# Author's review of his research, achievements and publications

**1.** Name: **Tomasz Bernard Dzido**

**2.** Obtained diplomas and academic degrees:

- M.Sc. in Mathematics, the University of Gdańsk, Faculty of Mathematics and Physics, 2000;
- Ph.D. in Mathematics, the University of Gdańsk, Faculty of Mathematics, Physics and Informatics, 2006.

**3.** Employment in scientific institutions:

- Assistant Lecturer at the University of Gdańsk, Faculty of Mathematics, Physics and Informatics, Institute of Mathematics in 2000–2006;
- Assistant Professor at the University of Gdańsk, Faculty of Mathematics, Physics and Informatics, Institute of Mathematics in 2006–2008;
- Assistant Professor at the Gdańsk School of Banking since 2006;
- Assistant Professor at the University of Gdańsk, Faculty of Mathematics, Physics and Informatics, Institute of Informatics since 2008.

**4.** Scientific achievements resulting from Article 16 Paragraph 2 of the Act of 14 March 2003 on Academic Degrees and Title and on Degrees and Title in the Field of Art is a series of publications under the title:

**"Ramsey-type problems for chosen classes of graphs"**.

List of publications included in the above-mentioned achievement:

[A1] T. Dzido, R. Fidytek, On some three-color Ramsey numbers for paths and cycles, Discrete Mathematics 309 (2009) 4955–4958.

[A2] T. Dzido, A note on Turán numbers for even wheels, Graphs and Combinatorics 29 (2013) 1305–1309.

[A3] J. Dybizbański, T. Dzido, On some Ramsey numbers for quadrilaterals versus wheels, Graphs and Combinatorics 30 (2014) 573–579.

[A4] J. Dybizbański, T. Dzido, S. Radziszowski, On some Zarankiewicz numbers and bipartite Ramsey numbers for quadrilateral, Ars Combinatoria 119 (2015) 275–287.

[A5] J. Cyman, T. Dzido, J. Lapinskas, A. Lo, On-line Ramsey numbers for paths and cycles, Electronic Journal of Combinatorics 22 (2015) #P1.15.

What follows is the discussion of the results of the above-mentioned publications.

## Introduction

Ramsey's proof of what later would become known as the Ramsey Theorem was published shortly after his death in 1930 [38]. One of its possible contemporary formulations in the language of graph theory states that for all $k \in N$, there exists $t \in N$ such that any red-blue edge coloring of a clique $K_t$ contains a monochromatic clique of order $k$. We call the least such $t$ the $k^{th}$ *Ramsey number*, and denote it by $r(k)$. The first paper to present the values for Ramsey numbers was published in 1955 by Greenwood and Gleason [20]. Surprisingly, Ramsey himself did not study such types of problems.

Studying Ramsey numbers for graphs other than complete graphs became popular quite soon. The *two-color Ramsey number* for graphs $G$ and $H$, denoted by $R(G, H)$, is the least $t$, such that any edge coloring of graph $K_t$ using two colors, say red and blue, contains a red copy of $G$ or a blue copy of $H$. By analogy we can define Ramsey numbers for more graphs and colors, as well as for hypergraphs. Numerous varieties of non-classical Ramsey numbers have been defined in later years. For example: bipartite, planar, on-line, induced, local, diagonal, geometric, rainbow, linear and other. Radziszowski's frequently updated "Small Ramsey Numbers" [37] provide an overview of the known classical Ramsey numbers. Many publications focus on the asymptotic approach towards the topic. Unfortunately, for many years the progress has been very slow as for the exact values of many types of Ramsey numbers. As an example, let us mention that the latest exact value of the two-color Ramsey number for two complete graphs was found by McKay and Radziszowski in 1995 and equals $r(4, 5) = 25$. Therefore, for 20 years there has been no improvement, despite the large popularity of this subject matter. In the case of three-color numbers of this type, Greenwood and Gleason published a result of $r(3, 3, 3) = 17$ in 1955 [20]. Only in 2014 Codish, Frank, Itzhakov and Miller [13] reported a second result, namely $r(3, 3, 4) = 30$. Tightening the bounds on Ramsey numbers had lasted for many years, with the most progress being done by Piwakowski and Radziszowski in 1998 [34] ($30 \leq r(3, 3, 4) \leq 31$). The history of research on Ramsey numbers seems to indicate that the difficulty of computing or estimating $R(G, H)$ increases with the density of edges in $G$.

