Paweł Barbarski

Various generalizations and applications of the Sharkovsky Theorem on coexistence of periodic orbits of continuous mappings

doctoral dissertation
in the form of a thematically consistent set of published articles

Supervisor: dr hab. Piotr Szuca, prof. UG
Auxiliary supervisor: dr Nikodem Mrożek
Scientific discipline: mathematics

University of Gdańsk
Faculty of Mathematics, Physics and Informatics
Institute of Mathematics

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Contents

Abstract in English

author: **Paweł Barbarski**
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The Sharkovsky Theorem states what are the possible sets of periods of periodic points of a continuous real function. Thus, it belongs to the area of discrete dynamical systems (combinatorial dynamics), but also to the real functions theory. Nevertheless, multiple directions of generalizations and applications of the theorem go beyond those areas. Here we concentrate on two directions of those, thus the results of the dissertation can be divided into two categories.

Firstly, we formulate some new randomized Sharkovsky-type results and show their applications. We can extend a discrete dynamical system with an additional dimension, and assume some measurability conditions on it, while preserving some continuity conditions on the original dimension. In such a way we arrive at a notion of a random operator and its random orbit. In the dissertation we generalize some known randomization results. We achieve that by significantly generalizing the characterization results on the random operators (the so called "transformation to the deterministic case" method), through using stronger measurability arguments, including selection theorems. Furthermore, we formulate some new kind of Sharkovsky Theorem analogues and use the results in the area of random differential inclusions.

Secondly, we analyze algebraic properties of functions fulfilling the assertion of the Sharkovsky Theorem. We solve some published problem in that area, actually generalizing it significantly. The solution of the problem is the characterization of all continuous functions belonging to an Aumann ring generated by any given Darboux function. We arrive at the final solution through multiple steps providing solutions to some simpler cases of the problem, and through some real functions theory reasoning.

Streszczenie w języku polskim

autor: **Paweł Barbarski**
tytuł: **Różne uogólnienia i zastosowania twierdzenia Szarkowskiego o współwystępowaniu orbit okresowych odwzorowań ciągłych**

Twierdzenie Szarkowskiego określa możliwe zbiory okresów punktów okresowych ciągłych funkcji rzeczywistych. Należy ono zatem do teorii dyskretnych układów dynamicznych (dynamika kombinatoryczna), ale również do teorii funcji rzeczywistych. Jednakże różne kierunki uogólnień i zastosowań twierdzenia wykraczają znacząco poza te dziedziny. Tutaj koncentrujemy się na dwóch kierunkach, i stąd wyniki rozprawy można podzielić na dwie kategorie.

Po pierwsze, formułujemy pewne nowe wyniki z zakresu randomizacji twierdzenia Szarkowskiego i pokazujemy ich zastosowanie. Dyskretny układ dynamiczny można rozszerzyć o dodatkowy wymiar, zakładając na nim pewnego rodzaju mierzalność przy jednoczesnym zachowaniu pewnej ciągłości na pierwotnym wymiarze układu. W ten sposób możemy zdefiniować pojęcie operatora losowego i jego orbity losowej. W rozprawie uogólniamy pewne znane wyniki dotyczące randomizacji. Wyniki udaje nam się osiągnąć poprzez znaczące uogólnienie charakteryzacji operatorów losowych (metoda tzw. „sprowadzenia do przypadku deterministycznego”), w dowodach wykorzystując pewne fakty na temat mierzalności, w szczególności twierdzenia selekcyjne. Ponadto, formułujemy pewne nowe analogi twierdzenia Szarkowskiego oraz stosujemy nasze wyniki w obszarze zrandomizowanych inkluzji różniczkowych.

Po drugie, analizujemy właściwości algebraiczne funkcji spełniających tezę twierdzenia Szarkowskiego. Rozwiązujemy pewien opublikowany problem w tej dziedzinie, a właściwie jego znacząco uogólnioną wersję. Rozwiązanie problemu to charakteryzacja wszystkich funkcji ciągłych należących do pierścienia Aumanna generowanego przez dowolną daną funkcję typu Darboux. Do ostatecznego rozwiązania problemu doprowadza nas szereg kroków pośrednich, w których rozwiązujemy pewne prostsze przypadki problemu oraz stosujemy rozumowanie z zakresu teorii funkcji rzeczywistych.

