. The differences in predictions for the extra no-detection outcome and the binning models for $m < 5$ are shown and discussed in figure 5. The obtained results show that increasing the number of settings significantly enhances the probability of violation. For instance, with $m = 2$ settings, the probability grows slowly and reaches the previously discussed maxima, whereas for more than two settings a rapidly growing sigmoid-type curve appears from the data. The breakpoint moves toward lower η 's, and the slope increases with the number of added settings, causing the $\mathcal{P}_V$ to grow exponentially past the critical efficiency for higher m 's. For $m > 4$ the probability of violation saturates to $\approx 100\%$ and this process occurs faster the greater the m . In the limit of $m \rightarrow \infty$, the expected behaviour follows a step function past the critical efficiency that immediately reaches a certain violation [34]. This is intuitively due to the certainty of sampling the optimal CHSH measurement angles. We thus conclude that typicality is preserved even when our detection efficiency is limited. Adding new measurement settings causes $\mathcal{P}_V$ to eventually reach $\approx 100\%$. Such asymptotic behaviour can be analysed in terms of statistical significance through, for example, the Wilson technique [55]. Given the confidence level of 95% and the observed $\mathcal{P}_V = 100\%$, the lower bound on the estimated probability is 99% with ≈ 270 events and 99.9% with ≈ 2700 events. For a confidence level of 99% it changes to ≈ 460 and ≈ 4600 respectively. Clearly, a few thousand samples are enough to infer the $\mathcal{P}_V > 99\%$.

An interesting observation from this data is that our detection capabilities can be compensated by performing more measurements. Naturally, this holds as long as we are past the critical efficiency below which the probability is always 0. This observation is consistent with the previous results on Bell non-locality showing the advantage of multi-setting Bell inequalities, see e.g. [17, 56–61]. It is worth noting that the observed behaviour of the probability of interest is similar to the one describing $\mathcal{P}_V$ as a function of visibility for a two-qubit Werner state [30], which should follow the symmetric $\mathcal{P}_V^c$ (5) in the $m = 2$ case. Moreover, our results corresponding to the perfect detection efficiency are in agreement with the perfect visibility case. At last, note that for $m = 4$ there exists a tight Bell inequality that allows for a slightly lower detection efficiency of $\eta_{\text{crit}} = 0.8214$ for the two-qubit maximally entangled state, compared to the CHSH critical efficiency of $\eta_{\text{crit}} = 0.8284$ [62].

We further investigate the observation of efficiency compensation by calculating the minimal number of settings needed for a given detection efficiency to increase $\mathcal{P}_V$ up to the maximal value of $\approx 28.32\%$ from the two-setting perfect detection case. When increasing them from 2 to 3, the first value of η that allows one to achieve such a task is $\eta = 0.91$, at which $\mathcal{P}_V = 31.31\%$. For $m = 4$, the required efficiency decreases to $\eta = 0.88$ with the corresponding probability of 38.71%. Similarly, for $m = 5$ we have $\eta = 0.86$ with the probability of 38.78%. Finally, for $m = 6$, the efficiency needed to outperform the perfect detection scenario with two settings is $\eta = 0.85$ with the probability of violation of 40.55%. The next question we ask is, when does the $\mathcal{P}_V$ increase the most while increasing the number of settings. For this, we calculate the relative increase in the probability of violation for different choices of the number of settings as $(\mathcal{P}_{V_{m_1}} - \mathcal{P}_{V_{m_2}})/(\mathcal{P}_{V_{m_1}}) = \Delta\mathcal{P}_{m_1 \rightarrow m_2}$, where $\mathcal{P}_{V_{m_1}}$ and $\mathcal{P}_{V_{m_2}}$ are the violation probabilities for the fixed number of settings m_i . The acquired results are presented in table C1 in appendix C. The greatest relative jump appears when one changes the number of settings from $2 \rightarrow 3$ with a probability growth factor of up to 1404.25% for $\eta = 0.83$. Similarly, the transition from $3 \rightarrow 4$ shows initial growth of 429.41% at $\eta = 0.83$,

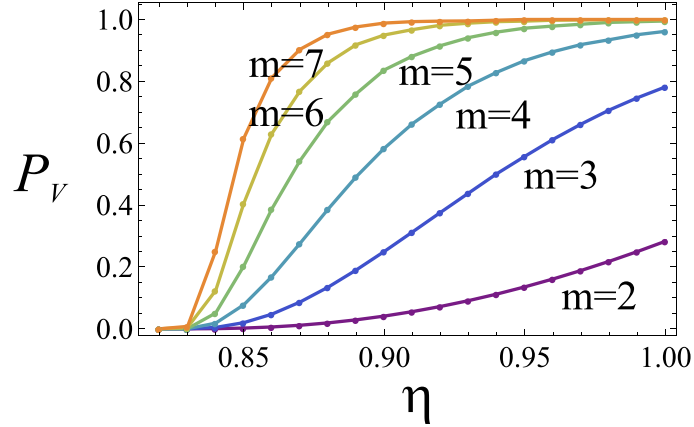


Figure 4. Probability of Bell inequality violation $\mathcal{P}_V$ as a function of symmetric detection efficiency η for a two-qubit maximally entangled state and various number of settings $m \in [2, 7]$. Each colour represents a different $m \in [2, 7]$. For $m = 2$ settings, the probability grows slowly with η . For $m \geq 3$ its behaviour can be described through a sigmoid-type curve. Its break point moves toward lower efficiencies, and the slope increases with m , causing the $\mathcal{P}_V$ to grow exponentially past the critical efficiency and then saturate to almost 100% for $m > 4$. The above results confirm the typicality of non-locality in the imperfect detection scenario and show how a drop in detection efficiency can be compensated with an additional setting.

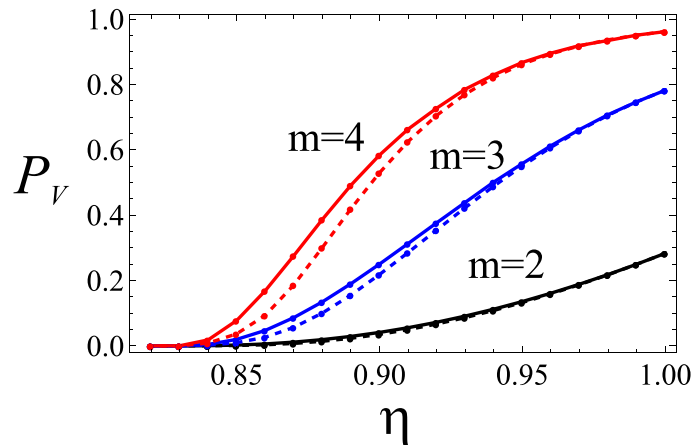


Figure 5. Probability of Bell inequality violation $\mathcal{P}_V$ for a two-qubit maximally entangled state and different models of imperfect detection efficiency in the symmetric scenario with m settings per party. Coloured points connected by solid lines correspond to the scenario where the no-detection event is treated as an extra outcome. Predictions of the binning model are plotted with dashed lines. In spite of the number of settings m , both models give similar results for efficiencies close to 1. Then, for smaller η they start to diverge and differ stronger the greater the m . In the scenario with $m = 2$, they overlap as the three-outcome approach does not provide any extra inequalities in this case [48]. In all the cases, the no-detection extra outcome model yields a greater (or equal) probability of violation.

which then drops to 22.95% at $\eta = 1$. These results highlight the benefit of adding more settings to our test when the detection efficiency available drops down.

5. Conclusions

The probability of Bell inequality violation can be heavily influenced by the detection efficiency of our set-up. In this paper, we have exploited this dependence in two detection efficiency models thoroughly by determining its values for several states and varying number of measurement settings. Our results contain analytical lower bounds as well as extensive numerical calculations based on linear programming. We have confirmed the basic intuition behind its behaviour and made new interesting observations. Any drop in detection efficiency for the two-qubit maximally entangled state above the critical one can be compensated by adding new random measurement settings to our Bell test. Preliminary data for the three-qubit GHZ and W states suggest that this behaviour extends to a multipartite scenario. Based on the same calculations, we confirmed the existence of typicality of non-locality in the case of imperfect detection efficiency by reporting that introducing additional random settings to the set-up causes the probability of violation to eventually

reach $\approx 100\%$. Furthermore, we took advantage of the generality of random measurements scenarios and determined the critical efficiencies η_{crit} for the three-qubit GHZ and W state, improving the previously best-known results.

These findings represent a significant step forward in understanding the practical limitations and possibilities of certifying non-locality, particularly in the context of long-distance Bell tests, such as those conducted via satellite-based communication. The obtained results and observations contain quantitative predictions that can help to improve current days and forthcoming experiments. Future work in this subject could focus on the higher number of particles, different states (including higher dimensions) for which the critical efficiency is unknown and genuine or network scenarios.

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: <https://arxiv.org/abs/2503.21519>.

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Appendix A. Derivation of the lower bound on the probability of violation

Here, we will derive the lower bound on the probability of local-realism violation $\mathcal{P}_V^c(\eta_A, \eta_B)$ presented in the main text. Since all maximally entangled states of two qubits are equivalent under local unitary transformations and give rise to the same statistics in the randomised scenario, we choose to work with the singlet state $|\psi^-\rangle$. The presented proof follows the lines of [29] and extends it to our setting. As noted in the main text we want to find $\mathcal{P}_V$ for the correlation part of (4) as it would provide us with a lower bound on the violation probability of an entire inequality. For convenience, we rewrite the inequality (2) with correlation functions. Now, the local bound is $L = 2$, and we are interested in observing

$$I' = \eta_A \eta_B [E(\vec{a}_1, \vec{b}_1) + E(\vec{a}_1, \vec{b}_2) + E(\vec{a}_2, \vec{b}_1) - E(\vec{a}_2, \vec{b}_2)] + 2(1 - \eta_A)(1 - \eta_B) > 2, \quad (\text{A.1})$$

as it is equivalent to $\tilde{Q} > 3$ in (4). Note that the $2(1 - \eta_A)(1 - \eta_B)$ term vanishes in the asymmetric case but can significantly improve the symmetric scenario results.

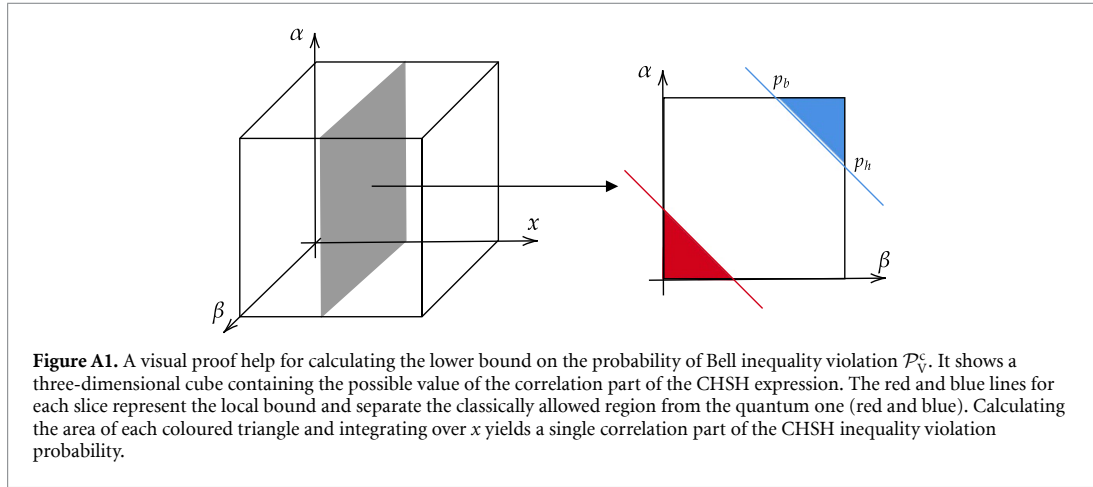
For a given random measurement directions $\vec{a}_i$ and $\vec{b}_j$ the correlation function of the examined state is given as $E(\vec{a}_i, \vec{b}_j) = -\vec{a}_i \cdot \vec{b}_j$. Defining $\vec{r} = \vec{b}_1 + \vec{b}_2$ and $\vec{r}_\perp = \vec{b}_1 - \vec{b}_2$, expression I' becomes

$$I' = \sqrt{2} \eta_A \eta_B |\sqrt{1 + x} \vec{a}_1 \cdot \hat{r} + \sqrt{1 - x} \vec{a}_2 \cdot \hat{r}_\perp| + 2(1 - \eta_A)(1 - \eta_B), \quad (\text{A.2})$$

where $\hat{\cdot}$ denotes a unit vector and $x = \vec{b}_1 \cdot \vec{b}_2$. The violation we are interested in now is $|\sqrt{1 + x} \vec{a}_1 \cdot \hat{r} + \sqrt{1 - x} \vec{a}_2 \cdot \hat{r}_\perp| > [2 - 2(1 - \eta_A)(1 - \eta_B)] / \sqrt{2} \eta_A \eta_B$. As argued in [29], from the rotational symmetry of the problem, directions of the unit vectors are irrelevant and to determine the probability of violation it is enough to sample $\alpha = \vec{a}_1 \cdot \hat{r}$, $\beta = \vec{a}_2 \cdot \hat{r}$, and x uniformly from $[-1, 1]$. Geometrically the variables α, β, x span a cube with vertices at ± 1 and a volume of 2^3 . Now, we ask what fraction of this cube leads to a violation of $I' < 2$. In the positive (I) and negative (II) domain of the expression within the modulus of I' , the violation condition defines two linear inequalities

$$\text{I} : \alpha > \frac{[2 - 2(1 - \eta_A)(1 - \eta_B)] / \sqrt{2} \eta_A \eta_B - \beta \sqrt{1 + x}}{\sqrt{1 - x}} \quad (\text{A.3})$$

$$\text{II} : \alpha < -\frac{[2 - 2(1 - \eta_A)(1 - \eta_B)] / \sqrt{2} \eta_A \eta_B + \beta \sqrt{1 + x}}{\sqrt{1 - x}}. \quad (\text{A.4})$$



Fixing x we can consider slices of a cube and examine the area of triangles defined by the above equations, see figure A1. Since they contribute in an equal manner, we will focus on the region I. The points defining the base and height of the triangle are given as $p_{b/h} = ([2 - 2(1 - \eta_A)(1 - \eta_B)] / \sqrt{2\eta_A\eta_B} - \sqrt{1 \mp x}) / \sqrt{1 \pm x}$. From that, the base length and its height are given as $1 - p_{b/h}$ respectively and yield the triangle area

$$S_{\Delta} = \frac{1}{2} \frac{(\sqrt{2}(\eta_B + \eta_A) - \eta_A\eta_B(\sqrt{1-x} + \sqrt{x+1} + \sqrt{2}))^2}{2\eta_A^2\eta_B^2\sqrt{1-x^2}}. \quad (\text{A.5})$$

The total area of two regions for which the violation occurs is equal to $2S_{\Delta}$. Due to the detection efficiency dependence, we need to include some additional constraints on the possible values of x , i.e. the range of x for which the triangle has a non-zero area. This condition is equivalent to $p_{h/b} \in (0, \pm 1)$. From now on, we will focus on two especially relevant cases, but a general reasoning is still possible to perform. These two cases are $\eta_A = 1, \eta_B = \eta_{\text{asym}}$ and $\eta_A = \eta_B = \eta_{\text{sym}}$ referred to as asymmetric and symmetric detection efficiencies in the main text. In the asymmetric case, x has to be contained within $\pm \sqrt{\eta_{\text{asym}}^2 - 1} / \eta_{\text{asym}}^2$ which leads to an upper bound on the critical efficiency of $\eta_{\text{asym}}^c = 1/\sqrt{2} \approx 0.7071$. In the latter the possible values of x are

$$\pm \left[\frac{(\eta_{\text{sym}}^2 - 2\eta_{\text{sym}} + 2) \sqrt{\eta_{\text{sym}}^4 + 4\eta_{\text{sym}}^3 - 8\eta_{\text{sym}}^2 + 8\eta_{\text{sym}} - 4}}{\eta_{\text{sym}}^4} \right] \text{ and } \eta_{\text{sym}}^c = \sqrt{5 + 4\sqrt{2}} - \sqrt{2} - 1 \approx 0.8503.$$

To obtain the minimal probability of a single CHSH inequality violation one has to integrate the expression for $2S_{\Delta}$ over x respecting the above bounds and divide it by the cube's volume. Before we do that, note that the two-setting two-outcome scenario is fully characterised by the set of four CHSH inequalities, and thus we can obtain the $\mathcal{P}_V^c$ by multiplying a single inequality violation probability by four. Performing the last two steps, we finally get the lower bounds presented in the main text.

Appendix B. Critical detection efficiency for the GHZ state in a two-setting three-party scenario

We present a symmetric Bell inequality for a three-party scenario with $m = 2$ and the critical efficiency of $\eta_{\text{crit}} = (\sqrt{10} - 1)/3 \approx 0.7208$. This threshold is lower than that of the Mermin setup [63], which requires $\eta_{\text{crit}} = 0.75$ (provided here for comparison). In both cases, the no-detection events are addressed by introducing an additional third outcome.

B.1. Inequality

We consider a Bell scenario with three observers, where each observer can choose between two measurements that yield three possible outcomes. In this case, the probability distribution is given by the set $\{P(a, b, c|x, y, z)\}$, where the measurement settings (inputs) take $x, y, z \in \{0, 1\}$ and the outcomes are $a, b, c \in \{+1, -1, 0\}$. A general three-party three-outcome Bell inequality can then be written as follows

$$I_{ABC} = \sum_{a,b,c} \sum_{x,y,z=0,1} B_{abcxyz} P(a, b, c|x, y, z) \leq 0, \quad (\text{B.1})$$

with a local bound of zero and with the indices a, b, c running over $\{+1, -1, 0\}$. However, by imposing the non-signaling conditions and the normalisation (i.e. the probabilities sum up to one for each triple of inputs

(x, y, z)), one may write a generic three-outcome Bell inequality in a form without explicitly including the third outcome (0). This is known as the Collins–Gisin format [64]:

$$\begin{aligned}
 I_{ABC} = & \sum_{a,b,c,x,y,z} \alpha_{abcxyz} P(a, b, c|x, y, z) \\
 & + \sum_{a,b,x,y} \beta_{abxy} P(a, b|x, y) + \sum_{a,c,x,z} \beta_{acxz} P(a, c|x, z) \\
 & + \sum_{b,c,y,z} \beta_{bcyz} P(b, c|y, z) + \sum_{a,x} \gamma_{ax} P(a|x) \\
 & + \sum_{b,y} \gamma_{by} P(b|y) + \sum_{c,z} \gamma_{cz} P(c|z) \leq 0,
 \end{aligned} \tag{B.2}$$

where the outcomes are restricted to $a, b, c \in \{\pm 1\}$ and the inputs are $x, y, z \in \{0, 1\}$.

For each triple of measurement settings (x, y, z) , we define a three-party correlation function and the sum of probabilities as

$$\begin{aligned}
 E_{xyz} &= \sum_{a,b,c=\pm 1} abc P(a, b, c|x, y, z), \\
 O_{xyz} &= \sum_{a,b,c=\pm 1} P(a, b, c|x, y, z).
 \end{aligned} \tag{B.3}$$

Similarly, the two-party quantities are defined by

$$\begin{aligned}
 O_{AB} &= \sum_{a,b=\pm 1} \sum_{x,y=0,1} P(a, b|x, y), \\
 O_{AC} &= \sum_{a,c=\pm 1} \sum_{x,y=0,1} P(a, c|x, z), \\
 O_{BC} &= \sum_{b,c=\pm 1} \sum_{y,z=0,1} P(b, c|y, z).
 \end{aligned} \tag{B.4}$$

Using a linear programming approach, we found a new tight facet-defining three-party Bell inequality, $\langle I_{ABC}^{(1)} \rangle \leq 0$, in the three-outcome probability space. This inequality, as we show below, provides a critical detection efficiency of $\eta_{\text{crit}} \approx 0.7208$ using a rotated version of the GHZ state and orthogonal qubit observables. In particular, we have the expression

$$I_{ABC}^{(1)} = I_{222} + J_{222} - K_{22}. \tag{B.5}$$

This Bell inequality $\langle I_{ABC}^{(1)} \rangle \leq 0$ is a special class of inequalities (B.2) with zero single-party marginal coefficients. The three separate terms that build up the expression are defined by

$$\begin{aligned}
 I_{222} &= 3E_{000} - \text{sym}[E_{001}] - 3\text{sym}[E_{011}] + E_{111}, \\
 J_{222} &= -O_{000} + 2\text{sym}[O_{001}] - \text{sym}[O_{011}] + 2O_{111}, \\
 K_{22} &= O_{AB} + O_{AC} + O_{BC},
 \end{aligned} \tag{B.6}$$

where we use here a compact notation

$$\text{sym}[E_{klm}] = \sum_{\pi(k,l,m)} E_{klm}, \tag{B.7}$$

for symmetrizing over different observers, with the sum taken over all permutations $\pi(k, l, m)$ of the indices (k, l, m) . For example, $\text{sym}[E_{011}] = E_{011} + E_{101} + E_{110}$, which corresponds to the permutations of $k = 0, l = 1, m = 1$.

We also provide a Mermin-type Bell expression, $\langle I_{ABC}^{(2)} \rangle \leq 0$, which leads to a critical detection efficiency of $\eta_{\text{crit}} = 0.75$. The Bell expression is given by

$$I_{ABC}^{(2)} = \tilde{I}_{222} + \tilde{J}_{222} - \tilde{K}_{22}, \quad (\text{B.8})$$

where

$$\begin{aligned} \tilde{I}_{222} &= E_{000} - \text{sym}[E_{011}], \\ \tilde{J}_{222} &= O_{111} + \text{sym}[O_{001}], \\ \tilde{K}_{22} &= (O_{AB} + O_{AC} + O_{BC})/2. \end{aligned} \quad (\text{B.9})$$

Notice that $\tilde{I}_{222}$ is a Mermin-type term [63], now defined in the space of three-outcome probability distributions.

B.2. Critical detection efficiency

Now, we compute η_{crit} corresponding to a quantum distribution generated using a rotated GHZ state and orthogonal observables in the Bell inequality (B.5). Note that the state below is equivalent to the original GHZ state under local unitaries and they are indistinguishable in the random measurements setting. The state under consideration is given by

$$|\text{GHZ}(\varphi)\rangle = \frac{|000\rangle + e^{i\varphi}|111\rangle}{\sqrt{2}}, \quad (\text{B.10})$$

where we fix $\varphi = \arctan(1/3)$, and every observer measures the same set of two observables:

$$A_0 = B_0 = C_0 = \sigma_x \text{ and } A_1 = B_1 = C_1 = \sigma_y. \quad (\text{B.11})$$

The three-party quantum correlations E_{xyz}^Q are then given by

$$\begin{aligned} E_{000}^Q &= \frac{3}{\sqrt{10}}, & E_{011}^Q &= E_{101}^Q = E_{110}^Q = -\frac{3}{\sqrt{10}}, \\ E_{111}^Q &= \frac{1}{\sqrt{10}}, & E_{100}^Q &= E_{010}^Q = E_{001}^Q = -\frac{1}{\sqrt{10}}. \end{aligned} \quad (\text{B.12})$$

For perfect detectors ($\eta = 1$) the third outcome never occurs, so that

$$O_{x,y,z}^Q = 1 \quad \text{for all } x, y, z \in \{0, 1\}. \quad (\text{B.13})$$

Furthermore, from the properties of the rotated GHZ state (B.10) and the fact that our observables are ± 1 -valued, one finds

$$O_{AB}^Q = O_{AC}^Q = O_{BC}^Q = 16 \times \frac{1}{4} = 4. \quad (\text{B.14})$$

By inserting these values into equations (B.12), (B.13), and (B.14), we obtain

$$I_{222}^Q = \sqrt{160}, \quad J_{222}^Q = 4, \quad \text{and} \quad K_{22}^Q = 12. \quad (\text{B.15})$$

Next, let us consider the case of finite efficiency of the detectors. In the symmetric scenario, i.e. $\eta_A = \eta_B = \eta_C = \eta$, the η -dependent quantities become

$$\begin{aligned} E_{xyz}^Q(\eta) &= \eta^3 E_{xyz}^Q, \\ O_{xyz}^Q(\eta) &= \eta^3 O_{xyz}^Q \end{aligned} \quad (\text{B.16})$$

for all $x, y, z \in \{0, 1\}$ and

$$\begin{aligned} O_{AB}^Q(\eta) &= \eta^2 O_{AB}^Q, \\ O_{AC}^Q(\eta) &= \eta^2 O_{AC}^Q, \\ O_{BC}^Q(\eta) &= \eta^2 O_{BC}^Q. \end{aligned} \quad (\text{B.17})$$

Plugging these values into equation (B.5) yields the quantum value as a function of η :

$$I_{Q,ABC}^{(1)}(\eta) = \eta^3 I_{222}^Q + \eta^3 J_{222}^Q + \eta^2 K_{22}^Q. \quad (\text{B.18})$$

For a non-zero quantum violation, this expression should exceed the local bound of zero. Hence, the critical detection efficiency η_{crit} is given by the condition $I_{Q,ABC}^{(1)}(\eta) = 0$. This leads to

$$\eta_{\text{crit}}^{(1)} = \frac{K_{22}^Q}{I_{222}^Q + J_{222}^Q} = \frac{\sqrt{10} - 1}{3} \approx 0.7208 \quad (\text{B.19})$$

for the Bell expression (B.5).

Concerning the Mermin-type expression (B.8), if we use the state (B.10) with $\varphi = 0$ and the observables given by equation (B.11), we have

$$\tilde{I}_{222}^Q = 4, \quad \tilde{J}_{222}^Q = 4, \quad \tilde{K}_{22}^Q = 6. \quad (\text{B.20})$$

Thus, the corresponding critical detection efficiency is

$$\eta_{\text{crit}}^{(2)} = \frac{\tilde{K}_{22}^Q}{\tilde{I}_{222}^Q + \tilde{J}_{222}^Q} = 0.75, \quad (\text{B.21})$$

This result has been proved to be optimal for the Mermin setup [16].

Appendix C. Probability of violation data tables

In table C1, we provide the data concerning the relative growth in the probability of violation for the maximally entangled two-qubit state and symmetric detection efficiency when increasing the number of random measurements from $m^A = m^B = m_1$ to m_2 . For a more detailed discussion, see section 4.

Table C1. Relative growth in the probability of violation for the maximally entangled two-qubit state and symmetric detection efficiency η when increasing the number of random measurements from $m^A = m^B = m_1$ to m_2 .