There are many branches of research, many open problems and many unknowns in Ramsey theory, some of which will be described in the following report. A vast number of accomplished researchers have been working on Ramsey numbers. As an example let us mention some of the most recent papers by the following scientists: Alon [4], Alon and Kostochka [2], [3], Baskoro *et al.* [5], Conlon and Sudakov [14], Nešetřil [30], Radziszowski [47], or West [52]. Soifer presents exciting developments in the history, results and people of Ramsey theory in his book [43], published in 2009.

Many interesting applications of Ramsey theory arose in the field of mathematics and computer science, these include results in number theory, algebra, geometry, topology, set theory, logic, information theory and theoretical computer science. The theory is especially useful in building and analyzing communication nets of various types. Ramsey theory has been applied by Frederickson and Lynch to a problem in distributed computations [19], and by Snir [45] to search sorted tables in different parallel computation models. The reader will find more applications in Rosta's summary titled "Ramsey Theory Applications" [40].

### Three-color Ramsey numbers for paths and cycles

In 2005 the author of this report started studying Ramsey numbers $R(P_i, P_j, C_k)$ and obtained first several values of numbers of this type [P1]. This work was later continued by many authors: Bielak [8], Omidi, Raeisi [31], Shao, Xu, Shi, Pan [41], to name just a few. In the 2009 paper [A1] (joint work with R. Fidytek) previously obtained results were generalized by the following theorem.

**Theorem 1 ([A1])** *Let $i, k, m$ be integers such that $m \geq 3$ is odd, $k \geq m$, and $k > \frac{3i^2-14i+25}{4}$ when $i$ is odd, and $k > \frac{3i^2-10i+20}{8}$ when $i$ is even. Then*

$$R(P_i, P_k, C_m) = 2\Big(k + \Big\lfloor \frac{i}{2} \Big\rfloor\Big) - 3.$$

The proof of this result uses a theorem linked with pancyclicity (Brandt Theorem [11]) as well as with the size of possibly largest cycles in a graph (Woodall Theorem [53] and Caccetta-Vijayan [12]). These methods have not been widely used previously, however many authors have used them later on to study other Ramsey numbers, e.g. in the paper by Broersma, Chen, Zhang [55] published in 2015. It is worth mentioning that theorem 1 does not consider all possible values of $i$, $k$, $m$, further results can be found in Omidi, Raeisi [31]. One may hypothesize that theorem 1 is true for other values of $i$, $k$, $m$. It does not, however, change the fact, that only in some cases do we know the values of $R(P_i, P_k, C_m)$ and it still remains an open problem in general. One of the possible solutions to this problem may be using three-color Ramsey numbers $R(P_i, P_j, P_k)$ and $R(C_i, C_j, C_k)$, which are also intensely studied (details can be found in Radziszowski's summary [37]).

### Turán numbers

Turán numbers are closely tied with Ramsey numbers and thus frequently used in theorem proofs. The *Turán number* $ex(n, G)$ is the maximum number of edges in any $n$-vertex graph that does not contain a subgraph isomorphic to $G$. A graph on $n$ vertices is said to be *extremal with respect to* $G$ if it does not contain a subgraph isomorphic to $G$ and has exactly $ex(n, G)$ edges. As early as in 1907 Mantel [29] calculated the value of $ex(n, K_3)$, although the definition of Turán numbers appeared only after 1941, when Turán solved the problem of values of $ex(n, G)$ for $G$ being a complete graph [51]. Ever since then many papers have been written on this topic. In the mid-70's the problem of Turán numbers for paths was finally solved (Faudree, Schelp [18]). Unfortunately, not much is known for cycles, there are still many open problems on this topic. Even in the case of the $C_4$ cycle values are known only for $n \leq 32$ (the last result being $ex(32, C_4) = 92$, obtained in 2009 by Shao, Xu and Xu [42]), whereas for larger $n$ only the upper or lower bounds are known. Many papers dealing with Turán numbers for cycles mention results only for $ex(n, C_k)$, where $k$ depends on $n$, take for example a paper by Bollobás [10], which contains the following theorem:

**Theorem 2 ([10])** *Assume that $2k-1 \geq \frac{1}{2}(n+3)$. Then*

$$ex(n, C_{2k-1}) = \binom{n-(2k-1)+2}{2} + \binom{(2k-1)-1}{2}.$$

Two problems have been stated in the [A2] paper. Can the results for cycles be transferred upon wheels? Don't the Turán numbers for wheels behave similarly to Turán numbers for other graphs?