List of articles in scope of the dissertation

The dissertation consists of the following published articles:

[A] Paweł Barbarski, *The Sharkovskiĭ theorem for spaces of measurable functions*, J. Math. Anal. Appl. **373** (2011), no. 2, 414–421.

[B] Jan Andres and Paweł Barbarski, *Randomized Sharkovsky-type results and random subharmonic solutions of differential inclusions*, Proc. Amer. Math. Soc. **144** (2016), no. 5, 1971–1983.

[C] Paweł Barbarski, *Continuous functions in rings generated by a single Darboux function*, Real Anal. Exchange **46** (2021), no. 1, 83–98

Articles A and C are an individual work of the author. Article B is a joint work, where the results were achieved through the discussion of the authors with their equal contribution.

Summary of the dissertation

1 Introduction

Taking any function from the real line to itself, especially a function belonging to a family defined by some continuity-related property, we can consider iterations of that function on a given point, and thus arrive at so called "discrete dynamical system". It's natural to ask about periodic points of such a system, those for which some iteration arrives at the starting point. One of the most inspiring theorems in the area of discrete dynamics of the real functions is the Sharkovsky Theorem. It actually gives a surprising relationship between existence of periods of periodic points of a continuous function. It was proven in 1964 ([13]), but did not gain a worldwide recognition until after the publication of an article by Li and Yorke ([10]), which independently proved a special case of that theorem. After that publication the original result of Sharkovsky became an inspiration for further research, and many articles on generalizations and applications of the theorem were published.

The classical path of generalization is substituting continuity with another related notion, usually revolving somehow around Darboux property (see [14], [11]), or generalizing the domain of the function, preserving somehow its continuum characteristic (see [12]), or preserving one-dimensional characteristic of the function itself (e.g. triangular maps in [6]). Looking on the families of functions fulfilling the assertion of the theorem can even lead to consider algebraic properties of such families (see [11]). Other interests focus on the Sharkovsky Theorem pattern with respect to mappings of other kind than classical functions. Multifunctions are most natural in that context and a few results goes that way ([2], [3]). The so called "randomization results" constitute another branch of findings (see [1], [7]). They actually extend the discrete dynamical system with a random (measurable) dimension and can be applied e.g. in the area of random differential equations (or inclusions).

Papers A and B consist of multiple results concerning the randomization of Sharkovsky-type theorems. As a tool for proving the randomized generalizations of the theorem we use the characterizations of random operators having a periodic random orbit, a tool firstly proposed and applied in [1]. These type of characterization is called transformation to deterministic case, since we express the existence of a random orbit through the existence of the "deterministic" orbits. They are also results on their own, since in some cases they have much more general assumptions than the actual Sharkovsky-type results. Some of the results both on characterization of random operators and the actual randomizations of Sharkovsky-type theorems are generalization of results in [1], some are independent findings. Furthermore, we state the analogues of Sharkovsky Theorem for spaces of measurable functions as a mapping domain, and applications of the results in random differential inclusions.

Paper C shows some of the Sharkovsky functions on the real line (or actually some function classes related to them) from algebraic perspective. It is a solution to the problem stated in [11], i.e. characterize all continuous functions belonging to each $\mathcal{A}$-ring of $\mathcal{S}$-function. $\mathcal{S}$-function is a notion related to first return continuity, whose precise definition is not needed here, since the paper C actually generalizes the problem to Darboux functions instead of $\mathcal{S}$-functions. Nevertheless, for all $\mathcal{S}$-functions the classical Sharkovsky Theorem holds (as proven in [11]), thus the results on that particular function class are a step forward to understand algebraic features of Sharkovsky functions.