η	$\Delta \mathcal{P}_{m_1 \rightarrow m_2} [\%]$			
	$2 \rightarrow 3$	$3 \rightarrow 4$	$4 \rightarrow 5$	$5 \rightarrow 6$
0.83	1404.25	429.41	159.26	542.86
0.84	775.00	309.15	178.23	142.75
0.85	742.06	290.52	161.59	99.66
0.86	695.78	256.83	129.67	62.76
0.87	652.37	219.14	96.83	41.46
0.88	602.53	187.41	73.31	28.22
0.89	555.17	158.45	54.57	20.87
0.90	507.93	133.35	43.33	13.49
0.91	463.27	111.80	32.94	9.71
0.92	421.10	93.18	25.85	7.21
0.93	381.43	78.88	19.86	4.95
0.94	346.40	65.41	15.65	3.51
0.95	311.13	55.49	12.06	2.34
0.96	280.05	46.27	9.25	1.93
0.97	250.78	38.85	7.14	1.38
0.98	224.10	32.12	6.00	0.88
0.99	199.02	27.32	4.41	0.59
1.00	176.23	22.95	3.47	0.45

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Sum of Entanglement and Subsystem Coherence Is Invariant under Quantum Reference Frame Transformations

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Recent work on quantum reference frames (QRFs) has demonstrated that superposition and entanglement are properties that change under QRF transformations. Given their utility in quantum information processing, it is important to understand how a mere change of perspective can produce or reduce these resources. Here we prove the existence of a QRF invariant which can be decomposed such that it captures the trade-off between entanglement and subsystem coherence: we demonstrate the invariance of the sum of entanglement and subsystem coherence for two pairs of resource quantifiers. Moreover, we find a weaker trade-off that holds for any possible pair of measures. Finally, we discuss the implications of this interplay for violations of Bell's inequalities, clarifying that for any choice of QRF, there is a quantum resource responsible for the violation. These findings contribute to a better understanding of the quantum information theoretic aspects of QRFs, offering a foundation for future exploration in both quantum theory and quantum gravity.

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Introduction—The principle of covariance, which states that physical laws remain unchanged under a reference frame transformation, forms our fundamental understanding of the physical world. However, a variety of measurable physical quantities, such as energy, or magnetic and electric fields, are frame dependent. Similar considerations arise when investigating the behavior of quantum systems relative to a *quantum reference frame* (QRF). For example, a system that is well localized in space in one QRF may be in a superposition of different locations in a different QRF [1]. The same applies to quantum entanglement, making it a frame-dependent notion in quantum theory. Since both coherence and entanglement are recognized as useful resources in quantum information science, the question naturally arises, *how can a mere change of perspective from one frame of reference to another create a useful resource?*

In this Letter we give a natural answer to this question by showing that, under QRF transformations, there exists an invariant which can be decomposed such that it captures a trade-off between the entanglement and subsystem coherence, in the basis associated with the symmetry group of the transformation. We show that, for certain entanglement and coherence quantifiers, their sum is preserved under QRF changes, for any pure bipartite state. Thus, any entanglement increase under a QRF transformation comes at the expense of a decrease in coherence, and vice versa. Furthermore, we show that this property does not hold for all quantifier pairs: fixing an entanglement (or coherence) quantifier, there is at most one coherence (or entanglement) quantifier that yields an invariant sum.

The apparent “creation” of coherence or entanglement under a QRF change can then be explained by there existing either one of these resources in the original frame, and the change of frame transforming one into the other. This is analogous to the case of electrodynamics, where a magnetic field can appear when applying a Lorentz boost, as a consequence of an electric field being present in the rest frame. As Maxwell demonstrated, they are two sides of the same coin, just described in different frames: they are different components of the electromagnetic tensor. Here we show that coherence and entanglement exhibit a similar behavior under QRF transformations. Moreover, just like in

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electromagnetism, where there is a quantity combining both magnetic and electric fields that is left invariant under Lorentz boosts (the free-field Lagrangian density), here we identify meaningful quantities that are left invariant under QRF changes.

To demonstrate the significance of the trade-off between coherence and entanglement in the field of quantum information, we discuss violations of Bell's inequalities and clarify the potential misunderstandings regarding "production" or "destruction" of resources through QRF transformations. Different QRFs induce different factorizations of the Hilbert space, both for the states and for the observables [2–6], such that, for example, entangled states measured by local operations (as in the Bell experiment) transform into product states and nonseparable operations. Thus the resource of entanglement is transferred from the state to the observables.

The study of various notions of reference frames in quantum theory has been pursued through different approaches within the contexts of superselection rules and quantum information protocols [7–10], quantum gravity [11–13], and symmetries [14,15]. During a more recent wave of research, several approaches to QRFs have been formulated [1,3–5,16–19] and applied at the interface of quantum physics and gravity [6,20–28].

One particular question we address in this Letter is relevant to most of these approaches. *If both superposition and entanglement are each frame-dependent notions on their own, is there any physical quantity involving the two that is invariant under quantum frame changes?* A first step to answering this question was taken by Savi and Angelo [29], who found that the so-called "total quantumness" of the system is conserved under QRF changes. However, the conservation of "total quantumness" was shown using only the unitarity of QRF transformations, and not their specific form; the same result thus holds for any possible unitary transformation. In the present Letter, we show that the specific form of (ideal) QRF transformations [1,16] leaves the sum of entanglement and subsystem coherence invariant. Despite different motivations, at the technical level our result is closely related to quantum resource theory findings in Refs. [30,31]. These results can be recovered by restricting to finite cyclic QRF transformations and vanishing initial entanglement, suggesting that the coherence-entanglement link identified in Refs. [30,31] admits an interpretation in terms of invariance under QRF transformations. See also Refs. [32,33] for related studies on trade-offs between single-system and bipartite properties in multiqubits systems.

The Letter is structured as follows. Sec. I introduces the essential features of QRFs with a simple qubit example. Sec. II presents the general definition of QRF transformations, and states our main results: a conservation theorem for two quantifier pairs, along with a weaker trade-off for any pair. Proof ideas are outlined, with details in

Supplemental Material [34]. Sec. III discusses Bell inequality violations in QRFs, and Sec. IV concludes with open questions.

A simple example—Let us start with a simple example to build some intuition for changes of classical and quantum reference frame (cf. [16], Ex. 1)). Consider three systems A , B , and C , which can, for now, be in two states, $\uparrow$ or $\downarrow$. We can conventionally choose that each system considers themselves to be in the state $\uparrow$, similarly to how an astronaut freely floating in space would consider the direction "up" to be aligned from their toes to their head. Take the classical systems to be in the state $\uparrow_A^{(A)} \uparrow_B^{(A)} \downarrow_C^{(A)}$ relative to A . Then, the corresponding state relative to B is $\uparrow_B^{(B)} \uparrow_A^{(B)} \downarrow_C^{(B)}$, while that relative to C is $\uparrow_C^{(C)} \downarrow_A^{(C)} \downarrow_B^{(C)}$. Note how the *relative* configurations between the systems are left invariant under these classical changes of frame.

Let us now extend this to the case of *quantum* reference frames (cf. [16], Ex. 2)). For this, take A , B , and C to be quantum systems with Hilbert space $\mathbb{C}^2$, and basis states $|\uparrow\rangle$ and $|\downarrow\rangle$. Correspondingly, if the state relative to A is $|\psi\rangle_{ABC}^{(A)} = |\uparrow\rangle_A^{(A)} |\uparrow\rangle_B^{(A)} |\downarrow\rangle_C^{(A)}$, then the state relative to B is $|\psi\rangle_{ABC}^{(B)} = |\uparrow\rangle_B^{(B)} |\uparrow\rangle_A^{(B)} |\downarrow\rangle_C^{(B)}$ and that relative to C is $|\psi\rangle_{ABC}^{(C)} = |\uparrow\rangle_C^{(C)} |\downarrow\rangle_A^{(C)} |\downarrow\rangle_B^{(C)}$. Thus, A , B , and C can be understood as qubits described *relative* to one another, meaning that $|\psi\rangle_{ABC}^{(C)}$ encodes the relative degrees of freedom of A and B with respect to C [1].

Take now the state

$$|\psi\rangle_{ABC}^{(C)} = |\uparrow\rangle_C^{(C)} \frac{|\uparrow\rangle_A^{(C)} + |\downarrow\rangle_A^{(C)}}{\sqrt{2}} |\uparrow\rangle_B^{(C)}. \quad (1)$$

We can infer the corresponding state $|\psi\rangle_{ABC}^{(A)}$ by applying the principle of *coherent change of reference frame*, namely in each branch we apply the classical reference frame transformation described above. The result is that the state relative to A is

$$|\psi\rangle_{ABC}^{(A)} = |\uparrow\rangle_A^{(A)} \frac{|\uparrow\rangle_B^{(A)} |\uparrow\rangle_C^{(A)} + |\downarrow\rangle_B^{(A)} |\downarrow\rangle_C^{(A)}}{\sqrt{2}}. \quad (2)$$

The transformation between Eqs. (1) and (2) is a QRF transformation and is illustrated in Fig. 1.

Note, furthermore, that the conventional state of the reference frame with respect to itself can be omitted, as it does not contain any information, being always $|\uparrow\rangle$. We can then represent states of two systems relative to the third in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Furthermore, for future convenience, we identify the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ with the computational basis $\{|0\rangle, |1\rangle\}$, since they are isomorphic. In this simplified and more abstract notation, we can write,

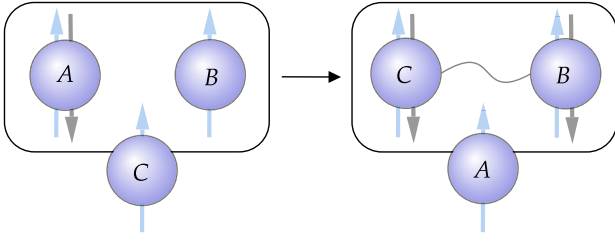


FIG. 1. Three qubits A , B , and C are described relative to one another. On the left, systems A and B are described relative to system C . While C is in the state $|\uparrow\rangle$ relative to itself, A is in a coherent superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ and B is in the state $|\uparrow\rangle$. Coherently changing reference frame to that of A , we find that B and C are in an entangled state. In general, superposition and entanglement depend on the QRF.

$$|\psi\rangle_{AB}^{(C)} = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} |0\rangle_B \quad (3)$$

and correspondingly

$$|\psi\rangle_{BC}^{(A)} = \frac{|0\rangle_B |0\rangle_C + |1\rangle_B |1\rangle_C}{\sqrt{2}}. \quad (4)$$

The transformation between the two states can be implemented with the unitary

$$S^{(C) \rightarrow (A)} = |0\rangle_{CA} \langle 0| \otimes \mathbb{1}_B + |1\rangle_{CA} \langle 1| \otimes \sigma_B^x, \quad (5)$$

meaning that the QRF transformation for these systems amounts to a quantum controlled flip, or CNOT, and can thus be seen as a refactorization of the Hilbert space. Note, however, that while it maps to a specific perspective relative to a single qubit, all measurements and operations are still performed in the laboratory frame.

While the state of A relative to C in Eq. (3) has some coherence (in the computational basis), the total state is factorized, that is, there is no entanglement between A and B relative to C . On the contrary, the state in the perspective of A in Eq. (4) is maximally entangled, and as such each subsystem is maximally mixed, meaning it has no coherence. Thus, the QRF transformation maps a factorized state with subsystem coherence to an entangled state with zero subsystem coherence. This is an instance of a generic feature of QRF transformations, which we prove as our main result: We show that the sum of entanglement and subsystem coherence is invariant under QRF transformations, for specific choices of entanglement and coherence quantifiers. Additionally, if there exists a frame in which either the coherence or the entanglement vanishes, a trade-off holds for any possible pair of measures.

Main results—Let us consider the most general ideal QRF transformation: Given a locally compact group G and a Hilbert space $\mathcal{H} = L^2(G) \otimes L^2(G)$, the most general QRF transformation can be written as [16]

$$S^{(C) \rightarrow (A)} = \int_G dg |g^{-1}\rangle_{CA} \langle g| \otimes U_B^\dagger(g), \quad (6)$$

with $\{|g\rangle\}_{g \in G}$ being the basis of $L^2(G)$, i.e., $\langle g|g'\rangle = \delta(g^{-1} \circ g')$, and $U(g)$ the left regular representation of G , such that $U(g)|g'\rangle = |g \circ g'\rangle$. In the case of discrete groups the integrals can be replaced by sums and the Hilbert space $L^2(G)$ with $\mathbb{C}[G]$. For instance, in the illustrative example above, the group is $G = \mathbb{Z}_2$ and $\mathcal{H} = \mathbb{C}[\mathbb{Z}_2] \cong \mathbb{C}^2$.

Given a generic bipartite pure state in the perspective of C , namely,

$$|\psi\rangle_{AB}^{(C)} = \int_G dg dg' \psi_{AB}^{(C)}(g, g') |g\rangle_A |g'\rangle_B, \quad (7)$$

we can calculate what the state in the perspective of A is, namely,

$$\begin{aligned} |\psi\rangle_{BC}^{(A)} &= S^{(C) \rightarrow (A)} |\psi\rangle_{AB}^{(C)} \\ &= \int_G dg dg' \psi_{BC}^{(A)}(g, g') |g\rangle_A |g'\rangle_B, \end{aligned} \quad (8)$$

with $\psi_{BC}^{(A)}(g, g') = \psi_{AB}^{(C)}(g^{-1}, g^{-1} \circ g')$.

Upon direct application of the definition, QRF transformations can be shown to be incoherent operations (see Supplemental Material A [34]); i.e., they cannot increase the coherence of a bipartite state—and in particular they preserve it. Nevertheless, when studying a subsystem only, QRF transformations can increase or decrease its coherence, as pointed out in the previous example. Hence, it is meaningful to characterize the amount of subsystem coherence before and after the QRF change, as well as the entanglement of the bipartite state, which can change as well.

To begin, we study entanglement entropy and relative entropy of coherence as our resource measures. The entanglement entropy is defined as $\mathcal{E}_e[|\psi\rangle_{AB}] = S[\rho_A]$, where $\rho_A = \text{Tr}_B[|\psi\rangle_{AB} \langle \psi|]$ is the reduced density matrix of the subsystem A and $S[\rho] = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy. The relative entropy of coherence is $\mathcal{C}_e[\rho] = S[\rho_d] - S[\rho]$, where ρ_d is the state obtained from ρ by keeping only the diagonal terms in the group element basis $\{|g\rangle\}_{g \in G}$. Intuitively, $\mathcal{C}$ quantifies how large the off-diagonal elements of ρ are in the given basis.

Given the entanglement in the perspective of C , i.e., $\mathcal{E}_e^{(C)} = \mathcal{E}_e[|\psi\rangle_{AB}^{(C)}]$, the coherence of subsystem A in the perspective of C , i.e., $\mathcal{C}_e^{(C)} = \mathcal{C}_e[\rho_A^{(C)}]$, where $\rho_A^{(C)} = \text{Tr}_B[\rho_{AB}^{(C)}]$, and analogous definitions for the perspective of A , namely $\mathcal{E}_e^{(A)} = \mathcal{E}_e[|\psi\rangle_{BC}^{(A)}]$ and $\mathcal{C}_e^{(A)} = \mathcal{C}_e[\rho_C^{(A)}]$, the following holds:

$$\mathcal{C}_e^{(C)} + \mathcal{E}_e^{(C)} = \mathcal{C}_e^{(A)} + \mathcal{E}_e^{(A)}. \quad (9)$$

The proof relies on showing that $\rho_A^{(C)}$ has the same diagonal elements as $\rho_C^{(A)}$ up to a permutation (see Supplemental Material B [34]), and hence the entropy of the diagonal states is the same before and after the transformation. Another consequence of the above is an additional conservation law, with a different pair of coherence and entanglement quantifiers: given the l^2 -norm of coherence [46] $\mathcal{C}_l[\rho] = \sum_{g \neq h} |\rho_{g,h}|^2$ and the linear entropy of the subsystem $\mathcal{E}_l[|\psi_{AB}\rangle] = 1 - \text{Tr}[\rho_A^2]$, it holds that

$$\mathcal{C}_l^{(C)} + \mathcal{E}_l^{(C)} = \mathcal{C}_l^{(A)} + \mathcal{E}_l^{(A)}. \quad (10)$$

The proofs of Eqs. (9) and (10) are detailed in Supplemental Material C1 and C2 [34]. We formulate our main result in the form of a theorem.

Theorem 1—The sum of entanglement and subsystem coherence $\mathcal{E} + \mathcal{C}$ is invariant under QRF transformations of pure bipartite states, for $(\mathcal{E}, \mathcal{C}) = (\mathcal{E}_e, \mathcal{C}_e)$ or $(\mathcal{E}_l, \mathcal{C}_l)$.

Our theorem demonstrates the invariance of the sum of entanglement and subsystem coherence under QRF changes, for any pure bipartite state. Moreover, in Supplemental Material C3 [34], we prove that the pairs of quantifiers in Theorem 1 cannot be chosen arbitrarily: for each entanglement (or coherence) quantifier, there is at most one coherence (or entanglement) quantifier such that their sum is conserved. This provides insight into which quantifiers can form a QRF invariant.

Theorem 1 entails a trade-off between the two quantities: a QRF transformation that increases entanglement only does so by consuming the existing coherence, and vice versa. In other words, it follows from Eqs. (9) and (10) that for both choices of quantifiers, when coherence increases under a change of frame, entanglement decreases, and vice versa, i.e., $\Delta \mathcal{C}_e \Delta \mathcal{E}_e \leq 0$ and $\Delta \mathcal{C}_l \Delta \mathcal{E}_l \leq 0$, with $\Delta \mathcal{C}_e = \mathcal{C}_e^{(A)} - \mathcal{C}_e^{(C)}$, and similarly for the others. Note, however, that even this weaker trade-off does not hold for any pair of coherence and entanglement quantifiers; a numerical example where $\Delta \mathcal{C}_l \Delta \mathcal{E}_e \geq 0$ is given in Supplemental Material C4 [34]. This means that there is a pair of measures such that coherence and entanglement can both increase under a QRF change. This is connected to the property of different entanglement and coherence measures not being monotones with respect to each other (see Refs. [52,53]): they do not order states such that, given two different measures $\mathcal{C}_1$ and $\mathcal{C}_2$, for any two pairs of states such that $\Delta \mathcal{C}_1 < 0$, it is also true that $\Delta \mathcal{C}_2 < 0$.

Nevertheless, it is still possible to find a trade-off for any possible pair of measures, for some specific set of states. This can be formulated as follows.

Theorem 2—For any choice of entanglement measure $\mathcal{E}$ and coherence measure $\mathcal{C}$, if there is a QRF where a pure bipartite state has $\mathcal{E} = 0$ or $\mathcal{C} = 0$, under a QRF transformation it holds that $\Delta \mathcal{E} \Delta \mathcal{C} \leq 0$.

The proof is analogous to that of Theorem 1 and is detailed in Supplemental Material C4 [34].

In the more general case of imperfect QRFs, where different frame orientations are assigned nonorthogonal states, both theorems fail, as detailed in Supplemental Material D [34].

Bell tests in QRFs—Nonlocal quantum states violating Bell’s inequalities are a key quantum information resource [54], including applications in quantum cryptography [55–59], randomness generation [60–63], and communication [64–67]. As discussed above, a change of QRF can lead to a decrease in entanglement and hence in the degree of nonlocality of quantum states, at the expense of a corresponding increase in coherence. However, since nonlocality is independent of single-particle coherence, it may appear lost or gained under QRF changes. If this hypothesis is indeed correct, it would suggest that those QRFs in which states manifest nonlocality are “preferred,” which would challenge the view that all QRFs are equally legitimate perspectives.

In the following we will show that the aforementioned hypothesis is incorrect. A set of probabilities that violates Bell’s inequalities in one QRF will violate them in any other QRF, since probabilities are invariant under QRF changes. While the resource for this violation may be entanglement of states in one QRF, it can be the nonseparability of operations in another QRF. In this sense the QRF change can transfer the “nonlocality of states” in one QRF into the “nonlocality of operations” in another QRF.

Bell’s local causality is a condition on joint probabilities

$$p(a, b|x, y) = \sum_{\lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda), \quad (11)$$

where $p(\lambda) \geq 0$, $\sum_{\lambda} p(\lambda) = 1$ is a probability distribution of the “hidden variable” and a and b are outcomes in Alice’s and Bob’s laboratories when they choose settings x and y , respectively. The probabilities that satisfy Bell’s local causality satisfy Bell’s inequalities.

In quantum mechanics, Bell’s inequalities can be violated, which demonstrates the incompatibility of Bell’s local causality and quantum mechanics. To demonstrate the violation, Alice and Bob may share a pair of particles in an entangled quantum state and perform local operations on them. We refer to this as the “nonlocality of states”.

We now proceed to an analysis of how the resources required for a violation of Bell’s inequalities change under QRF transformations. We begin our analysis in QRF C. Suppose that in this frame the quantum state $\rho_{AB}^{(C)}$ of the composite system A and B is nonlocal in the sense that there are local POVM elements $A_{a|x}^{(C)}$, $B_{b|y}^{(C)}$ such that

$$\text{Tr} \left[\rho_{AB}^{(C)} \left(A_{a|x}^{(C)} \otimes B_{b|y}^{(C)} \right) \right] \neq \sum_{\lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda). \quad (12)$$

The change of QRF results in a refactorization of the Hilbert space from $\mathcal{H}_{AB}$ to $\mathcal{H}_{CB}$, encompassing both the states and the observables. In general, the initial entangled state in QRF C can be mapped to a separable one in QRF A while products of POVMs are in general mapped into nonseparable POVMs,

$$S^{(C) \rightarrow (A)} \rho_{AB}^{(C)} (S^{(C) \rightarrow (A)})^\dagger = \rho_C^{(A)} \otimes \rho_B^{(A)} \quad (13)$$

$$(S^{(C) \rightarrow (A)})^\dagger \left(A_{a|x}^{(C)} \otimes B_{b|y}^{(C)} \right) S^{(C) \rightarrow (A)} = D_{a|x, b|y}^{(A)}, \quad (14)$$

where, in general, $D_{a|x, b|y}^{(A)} \neq \sum_j q_j C_{j, a|x}^{(A)} \otimes B_{j, b|y}^{(A)}$, with $q_j \geq 0$ and $\sum_j q_j = 1$.

We note that as the change of QRF is a unitary transformation, the probabilities for measurement outcomes remain the same before and after the change. Hence, although the state in QRF A is a local one, the probabilities still display nonlocality since

$$\text{Tr} \left[\left(\rho_C^{(A)} \otimes \rho_B^{(A)} \right) D_{a|x, b|y}^{(A)} \right] \neq \sum_\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda). \quad (15)$$

This is because the operations in QRF A are in general not in a separable form. In fact, if a state in QRF C violates a Bell's inequality for some local POVMs, then the whole set of transformed POVMs in QRF A cannot be separable, because otherwise one would not violate Bell's inequalities in that frame. We refer to this violation as resulting from “nonlocality of operations”. We conclude that the change from one to another QRF results in a change from “nonlocality of states” to “nonlocality of operations”.

We illustrate our analysis with an example. Consider the test of Clauser-Horne-Shimony-Holt (CHSH) inequality [54,68]:

$$\sum_{x,y=0}^1 p(a \oplus b = xy|x, y) \leq 3, \quad (16)$$

where $p(a \oplus b = xy|x, y)$ is the probability to obtain outcomes $a \oplus b$ to be equal to the product xy , if the choice of the settings is the pair x, y . It is known that if Alice and Bob share an entangled state of qubits A and B , as described by a reference qubit C ,

$$|\psi\rangle_{AB}^{(C)} = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}, \quad (17)$$

and measure eigenprojectors of a product of spins $\sigma_{\vec{m}} \otimes \sigma_{\vec{n}}$ along directions $\vec{m}$ and $\vec{n}$, they can maximally violate the CHSH inequality by $2 + \sqrt{2}$.

Let us now analyze the CHSH test from the perspective of QRF A . Note that performing a QRF change to the frame

of A , the state of B and C becomes

$$|\psi\rangle_{BC}^{(A)} = |0\rangle_B \frac{|0\rangle_C + |1\rangle_C}{\sqrt{2}}. \quad (18)$$

As we can expect from our results, the entanglement in the perspective of C has been transformed into coherence in the perspective of A , and now Alice and Bob share a separable state. However, the observables also change. As an example, the observable $\sigma_x \otimes [(\sigma_x + \sigma_z)/\sqrt{2}]$, which is factorized in the perspective of C , here becomes non-separable, i.e., $(S^{(C) \rightarrow (A)})^\dagger \sigma_x \otimes [(\sigma_x + \sigma_z)/\sqrt{2}] S^{(C) \rightarrow (A)} = 1/\sqrt{2}(\sigma_x \otimes \mathbb{1} - \sigma_y \otimes \sigma_y)$, in agreement with the general analysis given above.

Conclusion—Quantum coherence and entanglement are fundamentally intertwined. In this Letter, we explored this connection in the framework of QRFs, focusing on bipartite pure states. We proved the existence of a QRF invariant and showed that it can be decomposed in a way that captures the connection between coherence and entanglement. This was explicitly demonstrated for two pairs of quantifiers, showing that the sum of entanglement and subsystem coherence is conserved under QRF changes, implying that an increase of coherence under QRF change must come at the expense of a decrease in entanglement, and vice versa. Furthermore, we showed that a weaker trade-off holds for any possible pair of measures when there exists a QRF in which either the coherence or the entanglement vanishes.

Finally, we discussed violations of Bell's inequalities in different QRFs and highlighted the role of different quantum resources for the violation in different QRFs. As an example, we show that the entangled states and separable observables that violate the CHSH inequality are mapped to factorized states and nonseparable observables that violate the same inequality.

In Supplementary Material [34], we prove all statements and theorems from the main text and investigate imperfect QRFs.

Future work could extend the present results to multipartite entanglement involving any number of parties. Moreover, generalizing our results beyond the case of locally compact groups to the group of diffeomorphisms holds significant potential for insights into scenarios involving quantum gravity. Our findings suggest that while entanglement and subsystem coherence individually remain invariant under a classical diffeomorphism, their combined sum is preserved under *quantum* diffeomorphisms. This extension would imply that $\mathcal{C} + \mathcal{E}$ is an observable within the framework of general relativity, opening up exciting avenues for exploring the implications of this insight in the realm of quantum gravity.