A *wheel* $W_n$ is a graph on $n$ vertices obtained from a $C_{n-1}$ by adding one vertex and making it adjacent to all vertices of the $C_{n-1}$. Let us also consider an $n$-vertex, complete, balanced, $d$-partite graph $T^{n,d}$ such that the cardinality of each of its partitions equals $\left\lfloor \frac{n}{d} \right\rfloor$ or $\left\lfloor \frac{n}{d} \right\rfloor + 1$.

The classical result by Simonovits [44] from 1968 states:

**Theorem 3 ([44])** *Let $G$ be a given graph such that $\chi(G) \geq d+1$ but there is an edge $e$ in it such that $\chi(G - \{e\}) = d$ ($G$ is color critical). Then there exists an $n_0$ such that if $n > n_0$ then $T^{n,d}$ is the only extremal graph with respect to $G$.*

It is known, that for sufficiently large $n$ there is $ex(n, W_{2k}) = ex(n, K_4)$. It has been proved in [A2] how large the $n$ should be. By using the Woodall Theorem [53] and by determining the properties of $ex(n, C_{2k-1})$, the following result has been obtained.

**Theorem 4 ([A2])** *For all $k \geq 3$ and $n \geq 6k-10$,*

$$ex(n, W_{2k}) = \lfloor \frac{n^2}{3} \rfloor.$$

One may hypothesize, that $ex(n, W_{2k})$ for other values of $n < 6k-10$ are also equal $\left\lfloor \frac{n^2}{3} \right\rfloor$. The only remaining open problem is to obtain a better estimate of $n$, for which the thesis of theorem 4 still holds. Smaller values of $ex(n, W_{2k})$ could certainly be calculated using computer simulations.

## Two-color Ramsey numbers for wheels and the cycle $C_4$

Surahmat *et al.* [48] showed in 2002 that $R(C_4, W_m) = 9, 10$ and $9$ for $m = 4, 5$ and $6$ respectively. Independently, Kung-Kuen Tse [50] showed that $R(C_4, W_m) = 10, 9, 10, 9, 11, 12, 13, 14, 16$ and $17$ for $m = 4, 5, 6, 7, 8, 9, 10, 11, 12$ and $13$, respectively. In 2005, Surahmat *et al.* [49] obtained the property stating that $R(C_4, W_n) \leq n + \lceil (n-1)/3 \rceil$.

Paper [A3] (co-authored by J. Dybizbański) presents a much better bound.

**Theorem 5 ([A3])** *For all integers $n \geq 11$*

$$R(C_4, W_n) \leq n + \lfloor \sqrt{n-2} \rfloor + 1.$$

Let $q$ be a prime power. The famous Erdős-Rényi graph $ER(q)$, constructed by Erdős and Rényi in 1962, has been described in detail by Parsons [33]. Using properties of the $ER(q)$ graph, the following result was obtained in the [A3] paper (joint work with J. Dybizbański):

**Theorem 6 ([A3])** *For $q \geq 4$ being a prime power*

$$R(C_4, W_{q^2+1}) = q^2 + q + 1.$$

It seems that the bound presented in theorem 5 can hardly be improved, which is supported by the paper of Wu, Sun and Radziszowski [46] published in 2015. The paper contains, among other results, exact values of $R(C_4, W_n)$ for $n = 35, 36, 37, 44$ which are exactly equal $n + \left\lfloor \sqrt{n-2} \right\rfloor + 1$. Further continuation of research on topics mentioned in [A3] can be found in papers by Zhang *et al.* [54] published in 2014 and by Radziszowski *et al.* [47] published in 2015.

Many open problems deal with wheels and cycles other than $C_4$. The most recent results on this topic can be found in the paper by Broersma, Chen and Zhang [55] published in 2015.