2 Preliminaries

2.1 Sets

By $\mathbb{N}$, $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{I}$ we denote the set of natural numbers, the set of real numbers, set $\mathbb{R} \cup \{-\infty, +\infty\}$, and the interval $[0,1]$, respectively. For finite $I \subseteq \mathbb{N}$, the symbol $\mathrm{LCM}\, I$ denotes the lowest common multiple of elements of I.

We fix the following denotation: (Ω, Σ) – a measurable space (i.e. a set with a σ-algebra of some of its subsets), $\mathcal{I}$ – a proper σ-ideal on Ω (i.e. $\Omega \notin \mathcal{I}$), μ – (if it exists) a complete σ-finite nontrivial measure on (Ω, Σ). We say that Σ is complete if such a measure exists. We say that Σ is nonatomic with respect to $\mathcal{I}$ if for every $A \in \Sigma$ such that $A \notin \mathcal{I}$, there is a splitting $A = B \cup C$ such that $B, C \in \Sigma$ and $B, C \notin \mathcal{I}$. For σ-algebras Σ_1 and Σ_2 by $\Sigma_1 \otimes \Sigma_2$ we mean the product σ-algebra. For a complete measure space (Ω, Σ, μ) we denote $\mathcal{N}(\Omega) = \{A \subseteq \Omega \colon \mu(A) = 0\}$. For $\Omega = \mathbb{R}^k$ or $\Omega = \mathbb{I}^k$ the symbol $\mathcal{L}(\Omega)$ denote the σ-algebra of Lebesgue measurable sets on Ω.

We recall the notion of a Suslin family. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences of natural numbers. For a family $\{A_s\colon s \in \mathbb{N}^{<\mathbb{N}}\}$ we put

$$\mathcal{A}\{A_s\colon s \in \mathbb{N}^{<\mathbb{N}}\} = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=0}^{\infty} A_{\sigma\restriction\{0,\dots,n-1\}}.$$

We say that a family $\mathcal{F}$ is a Suslin family if $\mathcal{A}\mathcal{F} = \mathcal{F}$, where $\mathcal{A}\mathcal{F} = \{\mathcal{A}\{A_s\colon s \in \mathbb{N}^{<\mathbb{N}}\}\colon A_s \in \mathcal{F}$ for $s \in \mathbb{N}^{<\mathbb{N}}\}$.

By (X, d) we always denote a metric space. A complete separable metric space is called a Polish space. A metric space which is a continuous image of a Polish space is called a Suslin space. For $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$ the symbol $\|\cdot\|$ denotes the Euclidean norm on X, and the symbol d denotes the Euclidean metric. The symbol $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets in X.

2.2 Functions

By B^A we denote the family of all functions $f\colon A \to B$. The image of a function f on a set $C \subseteq A$ is defined as $f(C) = \{f(c)\colon c \in C\}$. For families of function $\mathcal{R}_1$, $\mathcal{R}_2$ we denote $\mathcal{R}_1 \circ \mathcal{R}_2 = \{f \circ g\colon f \in \mathcal{R}_1, g \in \mathcal{R}_2\}$, where $f \circ g$ is a composition of functions f and g, i.e. $(f \circ g)(x) = f(g(x))$. By id_A we denote an identity function, i.e. $\mathrm{id}_A(x) = x$ for $x \in A$.

In the dissertation we consider real functions on a real line or compact interval. By $\mathscr{C}$, $\mathscr{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{P}$, $\mathcal{C}_{\mathcal{P}}$ we denote the family of constant functions, bounded functions, countinuous functions, Darboux functions, polynomials, and polynomially bounded continuous functions ($\mathcal{C}_{\mathcal{P}} = \{f \in \mathcal{C}\colon \exists_{h\in\mathcal{P}} |f| \leq h\}$), respectively.

A function $f\colon \mathbb{I} \to \mathbb{R}$ (or a function on any other compact interval) is called absolutely continuous when for each $\varepsilon > 0$ there is $\delta > 0$ such that for each pairwise disjoint family of intervals $(x_k, y_k) \subseteq \mathbb{I}$ for $k < n$, if $\sum_{k<n}(y_k - x_k) < \delta$, then $\sum_{k<n} |f(y_k) - f(x_k)| < \varepsilon$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous when each restriction of it to a compact interval is absolutely continuous.