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Appendix A: QRF transformations are incoherent operations

In this Supplemental Material we show that QRF transformations, defined in Eq. (6), are incoherent operations. We focus on the case of finite Hilbert spaces, to avoid possible divergences in the coherence quantifiers. Note, however, that the result also holds in the infinite dimensional case. Given a Hilbert space and an orthonormal basis $\{|i\rangle\}_{i=0,\dots,d-1}$, a state is called incoherent in that basis if it can be written as $\sigma = \sum_i \sigma_i |i\rangle\langle i|$ (see Ref. [1] for an overview). We refer to the set of incoherent states as $\mathcal{I}$. In turn, a CPTP map Λ is called incoherent if, given its Kraus decomposition, i.e. $\Lambda[\rho] = \sum_i K_i \rho K_i^\dagger$, each Kraus operator maps incoherent states to incoherent states, i.e. $\forall i \ K_i \mathcal{I} K_i^\dagger \subset \mathcal{I}$. As a simpler case, a unitary transformation is called incoherent if it maps incoherent states to incoherent states. We now show that this is the case for QRF transformations.

Let us consider a generic incoherent state in the perspective of C , i.e.

$$\rho_{AB}^{(C)} = \sum_{g,g'} P_{g,g'} |g\rangle\langle g| \otimes |g'\rangle\langle g'|. \quad (\text{A1})$$

The QRF transformation of Eq. (6) can be written in the discrete case as $S = \sum_g |g^{-1}\rangle\langle g| \otimes U(g^{-1})$. By direct application, and remembering that $U(g)|h\rangle = |g \circ h\rangle$, we find

$$S \rho_{AB}^{(C)} S^\dagger = \sum_{g,g'} P_{g,g'} |g^{-1}\rangle\langle g^{-1}| \otimes |g^{-1} \circ g'\rangle\langle g^{-1} \circ g'| \quad (\text{A2})$$

$$= \sum_{g,g'} P_{g^{-1}, g^{-1} \circ g'} |g\rangle\langle g| \otimes |g'\rangle\langle g'|, \quad (\text{A3})$$

where the second line has been obtained after two changes of variables. It is then clear that the resulting state is still incoherent, since the effect of the QRF transformation has only been the permutation of the diagonal elements.

This means that QRF transformations are indeed incoherent. By unitarity, they also have to preserve the coherence of the bipartite state [2]. This characterization provides a deep insight into the structure of the studied transformations. Within the QRF formalism, no additional coherence of the bipartite state can be created - *the total coherence is preserved* for any coherence measure. Furthermore, new possibilities for studying quantum resources associated with QRF transformations arise by using results from resource theory. One example could be the application of Strelstov's theorem [3], which concerns a wide family of coherence and entanglement measures based on distances, and allows one to upper bound the amount of entanglement present in the system by its coherence.

Note, however, that the coherence of each subsystem can change through a QRF transformation, and this is what is studied in the main text.

Appendix B: Conservation of diagonal elements up to permutation

1. Continuous case

We present here a lemma that leads to the main results of our work (Theorem 1 and Theorem 2). Given a group G , a QRF transformation S associated to the group, an initial and final state $|\psi\rangle_{AB}^{(C)}$, $|\psi\rangle_{BC}^{(A)} = S|\psi\rangle_{AB}^{(C)}$, and the respective reduced density matrices $\rho_A^{(C)} = \text{Tr}_B[|\psi\rangle_{AB}^{(C)}\langle\psi|]$, $\rho_C^{(A)} = \text{Tr}_B[|\psi\rangle_{BC}^{(A)}\langle\psi|]$, we show that the diagonal elements of the two reduced density matrices are the same up to permutation. We can express this statement as a set equality:

$$\left\{ \langle g | \rho_A^{(C)} | g \rangle \mid g \in G \right\} = \left\{ \langle g | \rho_C^{(A)} | g \rangle \mid g \in G \right\}. \quad (\text{B1})$$

This can be proven as follows. Let us start with the state in the perspective of C defined in Eq. (7). Tracing out subsystem B we get

$$\rho_A^{(C)} = \int dg \int dg' \int dh \psi_{AB}^{(C)}(g, h) \psi_{AB}^{*(C)}(g', h) |g\rangle\langle g'|. \quad (\text{B2})$$

The set of its diagonal elements is thus given by

$$D^{(C)} = \left\{ \langle g | \rho_A^{(C)} | g \rangle = \int dh |\psi_{AB}^{(C)}(g, h)|^2 \mid g \in G \right\}. \quad (\text{B3})$$

Similarly, for $\rho_C^{(A)}$ we obtain

$$D^{(A)} = \left\{ \langle g | \rho_C^{(A)} | g \rangle = \int dh |\psi_{BC}^{(A)}(g, h)|^2 \mid g \in G \right\}. \quad (\text{B4})$$

From the definition of a QRF transformation in Eq. (6) we have

$$\langle g | \rho_C^{(A)} | g \rangle = \int dh |\psi_{BC}^{(A)}(g, h)|^2 = \int dh |\psi_{AB}^{(C)}(g^{-1}, g^{-1} \circ h)|^2 = \int dh |\psi_{AB}^{(C)}(g^{-1}, h)|^2 = \langle g^{-1} | \rho_A^{(C)} | g^{-1} \rangle. \quad (\text{B5})$$

Since for each element of the group G there exists a unique inverse in G , then $D^{(C)} = D^{(A)}$, and Eq. (B1) is proven.

The above observation can be used to derive the conservation laws in the main text, and the derivation is discussed in detail in Supplemental Material C. Note that this result holds in the case of ideal reference frames, but there are also more general types of quantum reference frames for which this result does not hold, as discussed in Supplemental Material D.

2. Discrete case

We provide here an alternative proof for discrete groups, which highlights the role of permutations in QRF transformations. Let us again consider three parties, A , B and C , and the most general discrete pure state with respect to C :

$$|\psi\rangle_{AB}^{(C)} = \sum_{g,h} C_{g,h} |g\rangle_A |h\rangle_B. \quad (\text{B6})$$

Here, $C_{g,h}$ is such that the state is normalized, namely $\sum_g (CC^\dagger)_{g,g} = 1$. Applying the QRF transformation, we get the state with respect to A ,

$$|\psi\rangle_{BC}^{(A)} = \sum_{g,h} C_{g,h} |g^{-1}\rangle_C |g^{-1} \circ h\rangle_B = \sum_{g,h} C_{g^{-1}, g^{-1} \circ h} |g\rangle_C |h\rangle_B := \sum_{g,h} D_{g,h} |g\rangle_C |h\rangle_B. \quad (\text{B7})$$

Tracing out the second subsystem from the perspective of C , we get

$$\rho_A = \text{Tr}_B [|\psi\rangle_{AB}^{(C)} \langle \psi|] = \sum_{g,h} (CC^\dagger)_{g,h} |g\rangle \langle h|, \quad (\text{B8})$$

and the diagonal part $\rho_{A,d} = \sum_g (CC^\dagger)_{g,g} |g\rangle \langle g|$. Analogously, starting from the state with respect to A , performing a QRF change to C , and applying a partial trace, we get

$$\rho_{C,d} = \sum_g (DD^\dagger)_{g,g} |g\rangle \langle g|. \quad (\text{B9})$$

Here, we want to show that

$$(DD^\dagger)_{g,g} = (CC^\dagger)_{\pi(g)\pi(g)}, \quad \forall g, \quad (\text{B10})$$

where $\pi(\cdot)$ is a permutation of the elements of G . Note that, by the definition of QRF transformations, we have

$$D_{g,h} = C_{g^{-1}, g^{-1} \circ h}. \quad (\text{B11})$$

A QRF change thus consists of applying a permutation $P^{(i)}$ on the columns of the density matrix, such that each element g gets mapped to its inverse g^{-1} , and a controlled permutation on the rows, such that each row undergoes a different permutation (this is the effect of the controlled unitary on the second system). That is,

$$D = P^{(i)} \left(\sum_{j=1}^{|G|} A^{(j)} C P^{(j)} \right), \quad (\text{B12})$$

where $P^{(i)}$ and $P^{(j)}$ are permutations matrices, $A^{(j)} = \text{diag}(\{\delta_{jk}\}_{k=1,\dots,|G|})$ are the control elements, i.e. diagonal matrices with only one element equal to 1 and the other elements being zero. The permutation matrices are orthogonal, i.e. $PP^T = 1$, and the control matrices are symmetric, that is, $A = A^T$. Overall, this yields

$$DD^\dagger = P^{(i)} \left(\sum_{j,l} A^{(j)} C P^{(j)} P^{(l)T} C^\dagger A^{(l)} \right) P^{(i)T}. \quad (\text{B13})$$

Note that given matrices M and N , if $N = PMP^T$, then $N_{g,g} = M_{\pi(g),\pi(g)}$, that is, the permutation acts in such a way that the diagonal elements are permuted between each other. Hence we obtain

$$(DD^\dagger)_{g,g} = \left(\sum_{j,l} A^{(j)} C P^{(j)} P^{(l)T} C^\dagger A^{(l)} \right)_{\pi(g),\pi(g)}, \quad (\text{B14})$$

where the permutation $P^{(i)}$ is the one that sends each element to its inverse, namely, $\pi(g)$ is the inverse of g .

Furthermore, we use the property that when $A^{(j)}$ and $A^{(l)}$ are applied to the left and right of a matrix, they select one element, namely $A^{(j)} M A^{(l)} = M_{j,l} S^{(j,l)}$, where $S^{(j,l)}$ is the matrix that has a 1 in position j, l , and zero otherwise: $S_{a,b}^{(j,l)} = \delta_{j,a} \delta_{l,b}$. As a result,

$$(DD^\dagger)_{gg} = \sum_{j,l} (C P^{(j)} P^{(l)T} C^\dagger)_{j,l} S_{\pi(g),\pi(g)}^{(j,l)} = (C C^\dagger)_{\pi(g),\pi(g)}, \quad (\text{B15})$$

which proves our statement.

Appendix C: Proofs of theorems

1. Proof of Theorem 1: Entanglement entropy and coherence entropy

To formulate the first conservation relation we use the *entanglement entropy* and the *relative entropy of coherence*. Given a pure bipartite state $|\psi\rangle_{AB}$ and a density matrix ρ of arbitrary dimension, the measures are defined as [1, 4]

$$\mathcal{E}_e[|\psi\rangle_{AB}] = \mathcal{S}[\rho_A], \quad (\text{C1})$$

$$\mathcal{C}_e[\rho] = \mathcal{S}[\rho_d] - \mathcal{S}[\rho], \quad (\text{C2})$$

where $\mathcal{S}[\rho] = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy, $\rho_A = \text{Tr}_B[|\psi\rangle_{AB}\langle\psi|]$ is the reduced density matrix of subsystem A , and the subscript d denotes the diagonal part of the state in the group basis, that is, $\langle g|\rho_d|g'\rangle = \delta(g^{-1} \circ g') \langle g|\rho|g\rangle$.

If we calculate the coherence of the reduced density matrix $\rho_A^{(C)} = \text{Tr}_B[|\psi\rangle_{AB}^{(C)}\langle\psi|]$, and of $\rho_C^{(A)}$, we find

$$\mathcal{C}_e[\rho_A^{(C)}] + \mathcal{E}_e[|\psi\rangle_{AB}^{(C)}] = \mathcal{S}[\rho_{A,d}^{(C)}], \quad (\text{C3})$$

$$\mathcal{C}_e[\rho_C^{(A)}] + \mathcal{E}_e[|\psi\rangle_{BC}^{(A)}] = \mathcal{S}[\rho_{C,d}^{(A)}]. \quad (\text{C4})$$

We now want to show that these two quantities are equal, as in Eq. (9) in the main text. This happens when $\mathcal{S}[\rho_{A,d}^{(C)}] = \mathcal{S}[\rho_{C,d}^{(A)}]$, which can be proved as follows. By construction, $\rho_{A,d}^{(C)}$ has the same diagonal elements as $\rho_A^{(C)}$, while all the other elements are zero, hence the diagonal elements of $\rho_A^{(C)}$ are the eigenvalues of $\rho_{A,d}^{(C)}$ (and analogously for $\rho_C^{(A)}$ and $\rho_{C,d}^{(A)}$). Furthermore, the lemma of Eq. (B1) states that $\rho_A^{(C)}$ and $\rho_C^{(A)}$ have the same diagonal elements up to a permutation, and hence $\rho_{A,d}^{(C)}$ and $\rho_{C,d}^{(A)}$ have the same eigenvalues up to a permutation. Since the von Neumann entropy of a state depends only on the set of eigenvalues of the state, it follows that the two von Neumann entropies are the same, proving Eq. (9) in the main text.

2. Proof of Theorem 1: Linear entropy and l^2 coherence

As for the second conservation relation, we consider as coherence and entanglement quantifiers the l^2 -norm of coherence $\mathcal{C}_{l^2}$ and the linear entropy of the subsystem $\mathcal{E}_l$, respectively. Note that the latter can be treated as a rescaled square of the entanglement concurrence. Once again, we focus here on discrete groups and finite Hilbert spaces. These quantifiers are given by [1, 5]

$$\mathcal{E}_l[|\psi\rangle_{AB}] = 1 - \text{Tr}[\rho_A^2], \quad (\text{C5})$$

$$\mathcal{C}_{l^2}[\rho] = \sum_{i \neq j} |\rho_{ij}|^2. \quad (\text{C6})$$

The above l^2 coherence can be easily decomposed into two terms: $\mathcal{C}_{l^2}[\rho] = \sum_{i,j} |\rho_{ij}|^2 - \sum_i |\rho_{ii}|^2$. The first term is actually the state's purity $\text{Tr}[\rho^2]$ and the latter is sometimes referred to as the ‘‘classical purity’’. By the lemma of Eq. (B1), the second quantity is conserved under QRF transformations. In other words, the quantity

$$\mathcal{C}_{l^2}[\rho_A^{(C)}] - \text{Tr}[(\rho_A^{(C)})^2] = - \sum_{g \in G} |(\rho_A^{(C)})_{g,g}|^2 \quad (\text{C7})$$

is invariant under QRF transformations. Finally, adding 1 to both sides of the above equation and writing the same equation in the reference frame of A , one finds

$$\mathcal{C}_{l^2}[\rho_A^{(C)}] + \mathcal{E}_l[|\psi\rangle_{AB}^{(C)}] = \mathcal{C}_{l^2}[\rho_C^{(A)}] + \mathcal{E}_l[|\psi\rangle_{BC}^{(A)}], \quad (\text{C8})$$

which is Eq. (10) in the main text.

We have thus found two conservation relations, using two different pairs of measures for coherence and entanglement. Note, however, that it is not possible to find the same conservation law for any possible pair of measures: an example is given in the next section, where $\mathcal{C}_{l^2} + \mathcal{E}_e$ is not conserved, and not even the weaker condition $\Delta\mathcal{C}_{l^2}\Delta\mathcal{E}_e \leq 0$ holds.

Nevertheless, the lemma of Eq. (B1) can be used to prove other entanglement-coherence trade-offs, possibly in a more convoluted form. To give an example, it is possible to find another conservation relation for two qubits. Let us consider the geometric measure of entanglement [6] $\mathcal{E}_g[|\psi\rangle_{AB}] = 1 - \frac{1}{2} \left(1 + \sqrt{1 - 4\det(\rho_A)} \right)$ and the geometric coherence [3] $\mathcal{C}_g[\rho] = 1/2 \left(1 - \sqrt{1 - 4|\rho_{01}|^2} \right)$. We can write

$$\mathcal{E}_g[|\psi\rangle_{AB}^{(C)}] = 1 - \frac{1}{2} \left(1 + \sqrt{1 + 2\mathcal{C}_g[\rho_A^{(C)}] \left(1 - 2\mathcal{C}_g[\rho_A^{(C)}] \right) - 4\rho_{00}\rho_{11}} \right) \quad (\text{C9})$$

$$= 1 - \frac{1}{2} \left(1 + \sqrt{1 + 2\mathcal{C}_g[\rho_A^{(C)}] \left(1 - 2\mathcal{C}_g[\rho_A^{(C)}] \right) - I} \right), \quad (\text{C10})$$

where I is a QRF invariant, due to the lemma of Eq. (B1). Solving the above equation for I , one could arrive at yet another entanglement-coherence trade-off relation.

3. Uniqueness proof

We prove here that once an entanglement (or coherence) quantifier is chosen, there cannot be more than one coherence (or entanglement) quantifier such that their sum is QRF invariant. We focus on the proof with a fixed entanglement quantifier, but a completely analogous proof holds for the case with a fixed coherence quantifier.

Specifically, we prove that given $\mathcal{E}_A$ and $\mathcal{C}_A$ such that their sum is invariant under QRF changes, i.e.

$$\mathcal{E}_A + \mathcal{C}_A = k_A, \quad (\text{C11})$$

where k_A is a QRF invariant, then there cannot exist another $\mathcal{C}_B \neq \mathcal{C}_A$, such that $\mathcal{E}_A + \mathcal{C}_B = k_B$, with k_B a QRF invariant.

Let us prove this by contradiction: assuming that $\mathcal{C}_B$ exists such that $\mathcal{E}_A + \mathcal{C}_B = k_B$ entails that

$$\mathcal{C}_B = \mathcal{C}_A + K, \quad (\text{C12})$$

where $K = k_B - k_A$ is a QRF invariant.

From this relation, it follows that $\mathcal{C}_B$ is a coherence quantifier if and only if K is. But K is by assumption a QRF invariant, and since coherence changes under QRF transformations, K cannot be a coherence quantifier (for example, it can be non-zero for incoherent states). As a result, there cannot exist $\mathcal{C}_B \neq \mathcal{C}_A$ such that $\mathcal{E}_A + \mathcal{C}_B = k_B$, and this concludes the proof.

4. Counterexample for a generic inequality

We have shown that there are two conserved quantities under QRF transformations, namely Eqs. (9) and (10) in the main text. They imply that the following two inequalities hold: $\Delta \mathcal{C}_e \Delta \mathcal{E}_e \leq 0$ and $\Delta \mathcal{C}_{l^2} \Delta \mathcal{E}_l \leq 0$. This, however, does not mean that for any possible pair of measures, one can obtain an analogous inequality. For example, it does not hold true that for any state $\Delta \mathcal{C}_{l^2} \Delta \mathcal{E}_e \leq 0$. Here we provide a counterexample of a state and a QRF transformation for which the inequality is violated.

Let us consider the cyclic group $G = \mathbb{Z}_3$, a basis associated to the group elements $\{|0\rangle, |1\rangle, |2\rangle\}$, and a unitary representation of the group $U(g)$, acting as $U(g)|g'\rangle = |(g + g') \bmod 3\rangle$. A counter example is an initial state in the perspective of C , i.e. $|\psi\rangle_{AB}^{(C)} = \sum_{i,j=0}^2 \psi_{i,j}^{(C)} |ij\rangle_{AB}$, where the coefficients are (approximately) given by

$$\psi^{(C)} = \begin{bmatrix} +0.20 - 0.05i & -0.22 + 0.30i & -0.28 + 0.12i \\ -0.36 + 0.21i & -0.15 - 0.24i & +0.25 + 0.12i \\ -0.41 + 0.31i & +0.04 - 0.09i & -0.08 - 0.34i \end{bmatrix}.$$

After the QRF change, one finds the state

$$\psi^{(A)} = \begin{bmatrix} +0.20 - 0.05i & -0.22 + 0.30i & -0.28 + 0.12i \\ +0.25 + 0.12i & -0.36 + 0.21i & -0.15 - 0.24i \\ +0.04 - 0.09i & -0.08 - 0.34i & -0.41 + 0.31i \end{bmatrix}.$$

By explicit computation of the measures, one finds $\mathcal{E}_e^{(C)} = 0.76$, $\mathcal{C}_{l^2}^{(C)} = 0.23$, $\mathcal{E}_e^{(A)} = 0.69$, $\mathcal{C}_{l^2}^{(A)} = 0.17$, hence $\Delta \mathcal{E}_e \Delta \mathcal{C}_{l^2} = 0.004 > 0$.

5. Proof of Theorem 2

We provide here a proof of Theorem 2, which states that, given a bipartite pure state $|\psi\rangle_{AB}^{(C)}$ in the frame of C and its transformed version $|\psi\rangle_{BC}^{(A)} = S^{(C) \rightarrow (A)} |\psi\rangle_{AB}^{(C)}$ in the frame of A , for any possible choice of coherence and entanglement measures $\mathcal{C}$ and $\mathcal{E}$, if there exists a frame in which $\mathcal{C}[\rho_A^{(C)}] = 0$ or $\mathcal{E}[|\psi\rangle_{AB}^{(C)}] = 0$, with $\rho_A^{(C)} = \text{Tr}_B[|\psi\rangle_{AB}^{(C)}\langle\psi|]$ and similarly for $\rho_C^{(A)}$, then

$$\left(\mathcal{C}[\rho_C^{(A)}] - \mathcal{C}[\rho_A^{(C)}]\right) \left(\mathcal{E}[|\psi\rangle_{BC}^{(A)}] - \mathcal{E}[|\psi\rangle_{AB}^{(C)}]\right) \leq 0. \quad (\text{C13})$$

From now on, we will adopt the same convention of the main text, and call $\mathcal{C}^{(C)} = \mathcal{C}[\rho_A^{(C)}]$, $\mathcal{E}^{(C)} = \mathcal{E}[|\psi\rangle_{AB}^{(C)}]$ for short, and equivalently for A .

Let us start from the $\mathcal{C}^{(C)} = 0$ case. Since for any coherence measure it holds $\mathcal{C}^{(A)} \geq 0$, then we have to prove that $\mathcal{E}^{(C)} \geq \mathcal{E}^{(A)}$. We will do so by proving that $|\psi\rangle_{AB}^{(C)}$ can be converted to $|\psi\rangle_{BC}^{(A)}$ by a LOCC transformation. Let us consider the reduced density matrix $\rho_A^{(C)}$. Since $\mathcal{C}^{(C)} = 0$, it is diagonal and hence its vector of diagonal elements $\vec{d}^{(C)}$ coincides with its vector of eigenvalues $\vec{\lambda}^{(C)}$, which are the Schmidt coefficients of $|\psi\rangle_{AB}^{(C)}$. In an equation: $\vec{d}^{(C)} = \vec{\lambda}^{(C)}$. Moreover, by the Schur-Horn Theorem [7], for any Hermitian matrix, and in particular for $\rho_C^{(A)}$, it holds $\vec{d}^{(A)} \preceq \vec{\lambda}^{(A)}$, where $\preceq$ stands for weak majorization. Moreover, due to the lemma of Eq. (B1), we know that $\vec{d}^{(C)} = \vec{d}^{(A)}$, hence it must be that $\vec{\lambda}^{(C)} \preceq \vec{\lambda}^{(A)}$, where these vectors are the Schmidt coefficients of $|\psi\rangle_{AB}^{(C)}$ and $|\psi\rangle_{BC}^{(A)}$, respectively.

Finally, by Nielsen's theorem [8], since the Schmidt coefficients of $|\psi\rangle_{BC}^{(A)}$ majorize the ones of $|\psi\rangle_{AB}^{(C)}$, this means that $|\psi\rangle_{AB}^{(C)}$ can be transformed into $|\psi\rangle_{BC}^{(A)}$ by a LOCC, and hence $\mathcal{E}^{(C)} \geq \mathcal{E}^{(A)}$. This concludes the first part of the proof.

Let us investigate now the case where $\mathcal{E}^{(C)} = 0$. We want to show that coherence can only decrease, i.e. that $\mathcal{C}^{(A)} \leq \mathcal{C}^{(C)}$. To prove this, we show that the map between the initial subsystem and the final one is incoherent, when

the initial state is factorized. Given the reduced density matrices in the two perspectives, namely $\rho_A^{(C)}$ and $\rho_C^{(A)}$, these two states are related through a CPTP map $\phi[\rho_A^{(C)}] = \rho_C^{(A)}$ that corresponds to the QRF change. Explicitly the map is given by

$$\phi[\rho_A^{(C)}] = \text{Tr}_B [S \rho_A^{(C)} \otimes |\psi\rangle_B^{(C)} \langle\psi| S^\dagger], \quad (\text{C14})$$

since the initial state is factorized by assumption, and the form of S is given in Eq. (6). This map can be decomposed in terms of Kraus operators $E(g)$, satisfying $\phi[\rho] = \int dg E(g) \rho E(g)^\dagger$, where one can choose $E(g) = \langle g|S|\psi\rangle_B^{(C)}$.

Such an operation is incoherent iff, given any diagonal state $\rho_D = \int dh \rho_h |h\rangle\langle h|$, it remains diagonal after the action of each Kraus operator, i.e. $E(g)\rho_D E(g)^\dagger$ is diagonal $\forall g$. This can indeed be proven explicitly: From the expression of the Kraus operators and the definition of the QRF transformation, one can find

$$E(g) = \int dg' \langle g \circ g' | \psi \rangle_B^{(C)} |g'^{-1}\rangle\langle g'| \quad (\text{C15})$$

and replacing it in the incoherent operation condition, one gets

$$E(g)\rho_D E(g)^\dagger = \int dh \rho_h \langle g \circ h | \psi \rangle_B \langle \psi | g \circ h \rangle |h^{-1}\rangle\langle h^{-1}|, \quad (\text{C16})$$

which is clearly diagonal, as the proof requires. Hence, since ϕ is an incoherent operation, it cannot increase coherence, and it must be that $\mathcal{C}^{(A)} \leq \mathcal{C}^{(C)}$. As a consequence, it holds $\Delta\mathcal{C}\Delta\mathcal{E} \leq 0$, when $\mathcal{E}^{(A)} = 0$.

Appendix D: Imperfect quantum reference frames

In the main text, we have restricted our analysis to $L^2(G)$ quantum reference frames, i.e. QRFs that carry the regular representation of the symmetry group. These are referred to in the literature as perfect [9] or ideal [10] quantum reference frames. In the following, we show that Theorem 1 that we derived for ideal frames no longer holds for non-ideal QRFs, i.e. $\mathcal{C}_e + \mathcal{E}_e$ and $\mathcal{C}_l + \mathcal{E}_l$ are not conserved under an imperfect QRF change. This happens because the type of imperfect QRFs we discuss here is characterized by an overcomplete basis of group elements, and consequently the QRF transformation between any two perspectives is more involved.