## Zarankiewicz numbers and bipartite Ramsey numbers

Zarankiewicz numbers are the natural counterpart of Turán numbers for bipartite graphs. The *Zarankiewicz number* $z(m, n; s, t)$ is defined as the maximum number of edges in any subgraph $G$ of the complete bipartite graph $K_{m,n}$, such that $G$ does not contain $K_{s,t}$ as a subgraph. Zarankiewicz numbers and extremal graphs related to them have been studied by numerous authors, including Kövári, Sós, and Turán [26], Reiman [39], Irving [25] and Goddard, Henning, and Oellermann [21]. A compact summary by Bollobás can be found in [9].

The paper [A4] (joint work with J. Dybizbański and S. Radziszowski) considers only the case of avoiding quadrilateral $C_4$, i.e. case of $s = t = 2$. Consequently, throughout the remainder of this review Zarankiewicz numbers regarding this problem will be denoted by $z(m, n)$ or even $z(n)$, instead of $z(m, n; 2, 2)$ or $z(n, n; 2, 2)$ respectively.

Using the properties of a bipartite graph corresponding to a certain projective plane, it has been established in a paper by Kövári, Sós, Turán [26], that:

**Theorem 7 ([26])**

$$z(k^2 + k + 1) = k^3 + 2k^2 + 2k + 1.$$

In the [A4] paper (joint work with J. Dybizbański and S. Radziszowski) this research has been continued, yielding the following result:

**Theorem 8 ([A4])** *For prime powers $k$, for $0 \leq h \leq 4$, and for $n = k^2 + k + 1 - h$, there exist $C_4$-free subgraphs of $K_{n,n}$ of sizes establishing lower bounds for $z(n)$ as follows:*

$$z(k^2+k+1-h) \geq \begin{cases} k^3+2k^2 & \text{for } h=1, \\ k^3+2k^2-2k & \text{for } h=2, \\ k^3+2k^2-4k+1 & \text{for } h=3, \\ k^3+2k^2-6k+2 & \text{for } h=4. \end{cases}$$

The [A4] paper also contains an upper bound on the $z(k^2+k+1-h)$ number for $h = 1, 2, 3$. The following equations have also been proved:

**Theorem 9 ([A4])** *For any prime power $k$, and also for $k = 1$,*

$$z(k^2+k+1-h) = \begin{cases} k^3+2k^2 & \text{for } h=1, \\ k^3+2k^2-2k & \text{for } h=2, \\ k^3+2k^2-4k+1 & \text{for } h=3. \end{cases}$$

A result of $z(17) = 42$ has also been obtained. This means, that all values of $z(n)$ for $n \leq 21$ are known and the first open case for now is $z(22)$. The most recent results on various types of Zarankiewicz numbers can be found in the paper by Damásdi, Héger i Szőnyi [16].

Zarankiewicz numbers are useful in determining values or bounds for bipartite Ramsey numbers. The bipartite Ramsey number $b(n_1, \cdots, n_k)$ is the least positive integer $b$ such that any coloring of the edges of the complete bipartite graph $K_{b,b}$ with $k$ colors will result in a monochromatic copy of $K_{n_i,n_i}$ in the $i$-th color, for some $i$, $1 \leq i \leq k$. Many papers describing exact values and bounds of bipartite Ramsey numbers have been written. As it is in the case of classical Ramsey numbers, many problems remain open.

If $n_i = m$ for all $i$, then we will denote a bipartite Ramsey number by $b_k(m)$. Determining values even of such a number as $b_k(2)$ appears to be difficult. The only known exact results are $b_2(2) = 5$ (Beineke and Schwenk [7]) and $b_3(2) = 11$ (Exoo [17]). This fact has become an inspiration for finding results described in [A4].

By a slight modification of Lazebnik and Woldar's construction [28], the following inequality has been obtained in [A4]:

**Theorem 10 ([A4])** *For any prime power $k$, we have*

$$b_k(2) \geq k^2 + 1.$$

Another theorem presented in [A4] improves by one the upper bound on $b_k(2)$ for all $k \geq 5$, established by Hattingh and Henning in 1998 [24].

**Theorem 11 ([A4])** *For all $k \geq 5$,*

$$b_k(2) \leq k^2 + k - 2.$$

Using, among other results, the above-mentioned theorems for Zarankiewicz numbers, it has been obtained that:

**Theorem 12 ([A4])**

$$b_4(2) = 19.$$

The first open case of $b_k(2)$ is now for 5 colors. However, using theorems 10 and 11 we obtain its bounds.

