SUMMARY OF DOCTORAL DISSERTATION

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titled: "Connected components of the singular locus of the moduli space of compact Riemann surfaces."

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The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$, being the orbit space of the natural action of the mapping class group $\mathfrak{M}_g$ on the Teichmüller space $\mathcal{T}_g$ admits a structure of a complex orbifold of dimension $3(g-1)$. The canonical projection $\mathcal{T}_g \to \mathcal{M}_g$ is a branch covering whose branch set $\mathcal{S}_g$, called the singular locus, is composed of points corresponding to isomorphism classes of compact Riemann surfaces that admit nontrivial automorphisms. The subject of this dissertation is the connectedness and the structure and properties of connected components of the singular locus.

This dissertation is divided, excluding the introduction and the description of the results, into five sections.

The first section is devoted to presenting preliminary information needed for understanding the proofs presented in the remaining sections and to outlining the methodology that we use.

The subjects of the second section of the dissertation are so-called ordinary connected components of the singular locus, i.e., components that do not contain any point corresponding to a surface with an automorphism of order 2. One of the main results of this section is the theorem saying that every ordinary connected component is a Cornalba component. As a result, we showed that every such connected component is an irreducible algebraic subvariety of the moduli space. We further investigated the structure of ordinary connected components by proving that every equisymmetric stratum contained in such a component $\mathcal{C}$ is defined by an action of a cyclic p-group. Moreover, we showed that the prime number p is unique for $\mathcal{C}$, i.e., if two strata defined by actions of a p-group and a q-group are contained in $\mathcal{C}$, then $p = q$. We proved that ordinary connected components may be composed of arbitrarily many strata, for large enough genera. For a prime number $p \geq 5$ and an integer $n \geq 2$, we constructed an ordinary connected component composed of at least n strata defined by actions of p-groups. As a result, we proved that the minimal g for which an ordinary connected component composed of at least two strata exists is equal to 10.

In the third section, we present results concerning the structure of the exceptional connected component $\mathcal{C}_g^{2,3}$, which is the component containing points corresponding to surfaces admitting an automorphism of order 2 or 3. We introduce the Cornalba graph $\mathcal{G}_g$, whose vertices are in one-to-one correspondence with the Cornalba components contained in $\mathcal{S}_g$. Two vertices of $\mathcal{G}_g$ are connected by an edge if the intersection of the corresponding components is nontrivial. We show that the upper bound on the diameter of this graph does not depend on g and equals 5 for every genus. We define the reduced Cornalba graph $\widehat{\mathcal{G}}_g$, and we prove that it is a connected subgraph of $\mathcal{G}_g$, whose diameter is also smaller than 6. Furthermore, we show that this diameter equals 5 for infinitely many genera. For a subgraph $\mathcal{H}$ of the Cornalba graph, we introduce the bold hull $\overline{\mathcal{H}}$, and we show that $\mathcal{G}_g = \overline{\overline{\mathcal{G}_g(2)}}$, where $\mathcal{G}_g(2)$ is the subgraph spanned on vertices corresponding to actions of Z_2. We prove that the diameter of $\overline{\mathcal{G}_g(2)}$ is equal to 3 for infinitely many genera. As a result, we find necessary conditions for two vertices to be distant by 5 in $\mathcal{G}_g$.

The fourth section is devoted to the study of so-called finitely maximal actions of Z_p. Equisymmetric strata defined by such actions coincide with certain connected components, which make them particularly interesting for us. In the first subsection, we find the necessary and sufficient conditions for a signature $(h; p, \overset{r}{\ldots}, p)$ to correspond to a finitely maximal action. As a result, we classify these actions of Z_p whose extendability is determined solely by its signature. In the second subsection, we introduce a notion of the combinatorial density of a subset with respect to a filtration. We show that the set of finitely maximal actions is combinatorially dense in the set of all actions of groups Z_p with a fixed Teichmüller dimension, which, in some sense, means that the vast majority of actions are finitely maximal.

In the last section, we consider the problem of the connectedness of the singular locus. In [On the connectivity of branch loci of moduli spaces, *Ann. Acad. Sci. Fenn. Math.* **38** (1) (2013) 245–258.] G. Bartolini, A. F. Costa, and M. Izquierdo showed that $\mathcal{S}_g$ is connected only for $g = 3, 4, 7, 13, 17, 19, 59$. While trying to find a simpler proof of the existence of $\mathcal{C}_g^{2,3}$, we identified a gap in the proof of this theorem. As it turns out, the flaw cannot be fully repaired. We show that $\mathcal{S}_{149}$ is also connected. This observation leaves the possibility that the singular locus is connected for infinitely many genera. Using some classical results from analytic number theory, we prove that this is not the case. We conjecture that $\mathcal{S}_g$ is connected only for $g = 3, 4, 7, 13, 17, 19, 59$ and 149.