To see why, let us first give a brief introduction to the perspective-neutral approach to QRFs [10–16], which represents an alternative formalism for internal QRFs. Consider a system of interest S and a QRF R , both transforming under a given symmetry group G . The group is taken to be an arbitrary unimodular Lie group, that is, the left- and right-invariant Haar measures of the Lie group coincide. Given a kinematical description of both systems, one can assign a quantum state to S and R in the *kinematical* Hilbert space, which takes the product form $\mathcal{H}_{\text{kin}} = \mathcal{H}_R \otimes \mathcal{H}_S$. $\mathcal{H}_{\text{kin}}$ is assumed to carry a unitary tensor product representation of G , i.e. for all $g \in G$ and $|\psi_{\text{kin}}\rangle$, we have $U_{RS}(g)|\psi_{\text{kin}}\rangle = U_R(g) \otimes U_S(g)|\psi_{\text{kin}}\rangle$, where $g \mapsto U_R(g)$ and $g \mapsto U_S(g)$ are strongly continuous unitary representations. We shall require the existence of a *system of coherent states* $\{U_R, |\phi(g)\rangle_R\}$ such that G acts transitively on the coherent state system. That is, for all $g, g' \in G$, we have $|\phi(g'g)\rangle_R = U_R(g')|\phi(g)\rangle_R$, and, in particular, $|\phi(g)\rangle_R = U_R(g)|\phi(e)\rangle_R$. The coherent states $|\phi(g)\rangle_R$ are interpreted as the different possible orientation states of the G -frame, where g is called the *quantum frame orientation*. In order for the frame orientation states to admit a consistent probabilistic interpretation, the representation $U_R(g)$ and the seed coherent state $|\phi(e)\rangle$ are required to give rise to a *resolution of the identity*

$$\int_G dg |\phi(g)\rangle\langle\phi(g)|_R = n \mathbf{1}_R, \quad (\text{D1})$$

for $n > 0$ a constant.

An *ideal reference frame* is characterized by carrying the regular *representation* of G , in which case $\mathcal{H}_R \simeq L^2(G, dg)$. In this case, G acts regularly on itself, and the frame orientation states are orthogonal and thus perfectly distinguishable, i.e. $\langle g|g'\rangle = \delta(g^{-1} \circ g')$. A reference frame that does not carry the regular representation is referred to as *non-ideal*.

The physical Hilbert space contains all *physical* states, that is, all states invariant under the gauge group action U_{RS} . They are of the form $U_{RS}(g)|\psi_{\text{phys}}\rangle = |\psi_{\text{phys}}\rangle, \forall g \in G$. They can be constructed by applying the *coherent group averaging “projector”* $\Pi_{\text{phys}} := \int_G dg U_{RS}(g)$. Then, with a slight abuse of notation (see Ref. [10] for a detailed discussion), we have $|\psi_{\text{phys}}\rangle := \Pi_{\text{phys}} |\psi_{\text{kin}}\rangle$.

In order to reduce from a physical “perspective-neutral” state to an internal QRF perspective, one can apply the *Schrödinger reduction map* $\mathcal{R}_{\mathbf{S},R}(g) : \mathcal{H}_{\text{phys}} \rightarrow \mathcal{H}_{S,g}^{\text{phys}}$, which maps states from the physical Hilbert space to the description of the system S relative to the frame R being in orientation $g \in G$:

$$\mathcal{R}_{\mathbf{S},R}(g) = \langle \phi(g) |_R \otimes \mathbf{1}_S. \quad (\text{D2})$$

The image of this map is the *physical system Hilbert space* $\mathcal{H}_{S,g}^{\text{phys}}$, which contains all relative states of R that result from conditioning a gauge-invariant, physical state on the frame R being in orientation g . Thus, we can define *relative physical states* as $|\psi_S^{\text{phys}}(g)\rangle := \mathcal{R}_{\mathbf{S},R}(g) |\psi_{\text{phys}}\rangle$. The latter satisfy the covariance property $U_S(g') |\psi_S^{\text{phys}}(g)\rangle = |\psi_S^{\text{phys}}(g'g)\rangle$ and allow to rewrite physical states as

$$|\psi_{\text{phys}}\rangle = \int_G dg' |\phi(g')\rangle_R \otimes |\psi_S^{\text{phys}}(g')\rangle. \quad (\text{D3})$$

Finally, let us see how to change perspective from one frame to another. Consider two frame systems R_1 and R_2 , and the system of interest S . The kinematical Hilbert space is given by $\mathcal{H}_{\text{kin}} = \mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2} \otimes \mathcal{H}_S$ and the physical space is the image of the kinematical space under the coherent group averaging projector. To reduce to the perspective of the internal frame R_i being in orientation $g_i, i = 1, 2$, we apply the Schrödinger reduction map $\mathcal{R}_{\mathbf{S},R_i}(g_i)$. Since this map is invertible when its domain is restricted to $\mathcal{H}_{\text{phys}}$, it is now possible to change perspectives by passing via the perspective-neutral structure. The QRF change that maps the physical states of S relative to R_1 being in orientation g_1 to the states of S relative to R_2 being in orientation g_2 is given by

$$V_{R_1 \rightarrow R_2}(g_1, g_2) := \mathcal{R}_{\mathbf{S},R_2}(g_2) \cdot \mathcal{R}_{\mathbf{S},R_1}^{-1}(g_1), \quad (\text{D4})$$

mapping from $\mathcal{H}_{R_2 S, g_1}^{\text{phys}}$ to $\mathcal{H}_{R_1 S, g_2}^{\text{phys}}$.

Having reviewed the perspective-neutral approach to QRFs, let us, for simplicity, choose a concrete example for the symmetry group G , namely the smallest finite non-Abelian group S_3 . Take three systems A , B , and C , such that each has a Hilbert space $\mathbb{C}^3$ and carries the representation that maps S_3 onto the triangle. That is, S_3 acts as permutations on the three basis vectors $|i\rangle, i = 0, \dots, 2$, which form an orthonormal basis of $\mathbb{C}^3$. The coherent state system can be constructed from a general seed state $|\phi(e)\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$, where $\alpha, \beta, \gamma \in \mathbb{C}$. The conditions on the seed state are summarized as follows:

$$\begin{aligned} \alpha, \beta, \gamma \text{ are pairwise different,} \\ |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1, \\ \alpha\beta^* + \alpha^*\beta^* + \alpha\gamma^* + \alpha^*\gamma^* + \gamma\beta^* + \gamma^*\beta^* = 0. \end{aligned} \quad (\text{D5})$$

The physical Hilbert space for $N = 3$ systems for $G = S_3$ is five-dimensional. We can reduce into the perspectives of systems A and C by applying the respective Schrödinger reduction map $\mathcal{R}_{\mathbf{S},A}(e) = \langle \phi(e) |_A \otimes \mathbf{1}_{BC}$ or $\mathcal{R}_{\mathbf{S},C}(e) = \langle \phi(e) |_C \otimes \mathbf{1}_{AB}$.

Let us now give a concrete example in the case of imperfect frames for S_3 in which the conservation of $\mathcal{C}_e + \mathcal{E}_e$ and $\mathcal{C}_l + \mathcal{E}_l$ under changes of frame does not hold. Take the kinematical state of A , B , and C in $\mathcal{H}_{\text{kin}} = (\mathbb{C}^3)^{\otimes 3}$ to be

$$|\psi_{\text{kin}}\rangle_{CAB} = |0\rangle_C \otimes \frac{1}{\sqrt{3}} (i|0\rangle_A + |1\rangle_A - |2\rangle_A) \otimes |2\rangle_B. \quad (\text{D6})$$

Applying the group averaging projector $\Pi_{\text{phys}} = \frac{1}{\sqrt{|S_3|}} \sum_{g \in S_3} U_{CAB}(g)$, we get the corresponding physical state

$$\begin{aligned} |\psi_{\text{phys}}\rangle_{CAB} = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{3}} (i|002\rangle + |012\rangle - |022\rangle + i|110\rangle + |120\rangle - |100\rangle + i|221\rangle + |201\rangle - |211\rangle \\ + i|001\rangle + |021\rangle - |011\rangle + i|220\rangle + |210\rangle - |200\rangle + i|112\rangle + |102\rangle - |122\rangle)_{CAB}. \end{aligned} \quad (\text{D7})$$

Let us now reduce to the perspectives of systems C and A , respectively. We choose the following seed state:

$$|\phi(e)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle). \quad (\text{D8})$$

It is easy to check that this state satisfies conditions (D5). We apply the Schrödinger reduction map

$$\mathcal{R}_{\mathbf{s},C}(e) = \sqrt{3} \frac{1}{\sqrt{2}} (\langle 0| - i\langle 1|)_C \otimes \mathbb{1}_{AB} \quad (\text{D9})$$

to the physical state. Note that we added here a multiplicative factor of $\sqrt{3}$ in order to render the relative physical states normalized. We obtain the relative physical state of A and B , given C in orientation e :

$$\mathcal{R}_{\mathbf{s},C}(e)|\psi_{\text{phys}}\rangle_{CAB} = \frac{\sqrt{3}}{6} (i|00\rangle + i|01\rangle + |10\rangle - |11\rangle + 2|12\rangle - i|20\rangle + |21\rangle + (i-1)|22\rangle)_{AB}. \quad (\text{D10})$$

Similarly, we obtain the relative physical state of B and C , given A in orientation e :

$$\mathcal{R}_{\mathbf{s},A}(e)|\psi_{\text{phys}}\rangle_{CAB} = \frac{\sqrt{3}}{6} (2i|01\rangle + (-1-i)|20\rangle + (1+i)|21\rangle + 2|12\rangle)_{CB}. \quad (\text{D11})$$

We are now in the position to compare the diagonal elements of the reduced density matrices:

$$\rho_A^{(C)} = \text{Tr}_B[\rho_{AB}^{(C)}] = \frac{1}{12} (2|0\rangle\langle 0| + 6|1\rangle\langle 1| + 4|2\rangle\langle 2|) + \text{off-diagonals}, \quad (\text{D12})$$

$$\rho_C^{(A)} = \text{Tr}_B[\rho_{CB}^{(A)}] = \frac{1}{12} (4|0\rangle\langle 0| + 4|1\rangle\langle 1| + 4|2\rangle\langle 2|) + \text{off-diagonals}. \quad (\text{D13})$$

We can then see that after an imperfect QRF change the diagonal elements of $\rho_A^{(C)}$ and $\rho_C^{(A)}$ are not equal up to a permutation. This shows that the lemma of Eq. (B1) does not hold anymore for imperfect QRFs. As a consequence, one can check by direct computation that the conservation laws of Theorem 1 do not hold. In fact:

$$\Delta\mathcal{C}_e + \Delta\mathcal{E}_e = S[\rho_{A,d}^{(C)}] - S[\rho_{C,d}^{(A)}] = \left(\frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 6}{6} \right) - \log 3 \approx -0.09 \neq 0, \quad (\text{D14})$$

$$\Delta\mathcal{C}_{l^2} + \Delta\mathcal{E}_l = \sum_i \left(\left| \left(\rho_C^{(A)} \right)_{i,i} \right|^2 - \left| \left(\rho_A^{(C)} \right)_{i,i} \right|^2 \right) = -\frac{1}{18} \neq 0. \quad (\text{D15})$$

Let us now briefly consider the inequalities $\Delta\mathcal{C}_e\Delta\mathcal{E}_e < 0$ and $\Delta\mathcal{C}_{l^2}\Delta\mathcal{E}_l < 0$. While one can show that these inequalities are satisfied for ideal QRFs, as is done in the main text, this is generally not the case for imperfect QRFs that carry a coherent state system. Let us provide a counterexample here.

The general form of a reduced physical state is given by

$$\begin{aligned} |\psi\rangle_{BC}^{(A)} = & a \cdot (\alpha^*|00\rangle_{BC} + \beta^*|11\rangle + \gamma^*|22\rangle) \\ & + (1/\sqrt{2})[b \cdot (\alpha^* (|11\rangle + |22\rangle) + \beta^* (|00\rangle + |22\rangle) + \gamma^* (|00\rangle + |11\rangle)) \\ & + c \cdot (\alpha^* (|10\rangle + |20\rangle) + \beta^* (|00\rangle + |21\rangle) + \gamma^* (|11\rangle + |12\rangle)) \\ & + d \cdot (\alpha^* (|01\rangle + |02\rangle) + \beta^* (|10\rangle + |12\rangle) + \gamma^* (|20\rangle + |21\rangle)) \\ & + e \cdot (\alpha^* (|12\rangle + |21\rangle) + \beta^* (|02\rangle + |20\rangle) + \gamma^* (|01\rangle + |10\rangle))]. \end{aligned} \quad (\text{D16})$$

One counterexample for Theorem 1 and the above-mentioned inequalities is generated by the following parameters:

$$(\alpha, \beta, \gamma) = \frac{1}{\sqrt{3}}(i, 1, -1) \text{ and} \quad (\text{D17})$$

$$(a, b, c, d, e) = \left(\frac{39}{86} - \frac{29i}{69}, -\frac{77}{171} + \frac{37i}{85}, \frac{39}{89} + \frac{i}{43}, \frac{39}{272} - \frac{8i}{95}, -\frac{3}{41} - \frac{i}{513} \right), \quad (\text{D18})$$

for which

$$\Delta\mathcal{C}_e + \Delta\mathcal{E}_e = -0.138 \text{ and } \Delta\mathcal{C}_e\Delta\mathcal{E}_e = 0.005, \quad (\text{D19})$$

$$\Delta\mathcal{C}_{l^2} + \Delta\mathcal{E}_l = -0.453 \text{ and } \Delta\mathcal{C}_{l^2}\Delta\mathcal{E}_l = 0.005. \quad (\text{D20})$$

Regarding Theorem 2, let us briefly explain why the assumptions do not hold in the case of imperfect QRFs that carry a coherent state system. Note that the change of QRF in Eq. (D4) is a map from one physical system Hilbert space to another. Any state relative to one QRF is obtained from applying Π_{phys} to a kinematical state, which

strongly entangles the state, and conditioning on a frame state $|\phi(g)\rangle$. Using Eqs. (D2) and (D3), this gives rise to states of the form

$$|\psi\rangle_{BC}^{(A)} \equiv \mathcal{R}_{\mathbf{S},A}(g) \circ \Pi_{\text{phys}}|\psi_{\text{kin}}\rangle = \int dg' \langle\phi(g)|\phi(g')\rangle |\psi_{BC}^{\text{phys}}(g')\rangle \quad (\text{D21})$$

for states relative to A , and analogously for C . For any imperfect frame with overcomplete basis $\{|\phi(g)\rangle\}_{g \in G}$, that is, $\langle\phi(g)|\phi(g')\rangle \neq \delta(g'^{-1} \circ g)$, the relative state will still contain a certain degree of entanglement. We can thus not find separable states in $\mathcal{H}_{BC,g}^{\text{phys}}$ or $\mathcal{H}_{AB,g}^{\text{phys}}$. For the same reason, one can also not find any relative states with zero coherence. Thus, the statement of Theorem 2 does not apply to the case of such imperfect QRFs.

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Unmasking the polygamous nature of quantum nonlocality

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Quantum mechanics imposes limits on the statistics of certain observables. Perhaps the most famous example is the uncertainty principle. Similar trade-offs also exist for the simultaneous violation of multiple Bell inequalities. In the simplest case of three observers, it has been shown that if two observers violate a Bell inequality, then none of them can violate any Bell inequality with the third observer, a property called monogamy of Bell violations. Forms of Bell monogamy have been linked to the no-signaling principle, and the inability of simultaneous violations of all inequalities is regarded as their fundamental property. Here, we show that the Bell monogamy does not hold universally and that in fact the only monogamous situation exists for only three observers. Consequently, the nature of quantum nonlocality is truly polygamous. We present a systematic methodology for identifying quantum states, measurements, and tight Bell inequalities that do not obey the monogamy principle for any number of more than three observers. The identified polygamous inequalities enable any subset of $(N - 1)$ observers to reveal nonlocality, which is also shown experimentally by measuring Bell-type correlations of six-photon Dicke states. Our findings may be exploited for multiparty quantum key distribution as well as simultaneous self-testing of multiple nodes in quantum networks.

foundations of quantum mechanics | Bell inequalities | monogamy of Bell violations | quantum entanglement

Quantum nonlocality is one of the most intriguing features of quantum theory. Starting from its beginnings and the famous Einstein-Podolsky-Rosen (EPR) argument (1), through the first works of John Bell (2) to various experiments (3–12), it reveals the impossibility of a local-realistic description of quantum phenomena. Violation of a Bell inequality serves now not only as a fundamental test for the statements about the nature of reality but also finds applications in many areas of modern quantum technologies (13–19). One crucial concept in the study of quantum nonlocality is the monogamy principle, which states that it is impossible to simultaneously violate all k -partite ($k < N$) two-setting Bell inequalities among N different parties. The early findings by Scarani and Gisin (20), and later by Toner and Verstraete (21–23), reveal that a strict monogamy exists between violations of bipartite Bell inequalities, i.e., if two observers violate a Bell inequality, then none of them can violate any Bell inequality with the third observer [note that inequalities with more settings can be violated (24)]. A number of generalizations where quantum entanglement is shared in a multipartite scenario showed that more than one subset of parties can violate a multipartite inequality, but never all subsets of observers (25–31). Monogamy is a crucial aspect in the foundations of quantum mechanics and thus became the subject of many extensive studies linking it to no-signaling principle, secure communication, and randomness amplification (32–39). However, as shown here, the monogamy principle is not fundamental. Rather, it is a mere consequence of the specific mathematical structure of certain inequalities, which are not universal. To support our claim, we develop a systematic method to construct Bell inequalities among $N - 1$ observers that do not adhere to the monogamy principle for all $N > 3$. Furthermore, we provide an interesting minimalistic polygamous scenario based only on bipartite correlations between all pairs of observers. We show that the simultaneous violation of all inequalities is possible if the number of parties is $N = 18$. Recognizing the practical challenges associated with generating high-fidelity quantum states in experimental setups, we could still identify inequalities that are violated experimentally by six-qubit Dicke states. The polygamous nature of quantum nonlocality is therefore proven theoretically and confirmed in experiments.

Results

Three Parties and Strict Monogamy. We begin by recalling the standard results on Bell monogamy between three observers. Consider a scenario in which party A tries

Significance

The concept of monogamy in Bell inequality violations has been regarded as a fundamental feature of quantum theory. However, in our manuscript, we show that Bell monogamy is a mere consequence of the mathematical expressions used to describe these phenomena. We devise a systematic methodology providing examples of simultaneous Bell nonlocality detection in multipartite systems, revealing that the sole genuine monogamy is present only in systems involving three parties. These results have exemplary consequences for communication, security, and testing of quantum networks.

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to simultaneously violate the Clauser-Horne-Shimony-Holt (CHSH) inequalities (40) with parties B and C using a three-qubit quantum state. We denote the value of the CHSH-Bell parameters by $\mathcal{B}_{AB}$ and $\mathcal{B}_{AC}$, respectively. Quantum mechanics predicts that these parameters obey the relation (20, 21)

$$\mathcal{B}_{AB}^2 + \mathcal{B}_{AC}^2 \leq 8. \quad [1]$$

Note that if one of the inequalities is violated, e.g., $\mathcal{B}_{AB} > 2$, then the other one cannot be violated, $\mathcal{B}_{AC} < 2$. This is the statement of monogamy of Bell inequality violations. In the considered scenario, this relation is tight, in the sense that all the values of $\mathcal{B}_{AB}$ and $\mathcal{B}_{AC}$ that reach the bound are realized by quantum theory (21, 25). To see this, consider a pure state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\theta|110\rangle + \sin\theta|101\rangle + |011\rangle), \quad [2]$$

with $\theta \in [0, \pi/2]$ and the following measurement settings described by the local Bloch vectors

$$\begin{aligned} \vec{a}_1 &= \vec{x}, & \vec{b}_1 &= \vec{c}_1 = \frac{1}{\sqrt{2}}(\vec{x} + \vec{y}), \\ \vec{a}_2 &= \vec{y}, & \vec{b}_2 &= \vec{c}_2 = \frac{1}{\sqrt{2}}(\vec{x} - \vec{y}). \end{aligned} \quad [3]$$

Using them, one can evaluate the corresponding two-party Bell parameters as

$$\begin{aligned} \mathcal{B}_{AB} &\equiv E_{110} + E_{120} + E_{210} - E_{220} = 2\sqrt{2} \sin\theta, \\ \mathcal{B}_{AC} &\equiv E_{101} + E_{102} + E_{201} - E_{202} = 2\sqrt{2} \cos\theta, \end{aligned}$$

where E_{klm} stands for the correlation function (average of the product of local results) and index 0 indicates the party whose measurement outcomes are not included in computation of the correlations. This clearly realizes all the values in Eq. 1 that saturate the bound and gives rise to the circle depicted in Fig 1. The whole circumference is obtained without any permutation of observers and measurement outcomes.

However, this is the only scenario where all of the involved inequalities cannot be violated (see ref. 28 for the case of three

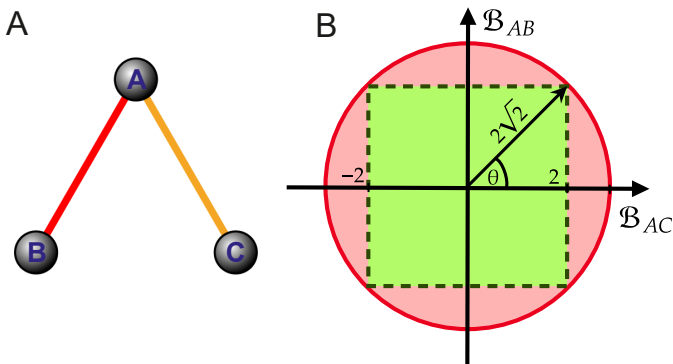


Fig. 1. Visual representation of monogamy between CHSH inequality violations for three parties. (A) Schematic of the arrangement where three observers (A , B , and C) try to violate two inequalities (red and orange edges). (B) Accessible values of Bell parameters $\mathcal{B}_{AB}$ and $\mathcal{B}_{AC}$ for parties AB and AC . According to local realistic models, the bound on each inequality is given by 2, and hence, its predictions are confined to the square with a side length of 4. From Eq. 1, quantum predictions lie within a circle of a radius $2\sqrt{2}$. The angle θ in Eq. 2 for which both inequalities are simultaneously maximized has been explicitly denoted. The principle of monogamy is clearly satisfied since such a state saturates both local realistic bounds and any attempt to violate one of the CHSH inequalities leads to classically achievable correlations on the other one.

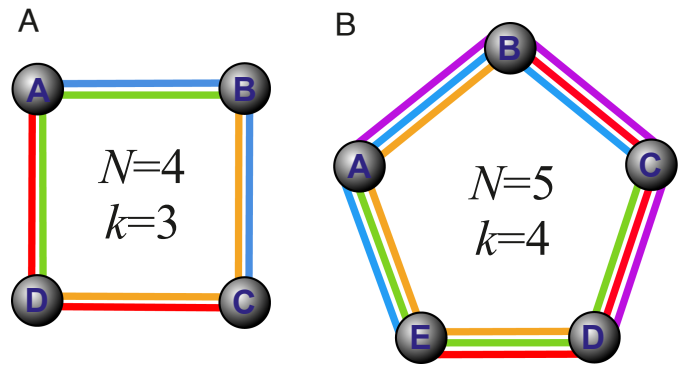


Fig. 2. Exemplary configurations showing Bell polygamy. Each color corresponds to a different choice of $N - 1$ observers taking part in the Bell test. Panels (A and B) picture the possible configurations of all three and four-qubit inequalities, respectively. According to our findings, in all such configurations with $N > 3$, one can simultaneously violate all N suitably designed Bell inequalities, each involving $k = N - 1$ observers.

inequalities). To demonstrate the polygamous nature of Bell nonlocality, we generalize the Toner-Verstraete scenario (21) to the case of N observers where each of them can perform measurements of two dichotomic observables $A_1^{(i)}, A_2^{(i)}$ ($i = 1, \dots, N$). Then, we analyze the simultaneous violation of Bell inequalities between $N - 1$ observers in all possible N configurations (Fig. 2). Later, we provide an inequality where each of the Bell tests includes measurements between pairs of observers only. Following these results, we present an inequality for $N = 6$ qubits and demonstrate that using an experimentally observed six-photon Dicke state one indeed can violate such an inequality for all six five-party subsystems simultaneously.

Polygamy of Mermin Inequalities. Our strategy to find Bell inequalities that do not satisfy the monogamy principle is by imposing permutation symmetry. Let N observers share an N -qubit quantum state that is permutationally invariant. They perform measurements of the same observables, i.e., $A_j^{(i)} = A_j^{(1)}$ ($j = 1, 2; i = 2, \dots, N$). In such a situation, if we find a Bell inequality that is violated by a subset of $(N - 1)$ observers, it immediately implies that the same inequality is violated by all other subsets with the same number of observers. This does not necessarily exclude correlation trade-offs, but it clearly rules out monogamy.

Let us now use this framework to show that the violation of multiparticle Mermin inequalities is polygamous. To demonstrate this, we consider a general permutationally invariant pure N -qubit state shared by all N observers

$$|\psi\rangle = \sum_{e=0}^N d_e |D_N^e\rangle, \quad [4]$$

where $|D_N^e\rangle$ is the N -qubit Dicke state with e excitations. Now, we form N subsets involving $(N - 1)$ parties, and for each of them evaluate the Mermin operator (41–43). For simplicity, justified by symmetry, all further calculations can be performed for observers labeled as $1, 2, \dots, N - 1$ only. The Mermin operator can be written as (43–45):

$$\begin{aligned} M_{N-1} &= 2^{N-2} \left(|GHZ^+\rangle_{N-1} \langle GHZ^+| \right. \\ &\quad \left. - |GHZ^-\rangle_{N-1} \langle GHZ^-| \right), \end{aligned} \quad [5]$$

where $|GHZ^\pm\rangle_{N-1} = \frac{1}{\sqrt{2}}(|0\dots 0\rangle_{N-1} \pm |1\dots 1\rangle_{N-1})$. For all local realistic models, the expectation value satisfies $\mathcal{M}_{N-1} = \langle M_{N-1} \rangle \leq 2^{(N-2)/2}$ (43). Rewriting [5] in terms of the Dicke states ($|0\dots 0\rangle_{N-1} = |D_{N-1}^0\rangle$, $|1\dots 1\rangle_{N-1} = |D_{N-1}^{N-1}\rangle$), we get

$$M_{N-1} = 2^{N-2}(|D_{N-1}^0\rangle\langle D_{N-1}^{N-1}| + |D_{N-1}^{N-1}\rangle\langle D_{N-1}^0|). \quad [6]$$

In order to calculate the expectation value of the Mermin operator, it is convenient to express the state [4] in the following way

$$\begin{aligned} |\psi\rangle &= d_0|D_{N-1}^0\rangle|0\rangle \\ &+ \frac{d_1}{\sqrt{N}}|D_{N-1}^0\rangle|1\rangle + d_1\sqrt{\frac{N-1}{N}}|D_{N-1}^1\rangle|0\rangle \\ &+ \dots \\ &+ \frac{d_{N-1}}{\sqrt{N}}|D_{N-1}^{N-1}\rangle|0\rangle + d_{N-1}\sqrt{\frac{N-1}{N}}|D_{N-1}^{N-2}\rangle|1\rangle \\ &+ d_N|D_{N-1}^{N-1}\rangle|1\rangle, \end{aligned} \quad [7]$$

where the dots signify the terms that do not contribute to the expectation value. Finally, we obtain

$$\begin{aligned} \mathcal{M}_{N-1} &= \langle \psi | M_{N-1} \otimes \mathbf{1} | \psi \rangle \\ &= \frac{2^{N-1}}{\sqrt{N}}(d_0d_{N-1} + d_1d_N). \end{aligned} \quad [8]$$

This value is maximal if $d_0 = d_{N-1} = 0$ and $d_1 = d_N = 1/\sqrt{2}$ or vice versa. Both solutions are locally unitarily equivalent, thus, we choose only the one stated explicitly above. The corresponding state is given as

$$|\psi_{\max}\rangle = \frac{1}{\sqrt{2}}(|D_N^1\rangle + |1\dots 1\rangle). \quad [9]$$

It leads to the maximal quantum value for the Mermin operator equal to $\mathcal{M}_{N-1}^{\max} = 2^{N-2}/\sqrt{N}$ and a violation factor of $2^{(N-2)/2}/\sqrt{N}$. Note that for $N = 3$ and $N = 4$ our result is consistent with (21) and (25), and the respective Mermin inequalities are not violated. However, already from $N = 5$ onward such a violation is possible and it increases exponentially with the number of qubits. Remarkably, in the limit of many particles, not only there is no monogamy of violations, but in fact every inequality is violated maximally.