**Theorem 13 ([A4])**

$$26 \leq b_5(2) \leq 28.$$

## On-line Ramsey numbers

The on-line Ramsey numbers were defined independently by Beck [6] and Kurek and Ruciński [27]. It is easiest to understand them by considering a game between two players, Builder and Painter, on an infinite set of vertices. In each round Builder joins two non-adjacent vertices with an edge, and Painter colors the edge red or blue. Builder's goal is to force the Painter to create a monochromatic copy of a previously chosen graph $H$ in as few rounds as possible. Painter's goal is to resist to do so for as long as possible. The *on-line Ramsey number* $\tilde{r}(H)$ is the minimum number of rounds it takes Builder to win, assuming that both Builder and Painter play optimally. An *asymmetric on-line Ramsey number* $\tilde{r}(G, H)$ is the minimum number of rounds it takes Builder to force the Painter to create a red copy of graph $G$ or a blue copy of graph $H$, again assuming that both Builder and Painter play optimally.

On-line Ramsey theory has been well-studied. The best known upper bound for $\tilde{r}(K_t)$ says that there is a positive constant $c$, such that $\tilde{r}(K_t) \leq t^{-c\frac{\log t}{\log\log t}} 4^t$ (Conlon [15]). The best lower bound $\tilde{r}(K_t) \geq \frac{r(t)-1}{2}$ is due to Alon (and was first published in a paper of Beck [6]). Conlon also proves in [15] that

$$\tilde{r}(K_t) \leq C^{-t}\binom{r(t)}{2}$$

for some constant $C > 1$ and infinitely many values of $t$.

For general graphs $G$, the best known lower bound for $\tilde{r}(G)$ is given by Grytczuk, Kierstead and Prałat [22]:

**Theorem 14 ([22])** *For graphs $G$, we have*

$$\tilde{r}(G) \geq \beta(G)(\Delta(G) - 1)/2 + e(G),$$

*where $\beta(G)$ denotes the vertex cover number of $G$.*

Various general strategies for Builder and Painter have been studied so far. Description of some of them can be found in the [A5] paper (co-authored by J. Cyman, J. Lapinskas and A. Lo).

Given the known bounds on $\tilde{r}(K_t)$, it is not surprising that determining the exact values of on-line Ramsey numbers exactly has proved even more difficult than determining the values of classical Ramsey numbers. Very few results are known, even for relatively small graphs. A significant amount of effort has been focused on the special case where $G$ and $H$ are paths. Grytczuk, Kierstead and Prałat [22] and Prałat [35, 36] have determined the values of $\tilde{r}(P_{k+1}, P_{l+1})$ when $\max\{k,l\} \leq 8$ (where $P_s$ is a path on $s$ vertices). Paper [22] gives the following bounds:

**Theorem 15 ([22])** *For all $k, l \in N$, we have $k+l-1 \leq \tilde{r}(P_{k+1}, P_{l+1}) \leq 2k+2l-3$.*

Using the $F-$blocking strategy described in [A5] (joint work with J. Cyman, J. Lapinskas and A. Lo), the following two results have been successfully obtained for the on-line Ramsey numbers. It is worth mentioning, that these are the first general results of this kind:

**Theorem 16 ([A5])** *For all $l \geq 2$, we have*

$$\tilde{r}(P_3, P_{l+1}) = \lceil 5l/4 \rceil.$$

*Also,*

$$\tilde{r}(P_3, C_l) = \begin{cases} l+2 & \text{if } l = 3, 4, \\ \lceil 5l/4 \rceil & \text{if } l \geq 5. \end{cases}$$

In addition, some bounds are given on $\tilde{r}(C_4, P_{l+1})$.

**Theorem 17 ([A5])** *For $l \geq 3$, we have $2l \leq \tilde{r}(C_4, P_{l+1}) \leq 4l - 4$. Moreover, $\tilde{r}(C_4, P_4) = 8$.*

Paper [A5] also improves inequalities stated in theorem 15 in some particular cases.

