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SUMMARY OF PHD THESIS OF PIOTR NOWAK-PRZYGODZKI "HOMOTOPY TYPE OF SPACES OF GRADIENT MAPS"

1. History of the issues discussed in thesis

In the eighties of the last century professor Kazimierz Gęba posed the following problem: is there a better invariant for gradient homotopies of gradient maps than the usual topological degree? In 1990 A. Parusiński [14] gave the negative answer. Namely, he proved that if two gradient vector fields on the unit disc D^n and nonvanishing in S^{n-1} are homotopic (have the same degree), then they are gradient homotopic.

In [8, 9] J.C. Becker and D.H. Gottlieb introduced a concept of a local map and an extremely useful generalization of the concept of homotopy called *otopy*. The main advantage of using these concepts lies in the fact that otopy connects local maps with not necessarily the same domain. It turns out that professor Gęba's problem solved by Parusiński can be stated while considering otopies between local maps as well.

Furthermore, it is worth pointing out that Parusiński's result constitutes a right introduction to studying homotopy type of considered spaces of maps due to the fact that it provides classification of path components of the space of gradient vector fields.

While working on the properties of spaces of local maps we were making use of approximation to the generic form of finite number of non-degenerate zeroes. This suggests indeed that there is a link with configuration spaces studied in [15] and [13] by G. Segal and D. McDuff. Their interest was inter alia homotopy type of these spaces.

One more motivation for studying (weak) homotopy type of continuous and gradient maps is a program sketched by M. Gromov in [12] conceived to establish relations between space of maps and its subspaces defined by some partial differential conditions. In particular, being gradient can be expressed by Schwarz condition.

It is worth mentioning as well that in [1] the authors have defined topological degree in the continuous and gradient case for G-equivariant local maps.

The second section of the summary consists of two thematic parts. Each part discusses two articles with a total of four (three published, one accepted), that were considered in the thesis.

In the third section we describe briefly two articles being under review that constitute continuation of our previous study. In this section I also mention open questions arising from the discussed subject.

All the above papers were written in collaboration with dr Piotr Bartłomiejczyk. Thus, I would also like to express my gratitude to dr hab. Grzegorz Graff, PG professor, for his scientific supervision.

2. Discussion on the results making up the thesis

2.1. **Gradient vector fields on the disc.** Parusiński's theorem mentioned above can also be formulated in the following way: the inclusion of the space of gradient vector fields into the space of all vector fields on D^n nonvanishing in S^{n-1} induces the bijection between the sets of path components of these function spaces.

The original proof of Parusiński's theorem is by induction on disc dimension. In dimension $n = 2$ there are particularly two cases: the degree of a field is equal or not equal to 1. First case, for which Parusiński has shown that any gradient field is gradient homotopic to identity or minus identity, is more difficult. In paper [4] we managed to fulfill a small gap proving (by two different methods) that identity and minus identity are also gradient homotopic. Our article provides a complete proof of the case $n = 2$ that emphasizes geometrical aspects of reasoning.

In the next paper [5] we develop these geometrical ideas which allows us to strengthen the result for $n = 2$. Namely, the mentioned inclusion is in fact a homotopy equivalence. Consequently, one can conclude that both spaces of vector fields (gradient and continuous) are homotopy equivalent to S^1.

To be precise, it was proven in [14] for degree different from 1. We carried out the proof for degree equal to 1 which turned out to be much more difficult.

2.2. **Gradient local maps.** Professor Gęba's question answered by Parusiński can be as well stated for a space of local maps: are two gradient local maps with the same degree gradient otopic?

Let us recall that a local map $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ is called proper if preimage of any compact set is compact under f. Let us introduce the following notation in dimension n:

- $\mathcal{F}[n]$ – otopy classes of local maps,
- $\mathcal{F}_\nabla[n]$ – gradient otopy classes of gradient local maps,
- $\mathcal{P}[n]$ – proper otopy classes of proper local maps,
- $\mathcal{P}_\nabla[n]$ – proper gradient otopy classes of proper gradient local maps.

The respective inclusions induce the following diagram:

$$\begin{array}{ccc} \mathcal{P}_\nabla[n] & \xrightarrow{\mathfrak{a}} & \mathcal{P}[n] \\ {\scriptstyle\mathfrak{b}}\downarrow & & \downarrow{\scriptstyle\mathfrak{c}} \\ \mathcal{F}_\nabla[n] & \xrightarrow{\mathfrak{d}} & \mathcal{F}[n]. \end{array}$$

Spaces of proper local maps were taken into consideration due to nice metrizable topology in which a given space is equipped (see [9]).

In [2] we showed that functions $\mathfrak{a}$ and $\mathfrak{b}$ are surjective, whereas functions $\mathfrak{c}$ and $\mathfrak{d}$ are bijective. Here, the main difficulty lays in proving a version of Hopf theorem for $\deg\colon \mathcal{F}_\nabla[n] \to \mathbb{Z}$.

In [3] we succeeded in strengthening essentially the above result showing that functions $\mathfrak{a}$ and $\mathfrak{b}$ are bijective as well. The general scheme of reasoning is similar in both papers. However, in the case of proper maps we encountered a series of technical difficulties that required introducing not only some new notions but also ideas.

3. Spaces of local maps - further perspectives

At the end I would like to say a few words about the articles that are submitted presently and continue the work on the results contained in my thesis.

In paper [6] we introduce a topology in the set of local maps and prove the exponential law for local and proper maps. Moreover we show that inclusion of the space of proper maps into the space of local maps is a weak homotopy equivalence. One should note as well that a weak homotopy type of $\mathcal{P}(n)$ is well known.

However in [7] we prove that the above spaces are not homotopy equivalent for $n > 1$. The case $n = 1$ still remains unsolved.

There are much more question marks concerning the subject of homotopy type. For example we can ask if inclusion of a space of gradient maps into a space of local maps is a (weak) homotopy equivalence. The same question can be posed assuming additionally that maps are proper.

In the context of the above problems the G-invariant case is also worth of interest. It opens a completely new field for research.

To sum up, the above topics seem to be interesting and promising. We are going to continue work in this direction with dr Bartłomiejczyk.

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