Furthermore, the polygamous property extends to complete sets of tight Bell inequalities, i.e., necessary and sufficient conditions for local realistic models of observed correlations. We show this for the Werner-Wolf-Weinfurter-Żukowski-Brukner (WWŻB) inequalities (46–48), which are complete for N -qubit two-setting correlation functions. For the proof, consider $(N-1)$ -qubit marginal of the state $|\psi_{\max}\rangle$. Surprisingly, it admits nonvanishing correlations in the xy plane for exactly the same observables as the Greenberger-Horne-Zeilinger (GHZ) state. However, while the GHZ correlations are all perfect, i.e., equal to ± 1 , the correlations of the marginals are reduced to $\pm 1/\sqrt{N}$. It was shown in ref. 48 that such states admit a local realistic model if and only if their squared correlations, measured for local observables in a plane of the Bloch sphere, are larger than 1. For the marginal state, this sum is equal to $2^{N-2}/N$ and is greater than 1 for $N > 4$. Note that for $N \leq 4$ the correlations do not violate any Bell inequality as they are compatible with the local hidden variable model constructed in ref. 48.

Polygamy for Four Parties. As shown in the previous section, it is not possible to violate all four three-qubit Mermin inequalities for observers (ABC, ABD, ACD, and BCD) in a four-party system (A, B, C, and D), where for simplicity $A^{(1)} \equiv A$, $A^{(2)} \equiv B$, etc. However, it is possible to find another set of two-setting Bell inequalities that have this feature.

Using an original method based on linear programming (*Methods*) we found a tight three-qubit inequality $\langle I_{ABC} \rangle \leq 6$, where

$$\begin{aligned} I_{ABC} &= 2 \text{sym}[A_1] - \text{sym}[A_1B_1] - \text{sym}[A_1B_2] \\ &+ \text{sym}[A_2B_2] + 2A_1B_1C_1 + \text{sym}[A_2B_1C_1] \\ &- 2 \text{sym}[A_2B_2C_1] - A_2B_2C_2. \end{aligned} \quad [10]$$

We use here a compact notation for symmetrizing over different observers

$$\text{sym}[A_kB_lC_m] = \sum_{\pi(k,l,m)} A_kB_lC_m, \quad [11]$$

where the sum is over all permutations of (k, l, m) , denoted as $\pi(k, l, m)$, assuming $A_0 = B_0 = C_0 = 1$, e.g., $\text{sym}[A_1B_1] = \text{sym}[A_1B_1C_0] = A_1B_1C_0 + A_1B_0C_1 + A_0B_1C_1 = A_1B_1 + A_1C_1 + B_1C_1$ being the permutations of $k = 1, l = 1, m = 0$. Analogous expressions can be formulated for I_{ABD} , I_{ACD} , I_{BCD} inequalities. This inequality belongs to the Śliwa's set (49). It can be directly verified that all four subsets of parties can simultaneously violate their inequalities using the four-qubit state of the form $|\psi\rangle = \cos\theta|D_4^1\rangle + \sin\theta|1111\rangle$. Note that this state lies in the same subspace as $|\psi_{\max}\rangle$ in Eq. 9. The maximal violation of $6.154 > 6$ is observed for $\theta = 0.144$, and observables lying in the xz plane, $A_i = B_i = C_i = D_i = \cos\phi_i\sigma_x + \sin\phi_i\sigma_z$ with $\phi_1 = 2.739$, $\phi_2 = 0.847$. It is worth noting that these inequalities can also be simultaneously violated ($6.064 > 6$) by the Dicke $|D_4^1\rangle$ state ($\theta = 0$) if the angles were chosen as $\phi_1 = 2.640$, $\phi_2 = 0.986$. We emphasize the relative simplicity, as every observer measures the same set of two observables. The above example shows that, already for a system of four particles, one can define inequalities involving three observers such that all of them are simultaneously violated.

Polygamy with Two-Body Correlators. The violation of monogamy is by no means limited to the case of higher-order correlations. Here, we focus on a minimalist scenario based on the measurements of all pairs of observers. Again, we approach this problem through the linear programming technique described in *Methods*. As a result, the $(N-1)$ -partite two-body Bell inequality was found to be $\langle I_{N-1} \rangle \geq 0$. The corresponding Bell operator is given as

$$\begin{aligned} I_{N-1} &= L + \alpha (\text{sym}[A_1] + \text{sym}[A_2]) \\ &+ \text{sym}[A_1B_1] + 2 \text{sym}[A_1B_2] + \text{sym}[A_2B_2], \end{aligned} \quad [12]$$

where L and α are defined by $L = 3((N-4)^2 + N - 2)$ and $\alpha = -3(N-4)$, respectively. Note that Eq. 12 fits into the permutationally invariant form of the two-body Bell expressions introduced in ref. 50. However, I_{N-1} differs from the two classes of Bell parameters defined in that paper.

Notably, it is easy to verify the validity of the local realistic bound of zero for Eq. 12 up to large N (e.g., up to 10^4). To do this, first note that we can restrict the parties to use deterministic strategies. For a given classical deterministic strategy, let $a_{++},$

a_{+-} , a_{-+} , and a_{--} denote the number of parties whose classical expectation values $\langle A_k^{(i)} \rangle$ (for $k = 1, 2$) are $\{1, 1\}$, $\{1, -1\}$, $\{-1, 1\}$, and $\{-1, -1\}$, respectively. By definition, we have $a_{++} + a_{+-} + a_{-+} + a_{--} = N$. Due to the symmetry of Eq. 12, every classical deterministic strategy can be mapped to the four-tuple $(a_{++}, a_{+-}, a_{-+}, a_{--})$. In this way, we reduce the 2^N -dimensional correlation space to the dimension of 3, and the number of classical deterministic strategies is reduced from 2^{2N} to $2(N^2 + 1)$, which is a significant reduction in the complexity of the problem (for more details, see ref. 50).

To determine the violation of Eq. 12 one has to minimize the Bell expression in Eq. 12 for the $(N - 1)$ -partite reduced state of some N -partite symmetric state, e.g., $|D_N^1\rangle$, using observables in the xz plane,

$$A_k = \cos \phi_k \sigma_x + \sin \phi_k \sigma_z, \quad [13]$$

where we assume that all observers measure the same pair of observables, i.e., $A_k^{(i)} = A_k$ for all i .

Since I_{N-1} in Eq. 12 is permutationally invariant, $|D_N^1\rangle$ is symmetric, and $A_k^{(i)} = A_k$, the problem of computing the quantum violation is greatly simplified. In fact, we only need to consider the following two-qubit reduced Bell operator

$$I_{A_1 A_2} = L \mathbf{1}_4 + \alpha(N-1)(A_1 \otimes \mathbf{1}_2 + A_2 \otimes \mathbf{1}_2) + \binom{N-1}{2}(A_1 \otimes A_1 + 4A_1 \otimes A_2 + A_2 \otimes A_2), \quad [14]$$

along with the reduced two-qubit state of $|D_N^1\rangle$:

$$\rho_2 = \frac{2}{N} |\psi_+\rangle\langle\psi_+| + \left(1 - \frac{2}{N}\right) |00\rangle\langle 00|, \quad [15]$$

where $|\psi_+\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$.

Now, by substituting $N = 18$ and plugging in the observables defined by the angles

$$\begin{aligned} \phi_1 &= \pi - \arcsin(21/22), \\ \phi_2 &= \arcsin(21/22). \end{aligned} \quad [16]$$

we obtain

$$\langle I_{A_1 A_2} \rangle_{\rho_2} = -\frac{4}{99} < 0. \quad [17]$$

Therefore, the two-body Bell inequality defined by [12] is clearly violated. This implies that the 18-qubit state $|D_{18}^1\rangle$ can simultaneously violate all 17-qubit two-body Bell inequalities and thus the monogamy principle does not hold. Note that here any exchange of information happens only between the pairs of observers.

Experimental Demonstration of Bell Polygamy. Although the polygamous character of Bell inequality violations is already present in four-qubit systems, an experimental demonstration of such phenomena would require an experiment with very low experimental errors. The critical visibility, i.e., the minimal occurrence probability of an entangled three-qubit state when mixed with white noise equals $6/6.154 = 0.975$. For this reason, we construct five-party inequalities for $N = 6$ observers with far less demanding visibility requirement to demonstrate experimentally that they are violated in all six five-party subsystems. Again, by

Table 1. Experimental polygamy of Bell nonlocality

Partition	ABCDE	ABCDF	ABCEF	ABDEF	ACDEF	BCDEF
Violation	6.315	6.204	6.479	6.307	6.146	6.128
SD	0.133	0.131	0.137	0.137	0.125	0.128

Simultaneous violation of all six five-party Bell inequalities [18], with the local realistic bound of 6, where each observer measures the same settings in the xz plane parameterized by $\phi_1 = 1.29244$ and $\phi_2 = 2.79605$. All inequalities are violated by at least one SD.

the linear programming method used before we arrive at the tight five-qubit inequality $\langle I_{ABCDE} \rangle \leq 6$ with

$$\begin{aligned} I_{ABCDE} &= -\text{sym}[A_1 B_1] - \text{sym}[A_2 B_2] \\ &\quad + \text{sym}[A_1 B_1 C_1 D_1] + \text{sym}[A_1 B_1 C_1 D_2] \\ &\quad - \text{sym}[A_1 B_2 C_2 D_2] + \text{sym}[A_2 B_2 C_2 D_2]. \end{aligned} \quad [18]$$

It is worth noting that although these inequalities are defined for five qubits (they constitute the facet of a five-party Bell-Pitovsky polytope), they do not involve five-qubit correlations. Interestingly, the reduced five-qubit state of the Dicke state $|D_6^3\rangle$,

$$\begin{aligned} \rho_{nc} &= \text{Tr}_F(|D_6^3\rangle_{ABCDEF}\langle D_6^3|) \\ &= \frac{1}{2}(|D_5^2\rangle_{ABCDE}\langle D_5^2| + |D_5^3\rangle_{ABCDE}\langle D_5^3|) \end{aligned} \quad [19]$$

does not contain any five-qubit correlations (51, 52). It can be verified that the inequality I_{ABCDE} , as well as all of the other inequalities obtained by exchanging the indices, are theoretically violated by the state [19] with a value of $7.8215 > 6$ for the settings $A_i = B_i = C_i = D_i = E_i = F_i = \cos \phi_i \sigma_x + \sin \phi_i \sigma_z$, with $\phi_1 = 1.26042$, $\phi_2 = 2.77953$. The maximal possible violation of Eq. 18 by any six-qubit state is comparable with the violation for the Dicke state and is equal to 7.8771. The critical visibility required to observe the violation of inequality [18] with the state ρ_{nc} in Eq. 19 is 76.71%.

We experimentally demonstrate the existence of polygamous Bell-type correlations using five-party subsystems of a six-qubit Dicke state $|D_6^3\rangle$ prepared with polarization entangled photons. For this purpose, we use a pulsed laser ($\lambda = 390$ nm) in an enhancement cavity to exploit the third order of spontaneous parametric down-conversion (SPDC) in a β -barium-borate crystal. By postselecting on six-fold detection events of this collinear type-II SPDC process, we obtain the desired polarization-encoded six-qubit Dicke state with three excitations. Due to the stochastic nature of SPDC the observed events are contaminated by even higher-order emissions resulting in a mixture of $|D_6^3\rangle$ with states of 6 photons traced from $|D_8^4\rangle$.

A detailed description of the experimental setup can be found in refs. 53 and 54. To obtain the density matrices of five-qubit marginals, we evaluate five-fold coincidence events for all subsets of five observers. This provides better measurement statistics compared with tracing out one qubit. All matrices were obtained using permutation-invariant tomography (55). Due to the different number of detection events and different efficiencies of individual detectors, the five-qubit reduced states are not identical but show small variations. The average fidelity (averaged over different five-party subsystems) to the ideal ρ_{nc} state in Eq. 19 is measured to be $91.2\% \pm 0.6\%$ (between $90.4\% \pm 0.2\%$ and $91.9\% \pm 0.2\%$). The five-qubit experimental states also show $4.9\% \pm 0.6\%$ average fidelity (between $4.6\% \pm 0.2\%$ and $5.1\% \pm 0.2\%$) to an even mixture of $|D_5^1\rangle$ and $|D_5^4\rangle$.

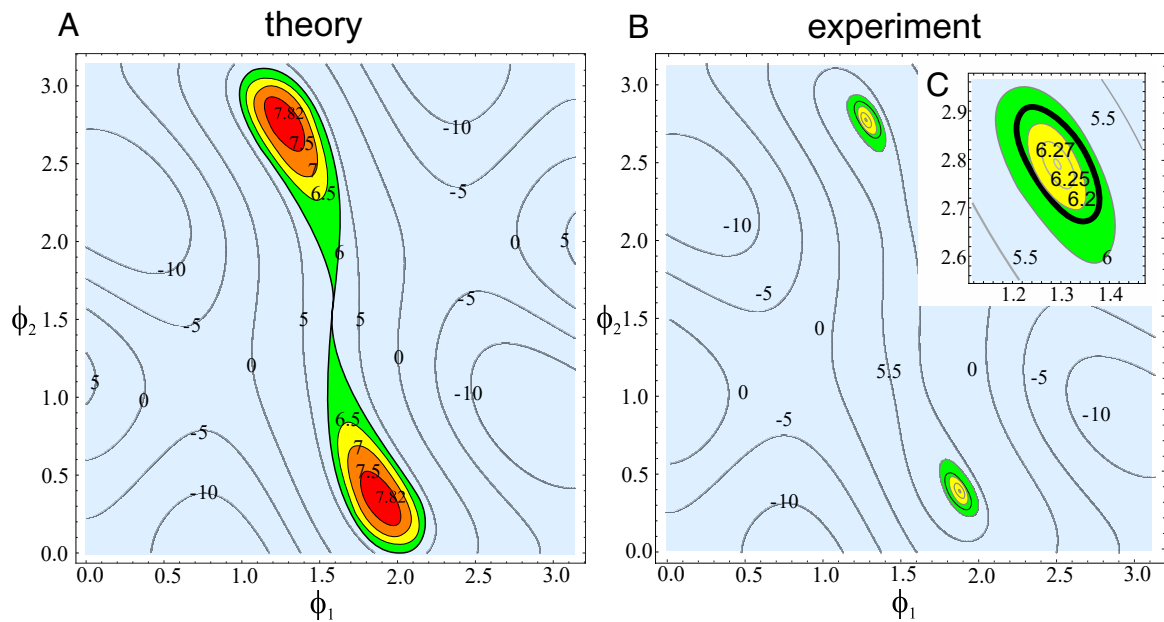


Fig. 3. The average violation of inequalities [18] plotted as a function of measurement angles ϕ_1 and ϕ_2 . Panel (A) shows the theoretical predictions arising from calculating the violation of inequalities using ideal state ρ_{nc} in Eq. 19. Due to the symmetry of the six-qubit Dicke state, the average violation is equivalent to the violation obtained by any set of five parties. The average violation of all inequalities based on the experimental data is presented in panel (B). Despite the apparent symmetry, due to the imperfections and statistics, not all inequalities are simultaneously violated in the colored region. It is the area inside the bold line in the inset (C) of panel (B) where all six inequalities are violated simultaneously.

states, originating in the higher-order SPDC emissions. This noise contributes negative values of the Bell parameter in Eq. 18 reducing the violations. Still, all six five-party Bell inequalities are simultaneously violated (Table 1) by at least one SD using the same settings parameterized by $\phi_1 = 1.29244$, $\phi_2 = 2.79605$. The observed average violation is $37.5787/6 > 6$.

Fig. 3 shows how the violation depends on the choice of measurement settings (the same for every observer) both for the theoretical prediction as well as for the experimentally prepared and measured state. A small, yet clearly visible, region of settings indicates where the simultaneous violation of all inequalities is indeed possible.

Discussion

We showed that quantum nonlocality can be shared among arbitrarily many parties if the number of observers is $N > 3$. Using an original linear programming method we found several interesting tight inequalities that do not satisfy the monogamy principle. These include polygamous inequalities for $N = 4$, inequalities based on two-body correlators for $N = 18$, and experimentally verified inequalities for $N = 6$, where all six five-party subsets of observers violated the Bell inequality simultaneously.

Our findings demonstrate that Bell inequality violation is an inherently polygamous phenomenon. In fact, only the scenario with three particles obeys monogamy relations. For more observers, any trade-offs between the values of Bell parameters result from a specific mathematical formulation of a given inequality. This conclusion has implications on all practical applications linked to Bell inequalities. First of all, security of quantum cryptography can be derived from Bell monogamy (35) and it is now evident that the safety becomes inequality-dependent. Simultaneous violation of all involved inequalities can be exploited for self-testing. To give a concrete example consider the Mermin inequality. One can verify that the violation

of a single inequality obtained by the $(N - 1)$ -qubit marginal of state $|\psi_{\max}\rangle$ derived here can also be obtained by a noisy $(N - 1)$ -qubit GHZ state. Therefore, the violation of a single inequality cannot be used for self-testing. However, it is easy to check that the noisy GHZ state cannot violate all inequalities. Adding the requirement of simultaneous violation we find that there is only a two-dimensional subspace spanned by the states discussed below [8] which achieves the maximum. This subspace can therefore be self-tested based on the polygamy of violations. Our final example is communication complexity problems. It is well known that violation of a Bell inequality can be exploited to save on communication of particular distributed-computing problems (17). The possibility of simultaneous violation by any subset of observers translates to the savings in communication for problems where part of the data will be obtained by a subset of computing stations but it is not known in advance which subset will be given the inputs. We hope these instances will stimulate further work on using the lack of monogamy, especially to test complex quantum networks.

Methods

Linear Programming Method for Constructing Bell Inequalities That Reveal Polygamy. We give a brief description of our method, which is based on linear programming. We illustrate it with permutationally invariant (PI) Bell inequalities with two dichotomic settings per party, and we focus here on three-party inequalities. However, this method can be easily generalized to arbitrary Bell inequalities (with any number of settings, parties, and outcomes).

We choose a PI four-qubit pure state $\rho_4 = |\psi\rangle\langle\psi|$. Let us denote its three-qubit reduced density matrix by ρ_3 . We focus on measurements in the xz plane of the Bloch sphere,

$$A_k = \cos \phi_k \sigma_x + \sin \phi_k \sigma_z \quad [20]$$

for $k = 1, 2$. It is also assumed that all three parties perform the same two measurements, that is, $A_k = B_k = C_k$. Let us define $A_0 = B_0 = C_0 = \mathbf{1}$.

Then we can write a generic two-setting PI Bell operator for three parties as follows:

$$I_{ABC} = \sum_{0 \leq k \leq l \leq m \leq 2} \alpha_{k,l,m} \text{sym}[A_k B_l C_m], \quad [21]$$

where as before $\text{sym}[X]$ means the symmetrization of the expression X , and $\alpha_{k,l,m}$ are the Bell coefficients. We identify $-\alpha_{0,0,0} = L$ with the local bound, and then the Bell inequality is given by $\langle I_{ABC} \rangle \leq 0$.

Our goal is to find the Bell coefficients $\alpha_{k,l,m}$ of the Bell inequality that give rise to the largest quantum per local bound Q/L , where the symmetrized quantum expectation values

$$E_Q(k, l, m) = \text{Tr}(\rho_3 \text{sym}[A_k \otimes B_l \otimes C_m]) \quad [22]$$

are completely determined by the state ρ_3 and the observables A_k, B_k , and C_k ($k = 1, 2$). On the other hand, a local deterministic strategy λ is given by a particular choice of $A_k = \pm 1, B_k = \pm 1$, and $C_k = \pm 1, k = 1, 2$ (and $A_0 = B_0 = C_0 = 1$). We compute the symmetrized products

$$E_\lambda(k, l, m) = \text{sym}[A_k B_l C_m] \quad [23]$$

for $0 \leq k \leq l \leq m \leq 2$, which amount to 10 components. There are 64 different λ strategies, however, due to the symmetry in Eq. 23 only $\binom{6}{3} = 20$ give distinct values in the set of polynomials [23].

Without loss of generality, we set $\alpha_{0,0,0} = -L = -1$ and solve the following linear programming task to obtain the maximum quantum per local ratio $(Q/L) = q + 1$ along with the Bell coefficients $\alpha_{k,l,m}$:

$$q \equiv \text{maximize} \sum_{0 \leq k \leq l \leq m \leq 2} \alpha_{k,l,m} E_Q(k, l, m), \quad [24]$$

such that

$$\sum_{0 \leq k \leq l \leq m \leq 2} \alpha_{k,l,m} E_\lambda(k, l, m) \leq 0 \forall \lambda, \quad [25]$$

where maximization takes place over $\alpha_{k,l,m}$, $0 \leq k \leq l \leq m \leq 2$ with the exception of $\alpha_{0,0,0} = -1$.

A solution $q > 0$ in the above optimization problem gives a PI Bell inequality with coefficients $\alpha_{k,l,m}$, which can be violated with the reduced three-qubit state ρ_3 and observables [20] specified by the angles ϕ_1 and ϕ_2 .

Data, Materials, and Software Availability. An exemplary code implementing the technique described in *Methods* and used to find the Bell inequalities presented in the article and data supporting Fig. 3B is available at Zenodo (56). The experimental data that support the findings of this study are available at Zenodo (57). All study data are included in the main text.

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PAPER

The fastest generation of multipartite entanglement with natural interactions

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E-mail: pawel.cieslinski@phdstud.ug.edu.pl and wieslaw.laskowski@ug.edu.pl**Keywords:** quantum entanglement, two-body Hamiltonians, multipartite correlations, 2-local Hamiltonians

Abstract

Natural interactions among multiple quantum objects are fundamentally composed of two-body terms only. In contradistinction, single global unitaries that generate highly entangled states typically arise from Hamiltonians that couple multiple individual subsystems simultaneously. Here, we study the time to produce strongly nonclassical multipartite correlations with a single unitary generated by the natural interactions. We restrict the symmetry of two-body interactions to match the symmetry of the target states and focus on the fastest generation of multipartite entangled Greenberger–Horne–Zeilinger, W, Dicke and absolutely maximally entangled states for up to seven qubits. These results are obtained by constraining the energy in the system and accordingly can be seen as state-dependent quantum speed limits for symmetry-adjusted natural interactions. They give rise to a counter-intuitive effect where the creation of particular entangled states with an increasing number of particles does not require more time. The methods used rely on extensive numerical simulations and analytical estimations.

1. Introduction

Entanglement is one of the central aspects of quantum mechanics. Its importance lies in the potential to revolutionise computing and communication as known today. Useful strongly entangled states are difficult to create and are usually obtained through a variety of at most two-qubit quantum gates [1–3]. However, using sequences of noisy gates decreases the fidelity with the target state exponentially with the depth of the circuit. One could solve this issue by preparing certain useful states with a *single* unitary acting on N qubits at the same time. Many-qubit gates have indeed been studied in various contexts including, e.g. quantum circuits construction [4], entanglement generation [5–7] and error correction [8, 9]. In a simple example, the three-qubit Greenberger–Horne–Zeilinger (GHZ) state could be created from $|000\rangle$ by applying a unitary $U = |GHZ\rangle\langle 000| + \dots$. Unfortunately, this approach can lead to non-zero 3-body interaction terms in the Hamiltonian generating the evolution, i.e. $\text{Tr}[i\hbar/t \log(U) \sigma_i \otimes \sigma_j \otimes \sigma_k] \neq 0$, where σ_l ($l = x, y, z$) is the Pauli matrix. While such interactions could be engineered in some systems [10–12], fundamentally physical couplings are pairwise. The question, therefore, arises whether quantum states with multipartite nonclassical correlations can be created through a single unitary evolution given by Hamiltonians with at most two-body terms. And further, how does this physically motivated constraint limit the speed of evolution?

We answer these questions here and structure the paper as follows. In section 2 we define the physical setting and symmetries of the problem. Next, in section 3 quantum speed limit (QSL) and normalisation of energy in the system are introduced. In section 4 we propose an intuitive estimation of energy standard deviation and use it to establish the minimal time for the creation of three-qubit GHZ and W states via

two-body interactions. Section 5 presents numerical results for the optimisation of fidelity between the target and evolved state that provides the minimal time and exact analytical form of the generating two-body Hamiltonians for $N = 3, \dots, 7$ qubits in GHZ and W states. We also apply these methods to produce absolutely maximally entangled state (AME) of 5 qubits and give the minimal time for which the fidelity maximisation exceeds 99%.