Let $\xi \sim \zeta$ hold for measurable $\xi, \zeta\colon \Omega \to X$ if and only if $\{\omega \in \Omega\colon \xi(\omega) \neq \zeta(\omega)\} \in \mathcal{I}$. We denote the set of equivalence classes of this relation by:

$$\mathfrak{M}_{(\Omega,\Sigma),\mathcal{I},(X,d)} = \{\xi\colon \Omega \to X\colon \xi \text{ is measurable}\}/\!\sim.$$

We will write $\mathfrak{M}$ instead of $\mathfrak{M}_{(\Omega,\Sigma),\mathcal{I},(X,d)}$ when Ω, Σ, $\mathcal{I}$, X, and d are clear from the context. When $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$, $\Sigma = \mathcal{L}(\Omega)$, $\mathcal{I} = \mathcal{N}(\Omega)$, and $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$, we also use the standard notation:

$$L^p(\Omega, X) = \left\{\xi\colon \Omega \to X\colon \xi \text{ is measurable} \wedge \int_\Omega \|\xi(\omega)\|^p d\omega < +\infty\right\}/\!\sim$$

for $p \in [1, +\infty)$ and

$$L^{+\infty}(\Omega, X) = \{\xi\colon \Omega \to X\colon \xi \text{ is measurable and bounded}\} /\!\sim.$$

The symbol $[\xi]$ stands for the class of equivalence of ξ.

A family of functions $\mathcal{R}$ is called an $\mathcal{A}$-ring (Aumann ring) if it fullfills the conditions:

- $\mathscr{C} \subseteq \mathcal{R}$,
- if $f, g \in \mathcal{R}$, then $f + g, fg \in \mathcal{R}$,
- if $(f_n)_n \subseteq \mathcal{R}$ and $f_n \rightrightarrows f$, then $f \in \mathcal{R}$,
- if $f, g \in \mathcal{R}$, then $\max(f, g), \min(f, g) \in \mathcal{R}$.

With respect to uniform convergence ($\rightrightarrows$) we recall classical result of Weierstrass (referred to as Stone-Weierstrass Theorem).

Proposition 2.1 (Weierstrass)**.** *For any continuous function $f\colon \mathbb{I} \to \mathbb{R}$ there is a sequence of polynomials $(P_n)_n$ such that $P_n \rightrightarrows f$.*

We denote preimage set of function f as $P_f = \{f^{-1}(\{y\})\colon y \in \mathbb{R}\} \setminus \{\emptyset\}$. P_f is a partition of the domain. We say that $P_1 \prec P_2$ for partitions P_1 and P_2 if for every $A \in P_1$ there is $B \in P_2$ such that $A \subseteq B$. We say that a partition is closed if it consists only of closed sets. We define the closure of a partition P (which is the smallest closed partition bigger than P with respect to relation $\prec$) as $\mathrm{cl}(P) = \{\mathrm{cl}_P(A)\colon A \in P\}$, where $\mathrm{cl}_P(A) = \bigcap\{A' \supseteq A\colon A' \in P' \text{ for some closed } P' \succ P\}$ for $A \in P$.

2.3 Multifunctions

We naturally identify a relation $\varphi \subseteq A \times B$ with a function $\varphi\colon A \to \mathcal{P}(B)$. If we want to emphasize the properties of φ as a subset of $A \times B$, we use the notion of a graph of φ, which is actually the same set as φ. Furthermore, if $\varphi \subseteq (A \times B) \times C$, i.e. $\varphi\colon A \times B \to \mathcal{P}(C)$, then for any $x \in A$ we define the relation $\varphi_x = \varphi(x, \cdot)$. A multifunction $\varphi\colon A \multimap B$ is a relation with nonempty values, i.e. $\varphi\colon A \to \mathcal{P}(B) \setminus \{\emptyset\}$, we identify a function $f\colon A \to B$ with the multifunction with one-element values fulfilling the condition $\varphi(x) = \{f(x)\}$ for every $x \in A$.