**5.** Author's other research achievements are incorporated in the following publications:

**Pre-doctoral journal papers:**

[P1] T. Dzido, Multicolor Ramsey numbers for paths and cycles, Discussiones Mathematicae Graph Theory 25 (2005) 57–65.

[P2] T. Dzido, A. Nowik, P. Szuca, New lower bound for multicolor Ramsey numbers for even cycles, Electronic Journal of Combinatorics 12 (2005) #N13.

**Postdoctoral journal papers:**

[P3] T. Dzido, M. Kubale, K. Piwakowski, On some Ramsey and Turán-type numbers for paths and cycles, Electronic Journal of Combinatorics 13 (2006) #R55.

[P4] T. Dzido, R. Zakrzewska, The upper domination Ramsey number $u(4,4)$, Discussiones Mathematicae Graph Theory 26 (2006) 419–430.

[P5] T. Dzido, R. Fidytek, The number of critical colorings for some Ramsey numbers, International Journal of Pure and Applied Mathematics 38 (2007) 433–444.

[P6] T. Dzido, R. Zakrzewska, The non-classical mixed domination Ramsey numbers, Australasian Journal of Combinatorics 45 (2009) 109–115.

[P7] T. Dzido, H. Furmańczyk, Altitude of wheels and wheel-like graphs, Central European Journal of Mathematics 8 (2010) 319–326.

[P8] J. Dybizbański, T. Dzido, On some Ramsey numbers for quadrilaterals, Electronic Journal of Combinatorics 18 (2011) #P154.

[P9] L. Boza, J. Dybizbański, T. Dzido, Three-color Ramsey numbers for graphs with at most 4 vertices, Electronic Journal of Combinatorics 19 (2012) #P47.

[P10] J. Cyman, T. Dzido, A note on on-line Ramsey numbers for quadrilaterals, Opuscula Mathematica 34 (2014) 463–468.

[P11] T. Dzido, K. Krzywdziński, On a local similarity of graphs, Discrete Mathematics 338 (2015) 983–989.

[P12] T. Dzido, K. Krzywdziński, Edit distance measure for graphs, to appear in Czechoslovak Mathematical Journal.

[P13] J. Dybizbański, T. Dzido, S. Radziszowski, On some values of three-color Ramsey numbers for paths, submitted.

[P14] T. Dzido, A. Jastrzębski, Turán numbers for odd wheels, submitted.

The content of all papers in which the author participated throughout his academic career can be divided into several fundamental topics.

## Classical Ramsey numbers

Apart from papers [A1] and [A3], author's scientific achievement regarding classical Ramsey numbers also comprises publications [P1-P3], [P5], [P8], [P9], [P13]. Papers [P1-P3] made up the core of author's doctoral thesis. Paper [P5] (co-authored by R. Fidytek) contains the list of all colorings critical for numbers $R(K_4, K_5 - e) = 18$ and $R(K_5, K_4 - e) = 15$ found using computer simulations. This may help the progress on tightening the bounds of $30 \leq R(K_5, K_5 - e) \leq 34$, which have not improved since 1992. Paper [P8] (co-authored by J. Dybizbański) contains the proof of $R(C_4, C_4, K_4 - e) = 16$, which finally closed one of the open problems stated by Arste, Klamroth, Mengersen in [1]. At the same time the value of $R(C_4, P_4, K_4 - e)$ was corrected from 10, as it was erroneously stated in [1], to 11. A vast number of new bounds on multicolor Ramsey numbers with quadrilateral $C_4$ were found using computer simulations. Paper [P9] (joint work with L. Boza and J. Dybizbański) is a refreshed version of Arste, Klamroth, Mengersen [1] containing the summary of all values and bounds for three-color Ramsey numbers for small graphs as well as the solution to many open problems. Paper [P9] (joint work with J.Dybizbański and S. Radziszowski) contains previously unknown values of $R_3(P_8) = 14$ and $R_3(P_9) = 17$. These values confirm the well-known result $R(P_n, P_n, P_n) = 2n - 2 + (n \bmod 2)$, which was proved for sufficiently large $n$ in 2007 by Gyárfás, Ruszinkó, Sárközy and Szemerédi [23].