2. Physical setting

Consider a general time-independent Hamiltonian with at most two-body interaction terms

$$H = \sum_{i,j} \sum_{\mu_i, \mu_j} h_{\mu_i \mu_j} \sigma_{\mu_i}^{(i)} \sigma_{\mu_j}^{(j)} + b_{\mu_i} \sigma_{\mu_i}^{(i)}, \quad (1)$$

where the first sum includes the two-body interactions with strengths $h_{\mu_i \mu_j}$, $\sigma_{\mu}^{(i)}$ stand for Pauli matrices corresponding to the i th site ($\mu = x, y, z$), and b_{μ_i} is the strength of the on-site external field. The sums are over the particles coupled according to the interaction graphs presented in figure 1. We distinguish different ranges of interactions, $\mathcal{I}$. Interactions of range $\mathcal{I} = 1$ couple only nearest neighbours, of range $\mathcal{I} = 2$ couple more distant particles, or correspond to a more complex topology. The interactions of the highest rank $\mathcal{I} = N - 1$ are represented by the complete interaction graph. This is a very broad class of models. Hamiltonians that are diagonal in the Pauli decomposition can be found, e.g. in the Heisenberg-like models [13] and the charge or flux superconducting qubit models [14]. Non-diagonal symmetric elements can exist in the transverse exchange [15] as well as in the Kaplan–Shekhtman–Entin–Wohlman–Aharony interaction [16].

Throughout most of the paper, we consider Hamiltonians that are isotropic and symmetric (full permutation symmetry), i.e.

$$h_{\mu_i \nu_j} = h_{\mu_k \nu_l} = h_{\nu_k \mu_l} \quad \text{and} \quad b_{\mu_i} = b_{\mu_j}. \quad (2)$$

We will provide time bounds on entangled state generation that are optimal within this broad family and note that this symmetry matches the symmetry of the target states.

Our focus is on the GHZ, W, Dicke and AME states for up to seven qubits, but it is clear that the methods can be used for arbitrary states. In this study, only the five-qubit AME state was considered explicitly due to the fact that for $N = 3$ it coincides with the GHZ state and in the case of $N = 4$ as well as $N = 7$ it was proven that such states do not exist [17–19]. The states under consideration are therefore written as follows:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|0 \dots 0\rangle + |1 \dots 1\rangle), \quad (3)$$

$$|W\rangle = \frac{1}{\sqrt{N}} (|10 \dots 0\rangle + |01 \dots 0\rangle + \dots + |00 \dots 1\rangle), \quad (4)$$

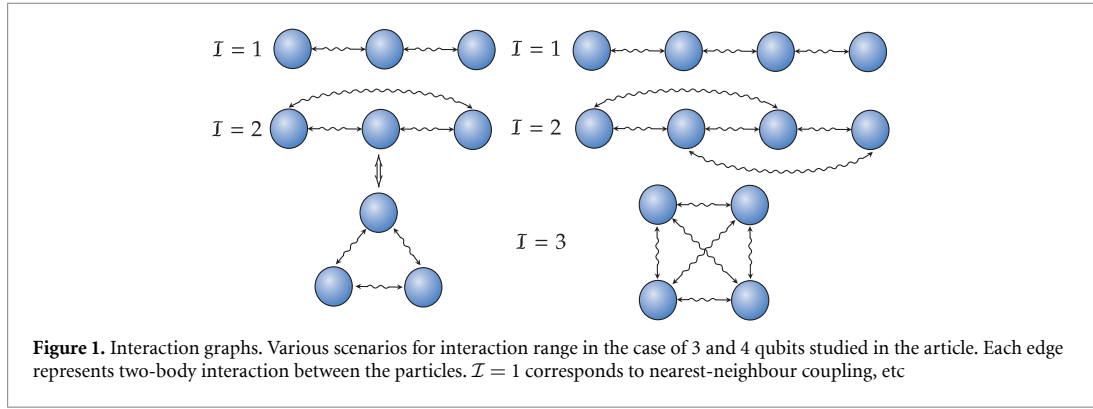
$$|D_N^k\rangle = \binom{N}{k}^{-1/2} \sum |s_1 \dots s_N\rangle, \quad s_i \in \{0, 1\}, \quad (5)$$

where the last sum is over all permutations of placing ‘1’ on k positions. The AME(5, 2) state examined in our work is given as

$$|AME(5, 2)\rangle = \frac{1}{\sqrt{8}} (|01111\rangle + |10011\rangle + |10101\rangle + |11100\rangle - |00000\rangle - |00110\rangle - |01001\rangle - |11010\rangle), \quad (6)$$

and was first constructed in [20]. It should be noted that we have chosen the common initial state $|0 \dots 0\rangle$ in all of the analysed cases, regardless of different fidelities with the target states.

Since we limit ourselves to the Hamiltonians with a certain symmetry, it is only possible to reach the states with the same invariance property. This fact can be easily seen using the symmetry operators $\mathcal{P}$, acting on pairs of particles, that leave H invariant, i.e. $\mathcal{P}H\mathcal{P}^{-1} = H$. If state $|\phi\rangle$ is invariant under $\mathcal{P}$ then $U|\phi\rangle = |\phi'\rangle$ has the same symmetry (the symmetry is preserved at all times). This comes from the fact that if H is invariant under $\mathcal{P}$ then by the series expansion U has the same property. Then clearly $[U, \mathcal{P}] = \mathcal{P}U - U\mathcal{P} = 0$ and $\mathcal{P}|\phi'\rangle = \mathcal{P}U|\phi\rangle = U\mathcal{P}|\phi\rangle = U|\phi\rangle = |\phi'\rangle$. The Hamiltonian corresponding to the complete interaction graph is invariant under all pairwise swaps of particles and imposes the restriction on unitarily available states to permutationally symmetric ones. Notice that such states are very common in quantum information [21–25].



3. Time-energy trade off—speed of evolution

Since H and t are linear in the exponent of the unitary evolution, one can always shorten or extend its duration by scaling the energy. It is the structure of energy levels, i.e. the variance or mean energy, that describes the ‘speed’ of evolution [26–28]. The result of Mandelstam and Tamm allows the calculation of the QSL for a given initial and final state based on the fidelity [26]

$$t \geq t_{\min} = \arccos(|\langle \psi(0) | \psi(t) \rangle|) \frac{\hbar}{\Delta H}, \quad (7)$$

where ΔH is the standard deviation of the energy operator in the initial state. The natural question arises, is the lower bound on this QSL tight for Hamiltonians with just one- and two-body terms? To study this problem we normalise the Hamiltonian in the following way

$$H \rightarrow \frac{H - \mathbb{1}E_{\min}}{\tilde{E}_{\max}}, \quad (8)$$

where $E_{\min}$ is the minimal eigenvalue of H and $\tilde{E}_{\max} = E_{\max} - E_{\min}$ is the maximal eigenvalue of $H - \mathbb{1}E_{\min}$. Note that this constraint does not fix the variance but the energy range available in the system. Thus, it can be seen as an energy bandwidth normalisation. A similar approach was recently pursued by Ness *et al* [29]. Normalisation endows us with operators H with eigenvalues $E_i \in [0, 1]$ in the natural units, which will be used throughout the rest of this paper.

4. Two-body interaction speed limits for three qubits

The key to our investigation of the effects of at most two-body interactions is hidden in the standard deviation of H . Indeed, the general speed limit (7) shows that in order to minimise the time required to produce a particular state, we should maximise the energy standard deviation. Intuitively, this quantity is maximised by the degenerate Hamiltonians because the energy levels can differ most in such cases and the eigenstates can be chosen such that the initial $|0 \cdots 0\rangle$ state admits the mean energy right in the middle between the energy levels.

This path is followed rigorously in appendix A.1 where we show that the Hamiltonian generating the W state can be chosen without any on-site terms and admits only two eigenvalues. According to our normalisation, the two energy levels are $E_0 = 0$ and $E_1 = 1$. This immediately leads to $\Delta H = \sqrt{p_1(1-p_1)}$, where p_1 is the probability that the initial state has energy E_1 . The maximal energy standard deviation is therefore $\Delta H = 1/2$ and the corresponding minimal time is given by

$$t_{\min}^{W,2} = \frac{\pi}{2} \frac{1}{1/2} = \pi, \quad (9)$$

where 2 in the superscript refers to isotropic and symmetric two-body Hamiltonians.

According to our analysis, there is no Hamiltonian with only two eigenvalues, within the considered class, that produces the GHZ state, see appendix A.2. The required energy spectrum has to involve at least five eigenvalues and the on-site terms turn out to be necessary. An exemplary Hamiltonian achieving the goal admits $\Delta H = 1/8$. This translates to

$$t_{\min}^{\text{GHZ},2} = \frac{\pi}{4} \frac{1}{1/8} = 2\pi. \quad (10)$$

This is similar to the geometrical approach in which the geodesics on two-dimensional subspace yield the fastest evolution [30, 31] while engaging more eigenstates leads to a slowdown [32]. Both of these times match our numerical results presented in section 5. An explicit form of the generating Hamiltonians is provided in appendix B. But before we get there let us compare these new two-body speed limits with the three-body ones emerging from equation (7).

4.1. Comparison with three-body interaction speed limit

A special case of three-body Hamiltonians are operators of the form $H = \sum_i \tilde{h}_i \sigma_i \otimes \sigma_i \otimes \sigma_i$, where $\tilde{h}_i$ give the strength of three-body interaction. Their spectrum consists of the minimal number of possible eigenvalues

$$E = \pm \sqrt{\tilde{h}_x^2 + \tilde{h}_y^2 + \tilde{h}_z^2}, \quad (11)$$

and therefore they lead to the evolution times

$$t_{\min}^{W,3} = \frac{\pi}{2} \frac{1}{1/2} = \pi, \quad (12)$$

$$t_{\min}^{GHZ,3} = \frac{\pi}{4} \frac{1}{1/2} = \frac{\pi}{2}, \quad (13)$$

which saturate the QSL. This is hardly a surprise since the optimal Hamiltonian contains N -body terms in general [30]. The GHZ Hamiltonian is purely a three-body operator, whereas the W Hamiltonian is the same as in the two-body interaction case. Accordingly, there is no speed increment due to tripartite coupling for the W state, but the GHZ state is produced four times faster.

The above predictions are consistent with the numerical calculations presented in the next section. For $N > 3$ we will proceed with numerics only due to the order of the characteristic polynomials found in higher dimensions. Note that even for $N = 3$ the polynomial is in general of the 8th order. The above estimations were possible due to the symmetry of H and focus on the number of its distinct eigenvalues.

5. Fastest entanglement generation for multipartite systems

We study unitary evolutions generated by Hamiltonians with at most two-body interaction terms through numerical computations. In order to answer our questions we maximise the fidelity between the target state $|\psi\rangle$ and the state produced at time t by different Hamiltonians:

$$\mathcal{F}(t) = \max_{h_{\mu\nu}, b_{\mu}} |\langle\psi|\exp(-iHt)|0\cdots 0\rangle|^2. \quad (14)$$

This procedure yields $t_{\min}$ for each target state and allows us to look into the ‘fastest’ Hamiltonians more closely. For the numerical optimisation, we performed parallel calculations using Wolfram Mathematica 13.0 and a random search-like algorithm written in Python. In both environments, the time variable was discretised into steps Δt ranging between 10^{-4} , when the fidelity grows rapidly, to 10^{-1} where higher temporal precision is not required. For each time t_k numerical maximisation of $|\langle\psi|\exp(-iHt_k)|0\cdots 0\rangle|^2$ is performed, with Hamiltonian re-normalisation after each choice of the parameters. In Mathematica, we used build-in numerical optimisation methods that included an evolutionary computation method called differential evolution and random search with 10^4 iterations. For the Python code, we sampled 100–1000 random initial parameters from a flat distribution and used the maximisation SciPy BFGS algorithm [33] to find the local maximum of fidelity for each sampling. Other optimisers from SciPy as well as genetic algorithms from PyGad library [34] were studied and the aforementioned one exhibited the best performance. All data sets acquired from both environments were then compared in order to find the maximal fidelity. According to the applied numerical methods, we consider fidelity approximation up to the sixth decimal place. Each point on the obtained plots in figures 2, 4 and 5, represents a different Hamiltonian that is optimal for a given time t . The final results on the two-body QSLs for the GHZ, W and AME (5, 2) states obtained in this section are gathered in table 1. For comparison, we also present there the Mandelstam–Tamm QSLs.

5.1. Three and four qubits

The two-body GHZ QSLs resulting from the optimisation are depicted in figure 2 and are consistent with the results obtained in the previous section. The three-body Hamiltonians for the GHZ state are, as expected, optimal and saturate the QSL. The complete interaction graph is leading to the smallest $t_{\min}$ in the two-body regime as it contains more interaction terms. The linear chain interaction graph ($\mathcal{I} = 1$) Hamiltonian

Table 1. Summary of two-body quantum speed limits for complete interaction graphs obtained in section 5. All the times except for the AME state (marked with a star) refer to fidelity equal to 1, up to the sixth decimal place. For the AME state we report the time corresponding to fidelity 0.99 due to an extremely long plateau afterwards, see the discussion in section 5.4. For comparison, the last column gives the Mandelstam–Tamm (MT) quantum speed limits. This bound is independent of the number of particles for the GHZ and W states. In the case of AME states, the MT quantum speed limit was given for $N = 5$.

State	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	MT
GHZ	$2\pi \approx 6.28$	$2\pi \approx 6.28$	$\frac{9\pi}{2} \approx 14.14$	$\frac{9\pi}{2} \approx 14.14$	$8\pi \approx 25.13$	$\frac{\pi}{2} \approx 1.57$
W	$\pi \approx 3.14$	$\frac{\sqrt{11}\pi}{\sqrt{2}} \approx 7.37$	$\frac{9\pi}{\sqrt{5}} \approx 12.64$	≈ 18.76	≈ 25.60	$\pi \approx 3.14$
AME	$2\pi \approx 6.28$	—	$\approx 10.72^*$	—	—	≈ 2.42

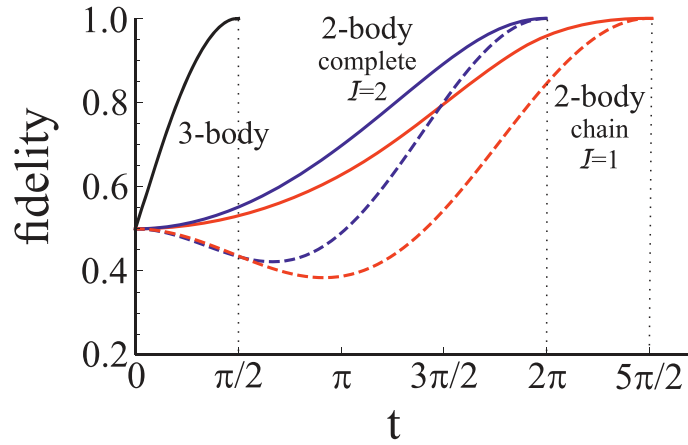


Figure 2. The fastest production of three-qubit GHZ state. The plot shows maximal fidelity $\mathcal{F}$ to the three-qubit GHZ state. The solid black line is for three-body interactions, the solid blue line for two-body interactions described by the complete interaction graph ($\mathcal{I} = 2$) and the solid red line is for the linear chain ($\mathcal{I} = 1$) scenario. All solid lines were obtained by optimising over Hamiltonians at every time instance. Exemplary optimal evolutions generated by a single Hamiltonian are plotted in dashed lines.

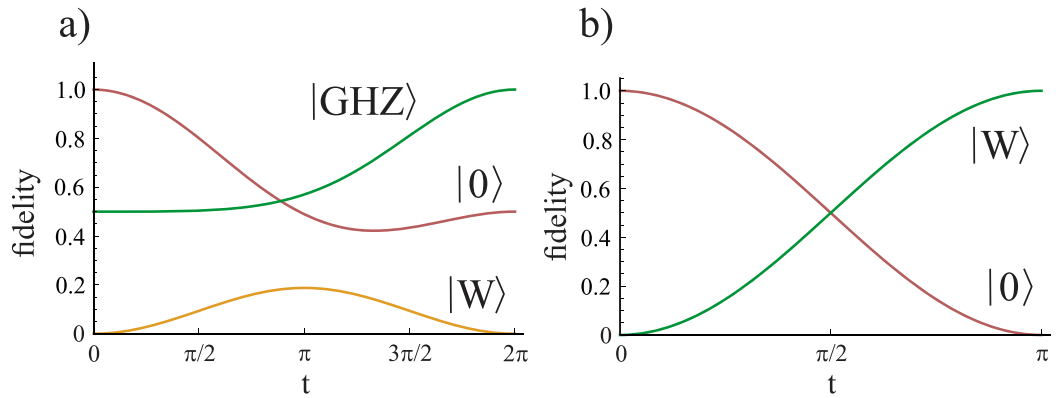
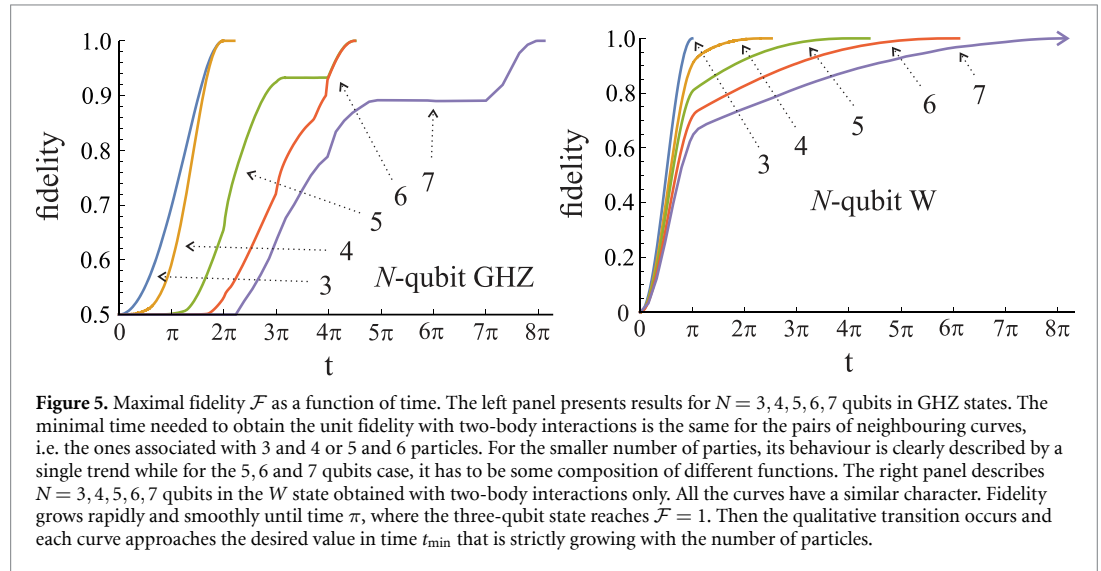
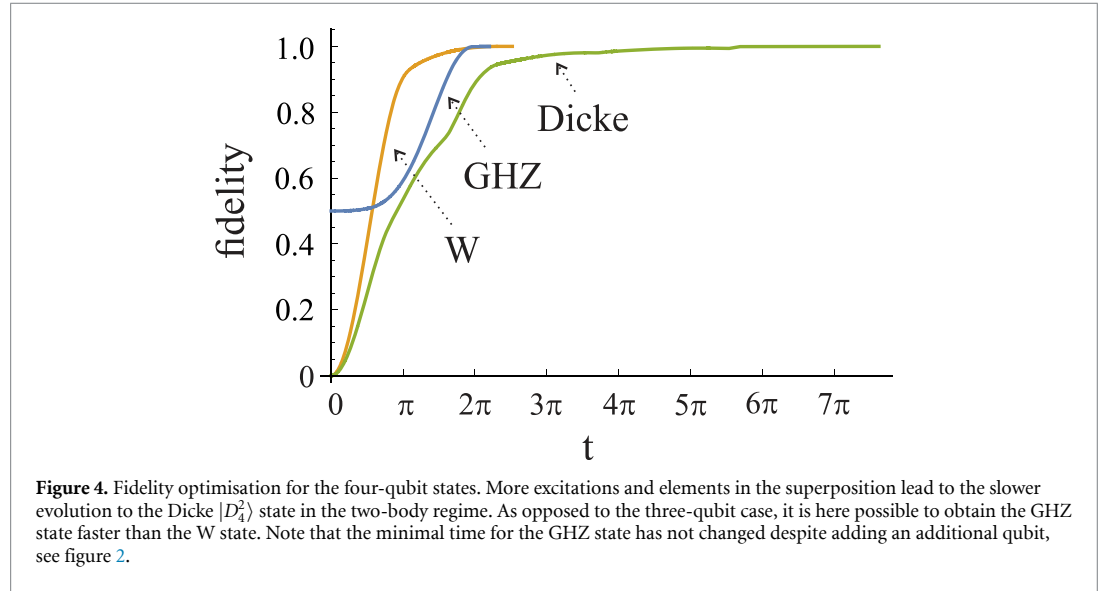


Figure 3. Change of fidelities during the optimal unitary evolution generated by Hamiltonians with at most two-body interactions for three-qubit (a) GHZ and (b) W state. Each curve is associated with a contribution from the corresponding state. For the GHZ, natural interactions incorporate additional W state components and thus lead to a slowdown. The saturation of QSL for the W state is a consequence of two-dimensional evolution.

performance is slower by the factor of $4/5$ ($t_{\min} = 5\pi/2$). Based on this we proceed with the complete interaction graph setting only in the next subsections.

The optimal unitaries generated by two-body Hamiltonians can incorporate more states than just the initial and target in the evolution and thus result in the slowdown. In the case of the three-qubit GHZ and W state, the change of fidelities associated with the corresponding states in time has been depicted in figure 3. As noted before, the optimal Hamiltonians, featuring a two-dimensional flow of fidelities between the initial and an appropriate orthogonal state, cannot be always realised without many-body terms. This statement is true for the GHZ state [30]. The time evolution of fidelities generated by at most two-body Hamiltonians for



this state leads to the emergence of non-zero contributions from the W state, see figure 3(a). Before fidelity with the target state reaches 1, the fidelities associated with the additional state have to be damped. This clearly needs some resources and is the reason for the slowdown. The low dimensional behaviour of fidelities in the two-body interactions regime for the W state can be observed in figure 3(b). As before, this confirms the analytical predictions and is the reason for the saturation of the general QSL in equation (7).

In the four-qubit scenario, there is a clear difference in the dynamics of fidelity optimisation for GHZ, W and two-excited Dicke state, see figure 4. While the QSL for the three-qubit W state was saturated with the two-body interaction in the four-qubit case it is not and the GHZ state can be achieved in a shorter time, see figure 4. The minimal time difference between one- and two-excited Dicke states is significant and results from the greater amount of flips in each of the basis states and the number of elements in the superposition. The computational difficulty of optimising the fidelity to the Dicke state was considerably greater than that of the W and GHZ states. It should be noted that, in the standard QSL, the minimum time required to obtain a perfect fidelity between the $|D_4^2\rangle$ and $|0000\rangle$ states is π , as they are orthogonal. However, we have observed that the minimal time required to achieve a perfect fidelity using two-body interactions is approximately 7.5 times greater.

5.2. GHZ states

The optimisation results obtained for $N = 3, 4, 5, 6, 7$ qubits in the GHZ state are shown in figure 5. An interesting effect associated with $t_{\min}$ for the neighbouring pairs of curves can be observed. The minimal time

needed to obtain the GHZ state with two-body interactions only is the same for 3 and 4 as well as 5 and 6 particles. This means that the amount of time (energy) needed for the creation of strong multipartite correlations is not strictly rising with the number of qubits. The observed pattern for the two-body GHZ QSL is extrapolated as follows:

$$t_{\min}^{\text{GHZ},2} = \pi \lceil N/2 \rceil^2 / 2, \quad (15)$$

where $\lceil N/2 \rceil$ gives the least integer that is not smaller than $N/2$. This agrees with the performed calculations and gives the expected behaviour of the time being grouped by pairs. This result can be easily comprehended in the sequential framework. For comparison, note that a product state of N qubits on which one applies $\lceil N/2 \rceil$ two-qubit and one-qubit gates is never genuinely N -party entangled, for $N > 3$. The time it takes to apply all these gates under the constraint that the total energy range (of the entire sequence) is $|E| = 1$ can be estimated as follows. Since there are $\lceil N/2 \rceil$ gates, the energy per gate is $E_1 = 1/\lceil N/2 \rceil$. The corresponding maximal energy standard deviation is $\Delta H = E_1/2$. The total time is therefore

$$t_{\text{seq}} \geq \lceil N/2 \rceil 2\lceil N/2 \rceil \pi / 4, \quad (16)$$

where $\lceil N/2 \rceil \pi / 2$ is the QSL bound for a single gate assuming the fidelity between the input and the output of the gate is equal to $1/2$, as one would have to create a global GHZ state. The shortest sequential time is, therefore, the same as $t_{\min}^{\text{GHZ},2}$ but we emphasise again that the sequence is not capable of producing genuine N -party entanglement.

To create a GHZ state of N -qubits in a sequential approach, a Hadamard gate followed by $N - 1$ CNOT gates are typically used. The Hamiltonian associated with the Hadamard gate at the time $t = 1$ is given by

$$H = \pi \begin{bmatrix} \frac{1}{4}(\sqrt{2} - 2) & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{4}(2 + \sqrt{2}) \end{bmatrix}, \quad (17)$$

which can be normalised by adding and dividing by π . This results in the desired superposition of two basis states at $t = \pi$. Similarly, the CNOT gate also has a normalised time of $t = \pi$. Hence, the minimal time required to generate the N -partite GHZ state after normalising the energy of the entire sequence is $t_{\min}^{\text{seq}} = N^2 \pi$. Recall that the minimum time obtained in the two-body regime, see equation (15), is less than or equal to $t_{\min}^{\text{seq}}$.

The natural effect of energy normalisation is the delay in a significant change of the initial state under unitary evolution. In a sense, it resembles the concept of static friction in classical mechanics, see figure 5. Due to the high fidelity between the initial and final state, this effect is clearly much more visible in the evolution to the GHZ states.