By a superposition of a function $f\colon B \to C$ with a relation $\varphi \subseteq A \times B$ we mean the relation $f \circ \varphi \subseteq A \times C$ defined for $a \in A$ by $(f \circ \varphi)(a) = f(\varphi(a))$, which is the image of f on a set $\varphi(a)$. By a product of relations $F_i \subseteq A \times B$

for $i \in I$ we mean the relation $\prod_{i\in I} F_i$ defined by $(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a)$ for $a \in A$.

We denote by

$$\varphi^-(B) = \{\omega \in \Omega \colon \varphi(\omega) \cap B \neq \emptyset\}$$

the large preimage of the set $B \subseteq X$ under the relation $\varphi \subseteq \Omega \times X$. A relation $\varphi \subseteq \Omega \times X$ is called measurable if $\varphi^-(F) \in \Sigma$ for every closed $F \subseteq X$. It is called weakly measurable if $\varphi^-(G) \in \Sigma$ for every open $G \subseteq X$. The paper [4] contains a thorough analysis of the notion of a measurable relation, and multiple facts from that paper are used in the proofs of the results of the dissertation.

We call a function $f \colon \Omega \to X$ a selector of a multifunction $\varphi\colon\ \Omega \multimap X$ and write $f \subseteq \varphi$ if $f(\omega) \in \varphi(\omega)$ for each $\omega \in \Omega$. The selection theorems are setting sufficient conditions for existence of a selector of a given mulitfunction.

We formulate the selection theorem of Kuratowski and Ryll-Nardzewski from [8] as follows:

Proposition 2.2 (Kuratowski, Ryll-Nardzewski)**.** *If a multifunction $\varphi\colon \Omega \multimap X$ is weakly measurable and has closed values, then it has a measurable selector $f \subseteq \varphi$.*

The next selection theorem follows directly from Lemma 2.3.2 in [5]:

Proposition 2.3. *Suppose that $p \in [1, +\infty]$, $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$, $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$. Let a multifunction $\varphi\colon \Omega \multimap X$ has a measurable graph, i.e. $\varphi \in \mathcal{L}(\Omega)\otimes\mathcal{B}(X)$ and there is $h \in L^p(\Omega, \mathbb{R})$ such that $\inf_{x\in\varphi(\omega)} \|x\| \leq h(\omega)$ for each $\omega \in \Omega$. Then φ has a measurable selector $f \subseteq \varphi$ such that $f \in L^p(\Omega, X)$.*

We state also a generalization of the Aumann-von Neumann selection theorem which is a simple consequence of Corollary to Theorem 7 in [9]:

Proposition 2.4 (Leese)**.** *Assume that Σ is a Suslin family and X is a Suslin space. Let a multifunction $\varphi\colon \Omega \multimap X$ has a measurable graph, i.e. $\varphi \in \Sigma \otimes \mathcal{B}(X)$. Then it has a measurable selector $f \subseteq \varphi$.*

Assume that X and Y are metric spaces. A relation $\varphi \subseteq X \times Y$ is called upper semicontinuous (u.s.c.) if $\varphi^-(F)$ is closed in X for every closed $F \subseteq Y$. It is called lower semicontinuous (l.s.c.) if $\varphi^-(G)$ is open in X for every open $G \subseteq Y$. If φ is both l.s.c. and u.s.c., then it is called continuous. In case of relations with one-element values (i.e. functions) the notion of a lower and upper continuity coincide with the continuity of a function. An extended analysis of the notion of continuity of relations can be found in Chapter 1 of the book [5].