## Ramsey numbers with domination parameters

Papers [P4] and [P6] link classical Ramsey numbers with parameters of graph domination. The *upper domination Ramsey number* $u(m, n)$ is the smallest integer $p$ such that every 2-coloring of the edges of $K_p$ with color red and blue, $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$. $B$ and $R$ are the subgraphs of $K_p$ induced by blue and red edges, respectively, and $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of a graph $G$. In [P4] (joint work with R. Zakrzewska) the reader will find proof of the result $u(4, 4) \leq 15$.

Paper [P6] (joint work with R. Zakrzewska) defines the *non-classical mixed domination Ramsey number* $v(m, G)$, which is the smallest integer $n$ such that in every 2-coloring of the edges of $K_p$ with color red and blue, either $\Gamma(B) \geq m$ or there exists a blue copy of graph $G$. Paper [P6] contains proofs of many values of $v(m, G)$, the most interesting of which is the proof of $v(3, K_6 - e) = 13$.

## Altitude parameter

Paper [P6] (joint work with H. Furmańczyk) deals with one of the parameters describing edge-ordering of graph (i.e. assigning consecutive numbers from 1 to $|E(G)|$

to edges of the graph), the altitude parameter to be exact. A one-to-one mapping $f$ from $E$ to the set of positive integers is called an *edge-ordering* of a graph $G = (V, E)$. A path of $G$ for which $f$ increases along the edge-sequence, is called an $f$-ascent of $G$. The height $h(f)$ of $f$ is the maximum length of an $f$-ascent. Denote the set of all edge-orderings of $G$ by $F$. Then, the *altitude* $\alpha(G)$ of a graph G is the number

$$\alpha(G) = \min_{f \in \mathcal{F}} h(f).$$

Paper [P7] shows, among other results, that $\alpha(W_n) = 3$ for all $n \geq 5$.

### On-line Ramsey numbers

Apart from [A5], author's scientific developments also enclose paper [P10] (joint work with J. Cyman), which contains the proofs of many exact values of on-line Ramsey numbers, the most interesting being the proof of $\tilde{r}(C_4) = 10$, which was accomplished without resorting to computer simulations.

### Similarity of graphs

Papers [P11] and [P12] (joint work with k. Krzywdziński) present exact values and bounds for various parameters describing graph similarity. The first of the above-mentioned papers introduces the $\eta(k, l)$ parameter, which is the smallest $n$ such that in any family of $l$ graphs on $n$ vertices there exists a $k$-similar pair of graphs. We say that two graphs $G$ and $H$, having the same number of vertices $n$, are *$k$-similar* if they contain a common induced subgraph of order $k$. It has been shown in the paper, that $\eta(k, 3) = r(k)$, therefore $\eta(k, l)$ can be treated as a generalization of Ramsey numbers. Moreover, a number of exact values as well as constructive and asymptotic bounds have been determined.

The second paper introduces an entirely different parameter, not linked to Ramsey numbers, i.e. $g(n, l)$, which is the biggest number $k$ guaranteeing that there exist $l$ graphs on $n$ vertices, each two having edit distance of at least $k$. By *edit distance* of two graphs $G, F$ we assume a number of edges needed to be added to or deleted from graph $G$ to obtain graph $F$. One of the more interesting results is the following:

**Theorem 18 ([P12])** *Let $l \in \{3, 4, 5, 6\}$. Then*

$$\frac{n^2}{4} - \frac{n}{2} \leq g(n, l) \leq \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor.$$

### Turán numbers

The last paper, i.e. [A5] (joint work with A. Jastrzębski) contains proofs of exact values and bounds on Turán numbers for odd wheels. It is the continuation of research presented in [A2], which deals with even wheels. Among other results, it contains exact values for $ex(n, W_5)$ and $ex(n, W_7)$, summarized by the following theorem.

**Theorem 19 ([P14])**

$$ex(n, W_5) = \begin{cases} \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor & \textit{if } n \neq 2 \textit{ mod } 4, \\ \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n-1}{2} \rfloor & \textit{if } n = 2 \textit{ mod } 4; \end{cases}$$

$$ex(n, W_7) = \begin{cases} \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 & \textit{if } n = 0 \textit{ mod } 2, \\ \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 2 & \textit{if } n = 1 \textit{ mod } 2. \end{cases}$$

Exact values for $ex(n, C_4)$ and $ex(n, C_6)$, i.e. for rims of wheels $W_5$ and $W_7$ remain unknown in general.

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