5.3. W states

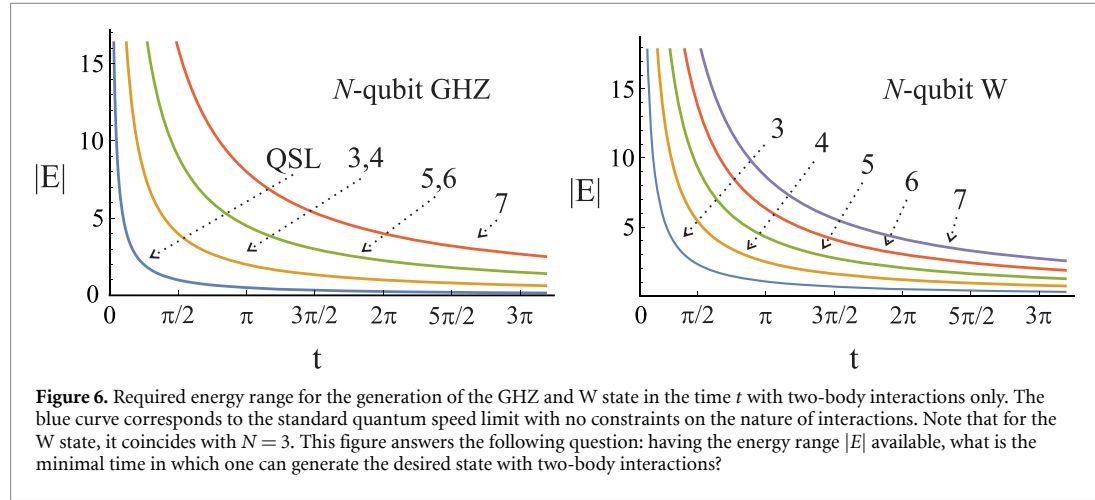
The maximisation results for the W states of $N = 3, 4, 5, 6, 7$ qubits are presented in the right panel of figure 5. Analogous to the GHZ case the minimal times are growing with the number of particles but do not join in pairs. The character of the optimised functions is similar for each curve.

The findings presented here can also be interpreted as optimal in the sense that they enable the production of the desired states in a shorter time compared to the conventional sequential approach. This has been already discussed in section 5.2 for the GHZ states. In the case of the W states, even for three qubits, the standard circuit consists of five gates, including one Hadamard and two CNOT gates discussed in the previous subsection. Recall that using our approach we have shown that for $N = 3$ the QSL can be saturated for the W state. This makes our argument even stronger.

Another interesting question, in terms of resources, concerns the achievable rate of producing a multipartite entangled state with the natural interaction, given an energy range $|E|$ that is either greater or smaller than 1. Our normalisation method allows us to extend or tighten the energy range of H by simply multiplying it by $|E|$. This results in eigenvalues $E_i \in [0, |E|]$, with $|E|$ being inversely proportional to time. The time-energy trade-off resulting from the presented QSLs for GHZ and W states has been presented in figure 6.

5.4. Five qubit AME state

A somewhat different example is provided by the AME state defined in equation (6). The isotropic and symmetric Hamiltonian evolution yields $\mathcal{F} \approx 0.1$ even for times $t \gg 1$. This poor fidelity is a consequence of the symmetry of that state. Although it is not completely symmetric, it remains invariant under $\mathcal{P}_{24}\mathcal{P}_{35}$,



where $\mathcal{P}_{ij}$ denotes a swap between the i -th and j -th particle. To achieve the AME state we, therefore, introduce a new symmetry of the Hamiltonian that aligns with the desired state's symmetry, i.e. the Hamilton operator should remain invariant under $\mathcal{P}_{24}\mathcal{P}_{35}$. This leads to 29 parameters that require optimisation, in contrast to the nine parameters involved in the previous case.

Our analysis of the speed of unitary evolution for the AME state shows that there exists a long-range plateau after achieving $\mathcal{F} = 0.99$. This implies that it takes considerably less time to evolve the state from 0%–99% fidelity than to enhance it by only 1%. Based on this observation and the number of parameters that require optimisation, we calculated the time for the fidelity to surpass 99%. The final result is $t_{99\%} = 10.72$, as presented in table 1.

5.5. Generating Hamiltonians

The numerical optimisation has allowed us to derive the analytical expressions for the Hamiltonians that generate the fastest evolution to the W and GHZ states. The explicit formulas, including normalisation factors, can be found in appendix B. We shall use the conventional periodic boundary conditions $\sigma_{\mu}^{(N+1)} = \sigma_{\mu}^{(1)}$. Based on our results, the optimal (unnormalised) Hamiltonian for the W states takes on a simple form

$$H = \sum_{i=1}^N \sum_{j=i+1}^N \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} - (N-3) \sigma_x^{(i)}. \quad (18)$$

For the GHZ state with an odd number of qubits, the general expression for the fastest two-body Hamiltonian is:

$$H = \sum_{i=1}^N \sum_{j=i+1}^N \sigma_x^{(i)} \sigma_x^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)} + 2\sigma_y^{(i)}. \quad (19)$$

Surprisingly, this Hamiltonian is realised by a spin-1/2 chain beyond the nearest-neighbour XXY coupling and magnetic field in the y direction.

Results for the GHZ state with an even number of qubits are presented in appendix B. As an example, the Hamiltonian for $N = 4$ is

$$H = \sum_{i=1}^4 \sum_{j=i+1}^4 \frac{1}{2} \left(\sigma_x^{(i)} \sigma_y^{(j)} + \sigma_y^{(i)} \sigma_x^{(j)} \right) + \sigma_y^{(i)} \sigma_y^{(j)} + \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) \sigma_z^{(i)} \sigma_z^{(j)}. \quad (20)$$

6. Conclusion

We considered the QSL for the generation of multipartite entangled states in the presence of physically motivated constraints. In particular, a natural constraint in multipartite systems is that the Hamiltonian contains at most two-body interaction terms. We showed that such Hamiltonians allow one to generate genuinely multipartite entangled states, such as W, GHZ, Dicke (up to fidelity 1) and AME(5, 2)

(up to fidelity 0.99), from an initial product state. Moreover, we have established the fastest times to generate them, taking into account the limitations imposed by the energy and type of interaction. The optimal Hamiltonians were also derived. We have found that the amount of time (energy) required for the creation of strong multipartite correlations does not have to rise strictly with the number of qubits. This phenomenon has been observed in the context of GHZ states, which are generated in the same minimal time for $N = 2, 3$ and also for $N = 4, 5$. We conjecture that this trend continues and we extrapolated the shortest time in a closed formula for any N . Our results demonstrate that in terms of time resources, our approach outperforms the sequential model. This sets the direction for fast entanglement generation with a lower noise level in specific tasks. Future research concerning this problem could concentrate on the analytical derivation of the suggested QSLs and a more detailed study of the generating Hamiltonian implementation.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Eigenvalues of 3-qubit system with 2-body interactions

A.1. Optimal creation of W state

Let us start with the Hamiltonian without one-body terms

$$H = \sum_{i=1}^N \sum_{j=i+1}^N \sum_{\mu_i, \mu_j} h_{\mu_i \mu_j} \sigma_{\mu_i}^{(i)} \sigma_{\mu_j}^{(j)}, \quad h_{\mu_i \nu_j} = h_{\mu_k \nu_l} = h_{\nu_k \mu_l}. \quad (\text{A.1})$$

The eigenvalues of the unnormalised isotropic and symmetric complete interaction graph Hamiltonian are

$$E = \begin{bmatrix} -h_{xx} - h_{yy} - h_{zz} \\ -h_{xx} - h_{yy} - h_{zz} \\ -h_{xx} - h_{yy} - h_{zz} \\ -h_{xx} - h_{yy} - h_{zz} \\ h_{xx} + h_{yy} + h_{zz} - 2F \\ h_{xx} + h_{yy} + h_{zz} - 2F \\ h_{xx} + h_{yy} + h_{zz} + 2F \\ h_{xx} + h_{yy} + h_{zz} + 2F \end{bmatrix}, \quad (\text{A.2})$$

where $F = [h_{xx}^2 + h_{yy}^2 + h_{zz}^2 + 3(h_{xy}^2 + h_{xz}^2 + h_{yz}^2) - h_{xx}h_{yy} - h_{xx}h_{zz} - h_{yy}h_{zz}]^{1/2}$. Such Hamiltonian has at least two energy levels. Two unnormalised eigenvalues occur for

$$h_{xx} = \frac{-h_{yy}h_{zz} + h_{xy}^2 + h_{yz}^2 + h_{xz}^2}{h_{yy} + h_{zz}},$$

and are given by $E = \{-\eta, -\eta, -\eta, -\eta, -\eta, -\eta, 3\eta, 3\eta, 3\eta\}$, where

$$\eta = (h_{yy}^2 + h_{zz}^2 + h_{xy}^2 + h_{yz}^2 + h_{xz}^2 + h_{yy}h_{zz}) / (h_{yy} + h_{zz}).$$

This choice of $h_{\mu\nu}$ preserves the hermiticity of H and allows the creation of the W state via the unitary evolution in the optimal time. One verifies that in this case $\Delta H = 1/2$ as reported in the main text.

A.2. Optimal creation of GHZ state

A general Hamiltonian (A.1) cannot produce the GHZ state via the unitary evolution due to vanishing overlap $\langle 111 | U | 000 \rangle$. Accordingly, we need at least one non-zero b_μ coefficient. Without loss of generality, we assume $b_y \neq 0$. The modified eigenvalues are

$$E = \begin{bmatrix} -h_{xx} - h_{yy} - h_{zz} + b_y \\ -h_{xx} - h_{yy} - h_{zz} + b_y \\ -h_{xx} - h_{yy} - h_{zz} - b_y \\ -h_{xx} - h_{yy} - h_{zz} - b_y \\ h_{xx} + h_{yy} + h_{zz} - \frac{1}{2}\sqrt{V} - \frac{1}{2}\sqrt{\tilde{V}} \\ h_{xx} + h_{yy} + h_{zz} - \frac{1}{2}\sqrt{V} + \frac{1}{2}\sqrt{\tilde{V}} \\ h_{xx} + h_{yy} + h_{zz} + \frac{1}{2}\sqrt{V} - \frac{1}{2}\sqrt{\tilde{V}} \\ h_{xx} + h_{yy} + h_{zz} + \frac{1}{2}\sqrt{V} + \frac{1}{2}\sqrt{\tilde{V}} \end{bmatrix}, \quad (\text{A.3})$$

where V and $\tilde{V}$ are complicated functions of all the parameters. The only possibility for an additional degeneracy, other than the one occurring for the first two pairs of energies, is when $\sqrt{V} = \sqrt{\tilde{V}}$. This gives at least five different energy levels which can be arranged for the energy standard deviation in the initial state equal to $\Delta H = 1/8$.

Appendix B. Explicit formulas for the generating Hamiltonians

B.1. Optimal Hamiltonians for W state

The following Hamiltonians H_N lead to the fastest evolution to the N -qubit W state in the complete graph configuration:

$$H_3 = \frac{1}{4\sqrt{3}} \left[\sum_{i=1}^3 \sum_{j=i+1}^3 \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} + 2\sqrt{3}\mathbb{1} \right] \quad (\text{B.1})$$

$$H_4 = \frac{1}{4\sqrt{22}} \left[\sum_{i=1}^4 \sum_{j=i+1}^4 \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} - \sigma_x^{(i)} + 2\sqrt{22}\mathbb{1} \right] \quad (\text{B.2})$$

$$H_5 = \frac{1}{36} \left[\sum_{i=1}^5 \sum_{j=i+1}^5 \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} - 2\sigma_x^{(i)} + 18\mathbb{1} \right] \quad (\text{B.3})$$

$$H_6 = \frac{1}{4\sqrt{3(41 + \sqrt{921})}} \left[\sum_{i=1}^6 \sum_{j=i+1}^6 \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} - 3\sigma_x^{(i)} - 2\sqrt{3(41 + \sqrt{921})}\mathbb{1} \right] \quad (\text{B.4})$$

$$H_7 = \frac{1}{32(16 + \sqrt{31})} \left[\sum_{i=1}^7 \sum_{j=i+1}^7 \sigma_x^{(i)} \sigma_z^{(j)} + \sigma_z^{(i)} \sigma_x^{(j)} - 4\sigma_x^{(i)} - 2(16 + \sqrt{31})\mathbb{1} \right]. \quad (\text{B.5})$$

B.2. Optimal Hamiltonians for GHZ state

Below are the explicit formulas for the Hamiltonians H_N generating the fastest evolution to the N -qubit GHZ states in the two-body interaction regime. All formulas, except the first one, pertain to the complete interaction graph:

$$H_3^{I=1} = \frac{1}{10} \left[\sum_{i=1,2} \left(\sqrt{2} \left(\sigma_x^{(i)} \sigma_x^{(i+1)} + \sigma_z^{(i)} \sigma_z^{(i+1)} \right) + \sigma_y^{(i)} \right) + 5\mathbb{1} \right] \quad (\text{B.6})$$

$$H_3 = \frac{1}{16} \left[\sum_{i=1}^3 \sum_{j=i+1}^3 \left(\sigma_x^{(i)} \sigma_x^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)} + 2\sigma_y^{(i)} \right) + 9\mathbb{1} \right] \quad (\text{B.7})$$

$$H_4 = \frac{1}{8\sqrt{2}} \left[\sum_{i=1}^4 \sum_{j=i+1}^4 \left(\frac{1}{2} \left(\sigma_x^{(i)} \sigma_y^{(j)} + \sigma_y^{(i)} \sigma_x^{(j)} \right) + \sigma_y^{(i)} \sigma_y^{(j)} + \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) \sigma_z^{(i)} \sigma_z^{(j)} \right) + \mathbb{1} \left(3 - 5\sqrt{2} \right) \right] \quad (\text{B.8})$$

$$H_5 = \frac{1}{36} \left[\sum_{i=1}^5 \sum_{j=i+1}^5 \left(\sigma_x^{(i)} \sigma_x^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)} + 2\sigma_y^{(i)} \right) + 20\mathbb{1} \right] \quad (\text{B.9})$$

$$H_6 = \frac{1}{36} \left[\sum_{i=1}^6 \sum_{j=i+1}^6 \left(- \left(\sigma_x^{(i)} \sigma_y^{(j)} + \sigma_y^{(i)} \sigma_x^{(j)} \right) - \sigma_z^{(i)} \sigma_z^{(j)} \right) + 21\mathbb{1} \right] \quad (\text{B.10})$$

$$H_7 = \frac{1}{64} \left[\sum_{i=1}^7 \sum_{j=i+1}^7 \left(\sigma_x^{(i)} \sigma_x^{(j)} - \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)} + 2\sigma_y^{(i)} \right) + 35\mathbb{1} \right]. \quad (\text{B.11})$$

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Exploring many-body interactions through quantum Fisher information

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The investigation of many-body interactions holds significant importance for both quantum foundations and information. Hamiltonians coupling multiple particles at once, beyond other applications, can lead to faster entanglement generation, multiqubit gate implementation, and improved error correction. As an increasing number of quantum platforms can enable the realization of such physical settings, it has become interesting to study the verification of many-body-interaction resources. In this work, we explore the possibility of higher-order coupling detection through the quantum Fisher information. For a family of normalized symmetric k -body Ising-like Hamiltonians, we derive bounds on the quantum Fisher information in product states. Due to its ordering with respect to the order of interaction, we demonstrate the possibility of detecting many-body couplings for a given Hamiltonian from the discussed family by observing violations of an appropriate bound. As a possible extension to these observations, we further analyze an example concerning three-body-interaction detection in the XY model.

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I. INTRODUCTION

Among many fascinating phenomena in physics, in recent years studying the nature of interactions has become not only a subject of fundamental research but also part of the current pursuit towards modern quantum technologies. Most current controllable quantum systems rely solely on the two-body interactions between particles [1–3]. Nevertheless, many-body interactions are often discussed in the context of effective models in low-energy physics [4–6]. These include studies of spin systems [7–11], extended Hubbard models describing ultracold atoms or molecules in optical lattices [12–24], quantum chemistry [25–28], and nuclear and particle physics [29–32]. Further applications of higher-order interactions can be found in entanglement generation [33–36] and error correction [37–39], among others [40–43]. Thus, the search for many-body interactions plays a significant role for both quantum foundations and future quantum technologies [44–49]. With rising demand for the implementation of many-body-interaction Hamiltonians, it has become interesting to study their verification [43]. A universal method solving this task could probe new physical effects and give insights into how to engineer the desired Hamiltonians. Furthermore, it would be useful for Hamiltonian-learning protocols because they often require prior knowledge of the maximal degree of an interaction graph (see, e.g., [50]). On the other hand, it would answer the question of whether the Hamiltonian is even worth learning if we are interested in its many-body-interaction properties. In this work, we demonstrate that the existence of genuine many-body interactions can be verified through

quantum Fisher information (QFI), thus possibly paving the way for a new area of nonlocal-interaction research.

QFI has been studied in various contexts [51–61], including quantum phase transitions [62–67] and, most notably, quantum metrology [68,69]. For a given Hamiltonian it allows one to find states that guarantee measurement precision beyond the classical limit [68,69]. An experimental measurement, or estimation, of QFI can be performed via several techniques. The proposed theoretical protocols take advantage of the dynamical susceptibility [70], projections onto the initial state (Loschmidt-echo protocol) [71], overlap detection [72], randomized measurements [73–75], and adiabatic perturbation theory [76]. A direct QFI (or its lower bound) measurement was performed in, e.g., [74,77,78].

In the standard metrological scenario, Hamiltonians under consideration are strictly local. However, many-body interacting systems were also examined in terms of improved scaling [69]. Here, to make our presentation as simple as possible, we study the family of k -local permutationally invariant Ising-like Hamiltonians of N particles. For them, we illustrate the main premise behind this paper, which is that QFI can be used to detect the interaction order. Based on this observation, we derive bounds on the Hamiltonians with at most two-body-interaction terms, showing the possibility of witnessing the presence of k -body interactions with product states (see Fig. 1). As a possible extension, we also discuss an example of a similar study for the XY model.

II. MOTIVATING EXAMPLE

First, we start with a simple example that motivates our work. Consider a system of three qubits on a triangle that interact in the σ_z direction. Fixing the order of interaction at

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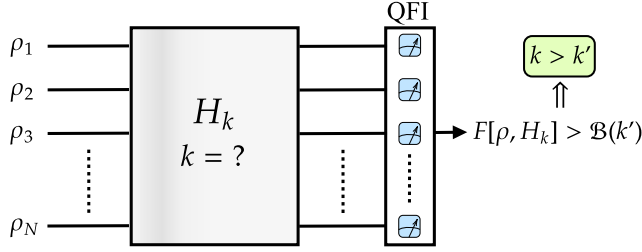


FIG. 1. Motivation. In this work, we explore the order of interactions, i.e., the maximal number of simultaneously coupled qubits in a Hamiltonian, through the use of quantum Fisher information. For symmetric Ising-like Hamiltonians H_k (7), we show that quantum Fisher information calculated in a product state is bounded with a bound dependent on k . This allows one to detect whether H_k manifests interactions beyond the k' -body case, where $k > k'$. The basic idea is as follows. Starting from an arbitrary separable state, we let the particles interact via H_k , and then we measure its quantum Fisher information. Based on our findings, we check whether the result is greater than the maximally allowed value for the chosen k' . If the answer is positive, we can claim that H_k contains terms of at least $(k' + 1)$ th order.

2, the two-body-interaction Hamiltonian is given as

$$H_2 = \frac{1}{2}(\sigma_z \otimes \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z \otimes \sigma_z + \sigma_z \otimes \mathbb{1} \otimes \sigma_z),$$

where, in order to keep the correspondence with standard metrological notation, the prefactor was chosen such that the maximal eigenvalue does not exceed $N/2 = 3/2$. On the other hand, focusing on only the three-body couplings, we get

$$H_3 = \frac{3}{2}(\sigma_z \otimes \sigma_z \otimes \sigma_z).$$

Now, let us consider the maximization of quantum mechanical variance $(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2$ over the set of pure product states. For H_2 the optimal product state, i.e., the state that maximizes the variance, is given as $|\psi_{\text{prod}}\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle)^{\otimes 3}$, where $|0\rangle$ and $|1\rangle$ are eigenstates of σ_z and $p = (3 + \sqrt{3})/6$. See the Appendix for direct calculations. The exact value of the maximal variance is $(\Delta H_2)_{\text{max}}^2 = 1$. It is straightforward to show that for trivial Hamiltonians with one-body terms the variance is given as $(\Delta H_1)_{\text{max}}^2 = N/4$. Consequently, for three qubits we get $(\Delta H_1)_{\text{max}}^2 = 3/4$, which is smaller than in the two-body case. Moving on to the higher-order-interaction Hamiltonian, the optimal state is *no longer* a tensor product of identical local states $|\psi\rangle^{\otimes 3}$. It might seem that the form of the state that maximizes the variance for the two-body interaction is trivially obtained from the symmetry, but it is not. In fact, the only condition arising from it is that $\langle \psi_1 \psi_2 \psi_3 | H | \psi_1 \psi_2 \psi_3 \rangle = \langle \psi_2 \psi_1 \psi_3 | H | \psi_2 \psi_1 \psi_3 \rangle$, with similar conditions for all other permutations of indices. This can be seen clearly in the three-body-interaction case. Examining the spectrum of H_3 , we can construct a product state which is a superposition of eigenstates associated with the maximal and minimal eigenvalues (see the Appendix). Namely, we have $|\psi_{\text{prod}}\rangle = |00\rangle \otimes 1/\sqrt{2}(|0\rangle + |1\rangle)$ and the corresponding $(\Delta H_3)_{\text{max}}^2 = 9/4$. Limiting ourselves to the product states being a tensor product of identical one-qubit states, we would arrive at the value accessible for $k = 2$. From these results, it

is apparent that

$$\begin{aligned} L_S &= (\Delta H_1)_{\text{max}}^2 < (\Delta H_2)_{\text{max}}^2 \\ &< (\Delta H_3)_{\text{max}}^2 \leftarrow L_H, \end{aligned}$$

where we denote the values of variances coinciding with the standard limit and the Heisenberg limit for one-body Hamiltonians and entangled states as L_S and L_H , respectively. Here, it is also worth noting that the ordering of maximal variance is not due to the possibility of obtaining higher eigenvalues by changing the interaction type or the number of terms in the Hamiltonian since they are all normalized in the same manner. This simple example shows that variance of a Hamiltonian calculated in a product state may be exploited as a tool for detecting many-body-interaction terms.

III. PHYSICAL MODEL AND PRELIMINARIES

In the following, to make our considerations more general, instead of the variance of a Hamiltonian, we will focus on the quantum Fisher information. For any Hermitian operator A and state ρ it is defined as

$$F[\rho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{(\lambda_k + \lambda_l)} | \langle k | A | l \rangle |^2, \quad (1)$$

with $|k\rangle$ and λ_k being the eigenvectors and eigenvalues of a density matrix ρ , respectively, and the sum being evaluated only when the denominator is different from zero. Most commonly, it is used in quantum metrology because it allows one to find a suitable state that one can use to boost phase-measurement sensitivity. It was shown that for local Hamiltonians QFI in product states is bounded by N [79]. Introducing entangled states moves this bound to N^2 , leading to the so-called Heisenberg limit [79,80]. Moreover, for any quantum state ρ a pure state $|\psi\rangle$ exists for which [68]

$$F[\rho, A] \leq 4(\Delta A)_{|\psi\rangle}^2 = F[|\psi\rangle, A]. \quad (2)$$

Hence, any maximization of QFI can be performed only among the pure states. This justifies the choice of variance and calculations presented in the previous section.

Our task is to construct a model and derive a bound on the quantum Fisher information in a *product state* that is an increasing function of an interaction order. Now, let us precisely define the notion of higher-order interactions. A given Hamiltonian

$$H_k = \sum_j h_j^{(k)} \quad (3)$$

is considered to be k -local if each $h_j^{(k)}$ acts nontrivially on at most k particles. Hamiltonians which act exactly on k or at most on K particles will be denoted as H_k and $\mathcal{H}^{(K)}$, respectively. By this definition, we will use the k -locality and k th order of interactions interchangeably. Measurements enhanced with Hamiltonians that are nonlinear in operators have been studied in terms of sensing and scaling in the past [69]. They include, for example, products of photon creation and annihilation operators [81–84], angular momentum and its components [85–88], many-body models [59,60,89–92], and, more generally, powers of the sum of local operators

[93], as well as the general k -body [94] and symmetric k -body-interaction Hamiltonians [93,95], with no comparison between entangled and separable states in the latter cases. However, in this work, we study a specific scenario which resembles the standard metrological approach but will be used in a far different context. For such means, we will consider only the symmetric Ising-like Hamiltonians. First, we define the auxiliary Hamiltonians that contain only the k -body-interaction terms as

$$H_k = \mathcal{N} \sum_{(i_1, \dots, i_k) \in G_k} \sigma_z^{i_1} \sigma_z^{i_2} \dots \sigma_z^{i_k}, \quad (4)$$

where $\mathcal{N}$ is a normalization constant and G_k is a fully connected interaction graph for k -body interactions; the summation is performed over all k -partite subsets of particles, making it permutationally invariant. In the case of $k = 1$ we retrieve the standard metrological Hamiltonians

$$H_1 = \frac{1}{2} \sum_{i=1}^N \sigma_z^i \quad (5)$$

if proper normalization is chosen. Another example can be given for $k = 2$ and $\mathcal{N} = J$, namely,

$$H_2 = J \sum_{i < j} \sigma_z^i \sigma_z^j. \quad (6)$$

Note that the above Hamiltonian represents a long-range-interaction Ising model on a complete interaction graph. For more examples see Secs. II and V. Now, a general symmetric Ising-like Hamiltonian containing at most k -body interactions can be constructed as

$$\mathcal{H}^{(K)} = \mathcal{N} \sum_{k \leq K} \alpha_k H_k, \quad (7)$$

where, again, $\mathcal{N}$ is a normalization constant and α_k are real numbers. Throughout most of this paper, we will focus on $\forall_k \alpha_k = 1$ and discuss its modifications in the examples. It is worth noting that all of the results presented here hold for any Hamiltonian equivalent under local unitary operations. This follows from the property of QFI which states that $F[\rho, U^\dagger A U] = F[U \rho U^\dagger, A]$ and local unitary invariance of entanglement. Hamiltonians from a different class will be discussed in Sec. VI.

Before we move on to our results, we need to specify a proper normalization for $\mathcal{H}^{(K)}$. Our motivation is to test the order of interactions present in the system. Moreover, we would like to compare our results with the metrological approach and stay consistent with its results. In order not to break the classical and Heisenberg scaling we choose to set the operator norm $\|\mathcal{H}^{(K)}\| = \max_\phi \|\mathcal{H}^{(K)}|\phi\rangle\| = N/2$. Note that we want the Hamiltonian to appear as if no k -body interactions were present in it. If we dropped this assumption, we would obtain the nonlinear Hamiltonians scalings $N^{\sim k}$ (see, e.g., [81–93]). Our approach leads to an upper bound on variance and QFI which cannot exceed N^2 for any k and quantum state $|\psi\rangle$, entangled or not. We implement this norm by setting $\mathcal{N}$ to $N/(2 \max_i |E_i|)$, where E_i is an eigenvalue of $\mathcal{H}^{(K)}$. As this normalisation is multiplication by a constant, there still is an

overlap between the previous research and our results. This will be commented on in Sec. V.

