Abstract

This doctoral dissertation focuses on the concept of certified dominating sets in graphs, introduced by Jerzy Topp et al. [26], in which each member vertex has either zero or at least two neighbors outside the set. This condition prevents the sets from being "uncertified" which could otherwise make them more vulnerable. The study employs a complex methodology, combining rigorous theoretical analysis, algorithm design, and practical applications. The majority of the work is devoted to the fundamental properties of the certified domination number, $\gamma_{cer}(G)$, and its relationship to the classical domination number, $\gamma(G)$, particularly in the case of $\mathcal{DD}_2$-graphs, which can be split into a dominating set and a 2-dominating set. The study provides structural characterizations of important graph classes, such as vertex-certified-domination-critical (γ_{cer}-critical) graphs, in which removing any vertex reduces the certified domination number, and minimal $\mathcal{DD}_2$-graphs. The latter concept, originally proposed by Jerzy Topp [84], has subsequently been extended to other variants of domination-type structures [56, 57]. A key result is a novel theorem stating that $\gamma(G) = \gamma_{cer}(G)$ if and only if the graph has a minimum dominating set D such that $V_G - D$ forms a 2-dominating set. The dissertation further establishes that deciding the inequality $\gamma(G) \neq \gamma_{cer}(G)$ is NP-hard, even for graphs with only one weak support vertex. To address this problem, a new linear-time algorithm for finding the smallest certified dominating set in trees is developed, along with a corrected and validated Integer Linear Programming (ILP) formulation for general graphs. Two new heuristic algorithms, "ClassicToCert" and "ApproxCert", are also proposed and systematically evaluated. Experimental results demonstrate that the constructive approach yields superior solutions compared to approaches that modify an existing dominating set. These theoretical and algorithmic contributions are motivated by practical applications, such as improving the efficiency of fire safety water networks and analyzing the structure of social networks. The dissertation concludes by presenting a new set of tools for analyzing graph domination and suggesting several promising directions for future research, including the study of k-certified domination and the development of parameterized algorithms.