IV. INTERACTION-DEPENDENT BOUNDS ON QFI FOR PRODUCT STATES

For the considered model it is possible to derive the explicit formulas for the eigenvalues based on its symmetry. For the k -local Hamiltonian defined in (7) and included normalization we get

$$\Omega_e^{N,K} = \sum_{k \leq K} \omega_e^{N,k}, \quad \omega_e^{N,k} = \sum_{j=0}^e \frac{N}{2} \frac{\binom{e}{j} \binom{N-e}{k-j}}{\binom{N}{k}} (-1)^j, \quad (8)$$

where e is the number of excitations, i.e., the number of $|1\rangle$ elements in the N -qubit state. Here, $\omega_e^{N,k}$ represents the eigenvalues of a Hamiltonian with only k -body terms.

As a first step, we will limit ourselves to a scenario with a fixed k , i.e., H_k . In such a case, the maximization of variance and hence the QFI (2) over pure product states can be performed as follows. Since variance is the function of the square modulus of amplitudes and Hamiltonian eigenvalues, we can consider only product states of the following form:

$$|\psi_{\text{prod}}\rangle = \bigotimes_i^N (\sqrt{p_i}|0\rangle + \sqrt{1-p_i}|1\rangle).$$

Using this parametrization and calculating the variance, we get

$$(\Delta H_k)^2 = \sum_{l_1, \dots, l_N=0,1} \prod_{i=1}^N p_i^{l_i} (1-p_i)^{1-l_i} (\omega_{l_1+\dots+l_N}^{N,k})^2 - \left(\sum_{l_1, \dots, l_N=0,1} \prod_{i=1}^N p_i^{l_i} (1-p_i)^{1-l_i} \omega_{l_1+\dots+l_N}^{N,k} \right)^2.$$

A necessary condition for the existence of a multivariable function extremum is the disappearance of its first derivatives. Taking derivatives of $(\Delta H)^2$ over p_i , we arrive at the system of N equations linear in p_k ,

$$\begin{aligned} 0 &= \frac{\partial (\Delta H_k)^2}{\partial p_k} \\ &= \sum_{l_1, \dots, l_N=0,1} (-1)^{1-l_k} \prod_{i \neq k} p_i^{l_i} (1-p_i)^{1-l_i} (\omega_{l_1+\dots+l_N}^{N,k})^2 \\ &\quad - 2 \left(\sum_{l_1, \dots, l_N=0,1} \prod_{i=1}^N p_i^{l_i} (1-p_i)^{1-l_i} \omega_{l_1+\dots+l_N}^{N,k} \right) \\ &\quad \times \sum_{l_1, \dots, l_N=0,1} (-1)^{1-l_k} \prod_{i \neq k} p_i^{l_i} (1-p_i)^{1-l_i} \omega_{l_1+\dots+l_N}^{N,k}, \end{aligned} \quad (9)$$

where the dependence on p_k is only in the term in the parentheses. Since we know all ω 's, this can be solved directly, and the resulting set of $(p_1, \dots, p_N)$ satisfying $0 \leq p_i \leq 1$ is the solution to our optimization problem. We present a step-by-step solution for $N = 3$ in the Appendix.

V. TESTING k -BODY INTERACTIONS

The most interesting case for higher k -body-interaction terms would be to find an explicit bound on QFI for $k = 2$. For fixed $k = 1$ we once again refer to the fundamental result obtained for local Hamiltonians [79]. One of its many consequences is the classical scaling with $\max_{|\psi_{\text{prod}}\rangle} 4(\Delta H_1)^2 = N$. It is straightforward to see that our approach is consistent with that. With direct solutions of (9) and numerical calculations, for $k = 2$ we notice that up to $N = 13$, the solution to the given optimization problem is obtained with $|\psi_{\text{prod}}\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle)^{\otimes N}$. For more discussion see the Appendix. This result is known to hold asymptotically when the number of particles is much greater than the interaction order, here $n \gg 2$ [93]. It is valid in our approach because a nonlinear Hamiltonian $(\sum_i H_1)^2$ for which this result was derived is equal to $N\mathbb{1} \cdots \mathbb{1} + NH_2$ and thus holds in our case. Furthermore, it is consistent with the results obtained for the Lipkin-Meshkov-Glick (LMG) model and the nearest-neighbor (as well as fully connected) Ising model with an interaction parameter smaller than its critical value (general parameter range) [91,92]. This will also be elaborated on later. For the given state, the variance reduces to

$$(\Delta H_2)^2 = \frac{4}{N-1} [-2N(2N-3)p^4 + 4N(2N-3)p^3 - N(5N-7)p^2 + N(N-1)p]. \quad (10)$$

By finding zeros of its derivative we get three unique roots, from which the one that maximizes the variance is given as

$$p_{\text{max}} = \frac{2N-3 + \sqrt{2N^2-7N+6}}{4N-6}. \quad (11)$$

The resulting maximal QFI in a product state, i.e., the solution to (9) obtained with the above p_{max} , for $k = 2$ is

$$F_{\text{max}}[|\psi_{\text{prod}}\rangle, H_2] = \frac{2N(N-1)}{2(N-2)+1}. \quad (12)$$

The results presented here coincide with the ones obtained for the LMG model in the limit of large coupling constant γ and the context of statistical speed (see the Supporting Information of [91]). This is due to the fact that once the coupling constant is large, the single-qubit terms in the LMG model can be neglected. Then, up to a constant, the LMG Hamiltonian is equivalent to H_2 studied here. Note that (12) is an increasing function of N which bounds, from the above, the one-body scaling. Explicit results for $k \leq 5$ and $N \leq 10$ are presented in Fig. 2. As expected, for $k = 1$ the maximal QFI scales as N [79]. For $k > 1$, the results clearly show that the maximal Fisher information in a product state is ordered with respect to the fixed interaction order k . This observation motivates our goal of testing the presence of k -body interactions with QFI and product states. We should also mention that for $k = N$ the QFI in a product state is maximal, as in [89], and a constant trend for $N < 2k$ is observed (see Fig. 2).

A natural extension of the above considerations is to fix the number of qubits N and study bounds on maximal QFI with Hamiltonians H_k containing at most k -body couplings. Technically, knowing the eigenvalues of such Hamiltonians [see (8)], we can once again maximize the variance and hence

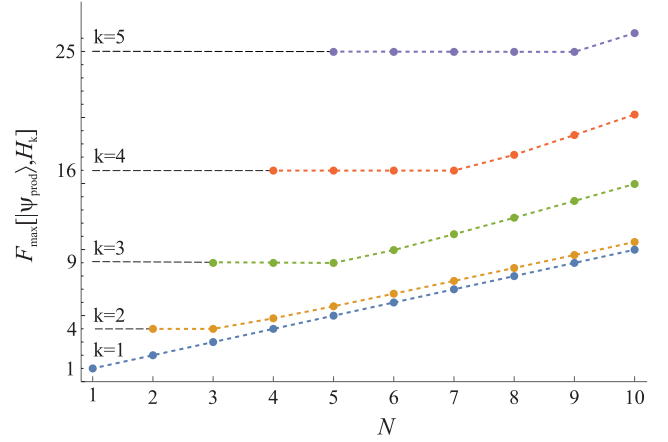


FIG. 2. Maximal quantum Fisher information of k -local Hamiltonian H_k in a product state as a function of the number of particles N . Numerical solutions of (9) for different choices of k are plotted in different colors, and dashed lines are guides to the eye. Values associated with $k = 1$ scale linearly with N [79], while for $k = 2$ their behavior is described by (12). From the plot, one can see that a clear ordering of the maximal QFI with respect to the fixed interaction order exists. This observation allows one to formulate criteria which would distinguish the minimal order of k -body interactions present in the Hamiltonian using separable states. According to our choice of normalization when $N = k$, Heisenberg scaling of N^2 is achieved as expected. Note that for $N < 2k$ the trend is constant. Then, once the number of particles is equal to $2k$ the QFI grows. This change in the behavior of QFI is necessary because it would be impossible for it to remain constant since it would eventually yield values smaller than the ones obtained with local Hamiltonians and would lead to a contradiction.

QFI directly. Here, however, we do not want to examine a sum of k -local Hamiltonians, but its behavior when the many-body couplings are varied. To detect interactions of order $k > 2$ it is always sufficient to violate the bound for $\mathcal{H}^{(2)} = H_1 + H_2$. Again, by solving (9) we can calculate the desired limits on QFI. As before, the optimal solution is found to be realized by $p_i = p$ for all i . For our problem, the exact calculations and numerical optimization for up to $N = 13$ give rise to the extrapolated pattern

$$\begin{aligned} \mathcal{B}_{1+2} &= \max_{|\psi_{\text{prod}}\rangle} F[|\psi_{\text{prod}}\rangle, \mathcal{H}^{(2)}] \\ &= \max \left(-\frac{16Np(p-1)}{(N+1)^2} \{ (N-2)^2 \right. \\ &\quad \left. + 2(N-1)p[(2N-3)p - (2N-5)] \} \right) \end{aligned} \quad (13)$$

and allow us to formulate the following criterion:

$$\text{if } F[\rho, \mathcal{H}^{(K)}] > \mathcal{B}_{1+2} \Rightarrow K \geq 3. \quad (14)$$

The polynomial to be maximized is fourth order, and a closed formula for the maxima can be found explicitly. However, due to its extensive structure, we chose to present the result in the above form. Explicit values of $\mathcal{B}_{1+2}$ for chosen N are shown in Table I. It is worth noting that $\mathcal{B}_{1+2}$ never exceeds the maximal QFI for only two-body interactions, and if needed, a stricter bound can be chosen. Furthermore, the presented

TABLE I. Explicit values of the maximal quantum Fisher information $\mathcal{B}_{1+2}$ (13) attainable in product states for a two-body Hamiltonian from the studied family. For each N , violation of this bound with any separable state yields the presence of at least three-body terms in the Hamiltonian.

	N									
	2	3	4	5	6	7	8	9	10	
$\mathcal{B}_{1+2}$	1.78	2.68	3.61	4.57	5.53	6.51	7.49	8.47	9.46	

approach, in principle, can be performed for any k . Nevertheless, we will focus on the first physically interesting scenario presented above.

A. Example

As an example, consider an Ising chain on a complete interaction graph with a uniform external field in the z direction ($k \leq 2$). Tuning the field according to the coupling strength, as well as normalizing an entire Hamiltonian, we get

$$\mathcal{H}_I = \mathcal{N}J \left(\sum_{i < j}^N \sigma_z^i \sigma_z^j + \sum_i^N \sigma_z^i \right), \quad (15)$$

where $\mathcal{N} = 1/J(N+1)$. Suppose now that this system contains some amount of three-body interactions of the same symmetry and the actual Hamiltonian is, up to normalization, $\mathcal{H}^{(3)} = \mathcal{H}_I + \gamma_3 H_3$. In Fig. 3 we plot the maximal QFI in product states for different N and varying γ_3 . Clearly, even for small γ_3 , higher-order interaction is detected. Moreover, let us examine a case in which we set different values of $\gamma_3 = \{0.1, 0.5, 0.7, 1\}$ and, for each choice, compute the QFI in a random pure three-qubit product state. In this scenario, it is also possible to violate $\mathcal{B}_{1+2}$. Namely, for 10^5 samples the

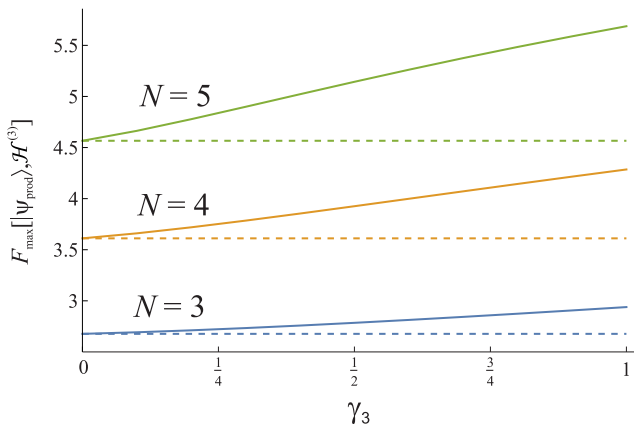


FIG. 3. Maximal quantum Fisher information in a product state for Hamiltonian $\mathcal{H}_I$ (15) with three-body contribution $\gamma_3 H_3$ for $N = 3, 4, 5$ and varying γ_3 (solid line). Bounds $\mathcal{B}_{1+2}$, which allow one to verify the presence of higher-order interactions, are plotted by dashed lines. One can see that it is possible to make statements about the order of interactions based on the presented results. If the bound of $\mathcal{B}_{1+2}$ is violated, then within the studied family of Hamiltonians, the interaction is at least 3-local.

estimated frequency of violation is $\{0.23\%, 4.3\%, 6.1\%, 8\%\}$ for the respective choices of γ_3 . Although, as expected, this frequency is small, it is still significant, and one can make conclusions about the present interaction type.

An interesting thing to comment on is the change in eigenlevels structure. In general, the state that maximizes variance is given as $1/\sqrt{2}(|E_{\min}\rangle + |E_{\max}\rangle)$. This, however, does not need to be a product state, and indeed, it is not in most cases. Nevertheless, if the order of interaction increases, the structure of eigenlevels changes, and Heisenberg scaling is available with product states. In fact, increasing γ_3 causes an attraction of the lowest energy levels, resulting in stronger degeneracy when all couplings are equal. The resulting system is effectively a two-level structure, and it is possible to form many product states which are a uniform superposition of two levels; hence, the maximal value of variance and QFI can be achieved.

The discussed protocol could be especially useful for verifying whether the many-body couplings have been engineered in a quantum simulator or another setup without direct comparison of evolution with a specific k -local Hamiltonian. The importance of such tasks was discussed in the Introduction.

VI. POSSIBLE EXTENSIONS

Here, we will briefly discuss the problem of k -body-interaction verification outside of the discussed class of Hamiltonians. Let us consider the long-range-interaction transverse-field XY model

$$\mathcal{H}_{XY} = \mathcal{N} \left\{ J \sum_{i < j} [(1 + \delta) \sigma_x^i \sigma_x^j + (1 - \delta) \sigma_y^i \sigma_y^j] + b \sum_i \sigma_z^i \right\},$$

where $\mathcal{N}$ is a normalization constant, J is an exchange constant, δ is an anisotropy parameter, and b stands for an external magnetic field. For a similar discussion on the transverse-field Ising model on a complete interaction graph see [92]. The above Hamiltonian differs significantly from the ones studied before, and in principle, the conclusions drawn in the previous sections may not hold.

While working with the XY model we need to specify the free parameters. We choose to set $J = 1$ and scan over different δ and b . The choice of J is arbitrary due to the normalization and scanning over different field and anisotropy values. The physical range of δ is $[-1, 1]$. For large values of the external transverse field, the interaction terms' contribution decreases, and the results should converge to the case of local Hamiltonians. Thus, we choose to restrict b to a significant region of $\pm\sqrt{J^2 + \delta_{\max}^2}$, i.e., $b \in [-2, 2]$. Performing a numerical optimization over the set of pure product states for the case of $N = 3$, we obtain the data plotted in Fig. 4(a), where the step of the parameter change is chosen to be $1/5$. Note that for $N = 3$ particles, the long-range-interaction term is equivalent to the periodic boundary condition and makes it more feasible experimentally. In general, we observe that the maximal QFI in a product state, i.e., $\mathcal{B}_{XY}$, is given as 5.97. This corresponds to $\delta = 0.6$ and $b = \pm 1.6$. For the $\delta = 0$ cut (the XX model) the maximal

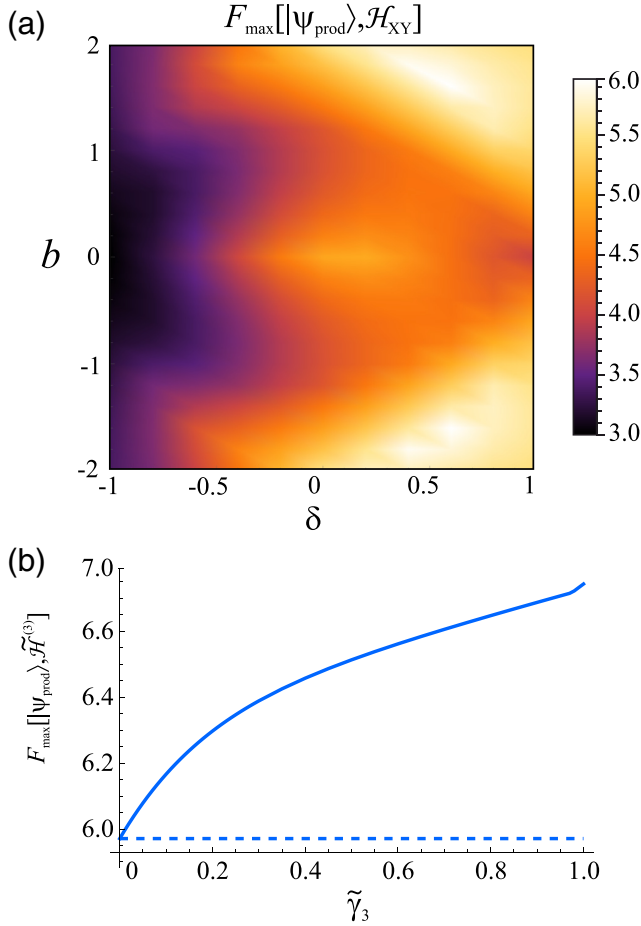


FIG. 4. (a) Maximal quantum Fisher information in a three-qubit product state for the normalized transverse-field XY Hamiltonian and different values of the external field b and anisotropy parameter δ . The free parameters were scanned within the region of ± 1 for δ and ± 2 for b with a step of $1/5$. The maximal QFI in a product state was found to be $\mathcal{B}_{XY} = 5.97$. For $\delta = 0$, the XX model, it yields a maximal value of 5.51. Since the QFI values plotted here do not saturate the upper bound of $N^2 = 9$, our reasoning can be used for higher-order-interaction verification. (b) Maximal quantum Fisher information in a product state for the normalized Hamiltonian $\tilde{\mathcal{H}}^{(3)} = \mathcal{H}_{XY} + \tilde{\gamma}_3 \tilde{H}_3$ with a varying $\tilde{\gamma}_3$ and the optimal XY model parameters $\delta = 0.6$ and $b = -1.6$ (solid line). The value of $\mathcal{B}_{XY}$ is plotted by a dashed line. Its violation allows one to verify that a three-body-interaction term is present in a system where one- and two-body terms are described with the XY model. Here, this is clearly possible.

QFI in a product state is found to be 5.51. Both of these numbers are essentially smaller than $N^2 = 9$, which is needed to verify the higher-order interactions. Later in this section we will consider $\mathcal{H}_{XY}$ with the optimal parameters $\delta = 0.6$ and $b = -1.6$.

First, we will examine a specific example of the many-body-interaction verification problem within the studied model. Then, a more general approach will be presented. Consider a three-body Hamiltonian $\tilde{H}_3$ which yields a QFI in a product state for the normalized $\tilde{\mathcal{H}}^{(3)} = \mathcal{H}_{XY} + \tilde{H}_3$ that violates $\mathcal{B}_{XY} = 5.97$. One possible choice is simply $\tilde{H}_3 =$

$\sigma_x^1 \sigma_x^2 \sigma_x^3$, as it leads to the maximal QFI in a product state for $\tilde{\mathcal{H}}^{(3)}$ equal to 6.74. Similar to what we did in the previous section, we vary the amount of the $\tilde{H}_3$ contribution as a function of the coupling strength $\tilde{\gamma}_3$. Once again, we conclude that the presence of higher-order interaction terms can be verified through QFI, as illustrated in Fig. 4(b).

Choosing the three-body Hamiltonian in the XY model is far more arbitrary. Hence, an additional interesting question arises. What if the underlying many-body-interaction Hamiltonian could contain any three-body terms? For this reason, let us consider a three-body Hamiltonian of the general form $\tilde{H}_3 = \sum_{i,j,k=1}^3 \gamma_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k$. Now, we introduce small random couplings by sampling γ_{ijk} from a uniform distribution on $(-\gamma_{\max}, \gamma_{\max})$ and choose $\gamma_{\max} = 0.5$. This leads to multiple violations of $\mathcal{B}_{XY}$ for $\mathcal{N}(\mathcal{H}_{XY} + \tilde{H}_3)$ in the fixed optimal XY -model product state with a frequency of 1.4%, estimated on 10^5 runs. Consequently, it shows the ability of a more general three-body-interaction verification within the XY model using the QFI.

VII. CONCLUSIONS

We examined the possibility of detecting higher-order interactions with the use of quantum Fisher information. For normalized symmetric Ising-like Hamiltonians, we showed that the maximal QFI in product states is ordered with respect to the fixed interaction order. Moreover, we calculated the maximal QFI obtained in the product states for the most natural scenario in which one- and two-body terms are present. This allowed us to verify the presence of at least three-body interactions in the chosen family of Hamiltonians through the violation of this bound. As a possible extension, we analyzed an example concerning the three-body-interaction verification in the XY model.

The considered problem has a strong foundational meaning because it paves the way for a better understanding of the nature of interactions. Furthermore, it encompasses some practical applications for Hamiltonian learning and could provide new perspectives for many-body-interaction engineering. As argued before, we emphasize that QFI can be measured experimentally via multiple techniques.

Future research on this problem could focus on generalizing this observation to an arbitrary Hamiltonian class. We note that a different approach that contains no assumptions on the symmetry and interaction strength is possible. However, it would require the possibility of the mean energy measurement in an arbitrary state and a presumably unknown Hamiltonian.

Note added. Very recently, we became aware that a similar problem is being examined independently in a different manner by Bluhm *et al.* (see [96] for their proposed solution to this task).

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APPENDIX: MAXIMAL QFI IN PRODUCT STATES

To perform an exact maximization of QFI for $N = 3$ and $k = 2$ let us consider the extremum conditions (9) explicitly:

$$G_{2,3}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$

$$G_{1,3}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$

$$G_{1,2}[1 + 2p_2(p_3 - 1) - 2p_3 + 2p_1(p_2 + p_3 - 1)] = 0,$$

where $G_{i,j} = 4(1 - p_i - p_j)$. Discarding the minimum solutions generated by eigenstates and limiting ourselves to $p_i \in [0, 1]$, from $G_{i,j}$ we get

$$p_1 = p_2 = p_3 = \frac{1}{2}, \quad (\text{A1})$$

which is a local minimum with QFI equal to 3. Assuming that the first term is nonzero, we obtain

$$p_1 = \frac{1 + 2p_2(1 - p_3) + 2p_3}{2(1 - p_2 - p_3)}. \quad (\text{A2})$$

This leads to QFI of 4 and can also be satisfied if all p_i are taken to be equal. Indeed, taking $p_i = p$ for all i , we get

$$F[|\psi\rangle^{\otimes 3}, H_2] = 2(-18p^4 + 36p^3 - 24p^2 + 3p), \quad (\text{A3})$$

with a maximum of 4 for $p = 1/6(3 + \sqrt{3})$. For the three-body-interaction Hamiltonian H_3 the calculations can be performed in an alternative manner. The spectrum of H_3 consists of only two levels. The energy of $-1/3$ is associated with the eigenstates $|111\rangle, |100\rangle, |010\rangle, |001\rangle$ and $|000\rangle, |110\rangle, |101\rangle, |011\rangle$ for the corresponding eigenvalue of $1/3$. The maximal algebraically allowed variance is given as the square of the energy bandwidth $(E_{\max} - E_{\min})^2$. Most often, it is associated with highly entangled states; for the studied family of Hamiltonians it is a Greenberger-Horne-Zeilinger state. Here, however, due to additional degeneracies arising from the three-body couplings (see the discussion at the end of Sec. V), a product state that saturates the variance can be constructed. Taking a uniform superposition of $|000\rangle$ and $|001\rangle$, we obtain $|0\rangle \otimes |0\rangle \otimes 1/\sqrt{2}(|0\rangle + |1\rangle)$. From the symmetry of the Hamiltonian any permutation of $1/\sqrt{2}(|0\rangle + |1\rangle)$ among $|0\rangle$ is an equally valid solution.

Now, we give another example for $N = 4$ and $k = 2$. Here, we want to solve the set of four equations arising from (9). We do not report their explicit forms here, but one can easily generate them by calculating the variance in a parameterized product state. We find the following families of solutions to the given problem:

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_2} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \quad (\text{A4a})$$

$$p_{i_1} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \quad p_{i_2} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \quad (\text{A4b})$$

$$p_{i_1} = p_{i_2} = \frac{1}{2} - \frac{1}{\sqrt{2}}, \quad p_{i_3} = p_{i_4} = \frac{1}{2} + \frac{1}{\sqrt{2}}, \quad (\text{A4c})$$

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{2}, \quad (\text{A4d})$$

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} - \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \quad (\text{A4e})$$

$$p_{i_1} = \frac{1}{2} - \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} - \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2} \quad (\text{A4f})$$

$$p_{i_1} = \frac{1}{2} + \frac{1}{\sqrt{10}}, \quad p_{i_2} = \frac{1}{2} + \sqrt{\frac{5}{2}}, \quad p_{i_3} = \frac{1}{2}, \quad p_{i_4} = \frac{1}{2}, \quad (\text{A4g})$$

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{2} \pm \frac{1}{\sqrt{10}}, \quad (\text{A4h})$$

where the trivial (eigenstate) solutions were discarded. These families are indexed with (i_1, i_2, i_3, i_4) , which take distinct values from $[1, 4]$. This resembles the symmetric structure of the studied Hamiltonians. One can check directly that the last solution with $\forall_i p_i = p$ gives rise to the highest QFI, as reported in the main text, i.e., $F_{\max}[|\psi_{\text{prod}}\rangle, H_2] = 24/5 = 4.8$.

For $N > 3$, solutions of the form (A2) do not guarantee the vanishing derivatives for all parameters, as shown for $N = 4$. Nevertheless, the more complicated solutions also contain the one for which the optimal state is a tensor product of single-qubit states (see the main text). To further check our results and examine $N \geq 5$, we found the upper bound on the QFI with standard numerical and symbolical optimization techniques built in Wolfram *Mathematica* (version 13.3). All of the results obtained by solving the extremum conditions are in agreement with the numerical calculations. It is worth noting that the problem under consideration is exactly solvable. One can express the variance of a Hamiltonian on two state copies, i.e., $\text{Tr}[(\mathbb{1} \otimes H^2 - H \otimes H)\rho']$, with $\rho' = \rho \otimes \rho$, which reduces it to a linear form. Constraints on the reduced states can ensure that the maximization is performed over the separable states ρ and hence solves the problem.

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