2.4 The Sharkovsky theorem

In description of discrete dynamics of multifunctions it is convenient to use the notion of a periodic orbit instead of the notion of a periodic point. A sequence $(x_i)_{i=0}^{k-1} \in A^k$ is called a k-orbit of the multimap $\varphi\colon A \multimap A$ if $x_{i+1} \in \varphi(x_i)$ for $i < k-1$, $x_0 \in \varphi(x_{k-1})$, and there is no $m < k$ such that $m \mid k$ and $x_{sm+i} = x_i$ for $i < m$ and $s < \frac{k}{m}$.

We recall the Sharkovsky ordering of the natural numbers:

$$\begin{array}{ccccccccc}
3 & \triangleright & 5 & \triangleright & 7 & \triangleright & 9 & \triangleright & \dots \\
2\cdot 3 & \triangleright & 2\cdot 5 & \triangleright & 2\cdot 7 & \triangleright & 2\cdot 9 & \triangleright & \dots \\
2^2\cdot 3 & \triangleright & 2^2\cdot 5 & \triangleright & 2^2\cdot 7 & \triangleright & 2^2\cdot 9 & \triangleright & \dots \\
 & & & & \vdots & & & & \\
\dots & \triangleright & 2^3 & \triangleright & 2^2 & \triangleright & 2 & \triangleright & 1
\end{array}$$

The original Sharkovsky theorem can be stated as follows:

Proposition 2.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has an n-orbit, then f has a k-orbit for every $k \triangleleft n$.*

The results of the dissertation use the following generalizations of Sharkovsky Theorem, which are simple corollaries of Theorem 6 from [2] and Theorem 2 form [3], respectively:

Proposition 2.6. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a l.s.c. multifunction with compact and connected values. If φ has an n-orbit, then φ has a k-orbit for every $k \triangleleft n$.*

Proposition 2.7. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be an u.s.c. multifunction with compact and connected values. If φ has an n-orbit, then φ has a k-orbit for every $k \triangleleft n$ with at most two exceptions.*

In particular, suppose that φ has an n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property.

1. *If $q > 3$, then φ has a k-orbit for every $k \triangleleft n$, except possibly for $k = 2^{m+2}$.*
2. *If $q = 3$, then at least one of the following two cases occurs:*
   - *(i) φ has a k-orbit for every $k \triangleleft n$, except possibly for $k = 2^{m+1}\cdot 3, 2^{m+2}$,*
   - *(ii) φ has a k-orbit for every $k \triangleleft n$, except possibly for $k = 2^{m+1}$.*
3. *If $q = 1$, then φ has a k-orbit for every $k \triangleleft n$.*

2.5 Random operators

We extend the definition of a random operator from [1] (Definition 3) to serve for a more generalized approach to randomization theorems.

Definition 2.8. A multifunction $\varphi\colon \Omega \times X \multimap X$ with closed values is called a random operator if it is weakly measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$. A random operator φ is called u.s.c. if φ_ω is u.s.c. for each $\omega \in \Omega$, it is called l.s.c. if φ_ω is l.s.c. for each $\omega \in \Omega$, and it is called continuous if φ_ω is continuous for each $\omega \in \Omega$.

We do the same with the notion of a random orbit. Definition 4 from [1] can be rephrased in the following way:

Assume that (Ω, Σ, μ) is a complete measurable space, $\mathcal{I} = \mathcal{N}(\Omega)$, and (X, d) is a Polish space. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$ where $\xi_i\colon \Omega \to X$ for $i < k$ is called a random k-orbit of the operator φ if

(a) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$ for $i < k-1$ and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,

(b) the sequence $(\xi_i)_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements (i.e. is not a concatenation of p identical m-orbits), where $mp = k$, even for almost all $\omega \in \Omega$.

We generalize it to the following form:

Definition 2.9. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$ where $\xi_i\colon \Omega \to X$ for $i < k$ is called a $(\Sigma, \mathcal{I})$-random k-orbit of the operator φ if

(a) $\Omega \setminus \{\omega \in \Omega\colon \forall_{i<k-1}\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega)) \wedge \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\} \in \mathcal{I}$,

(b) there is no $m < k$ such that $m \mid k$ and

$$\Omega \setminus \{\omega \in \Omega\colon \forall_{\substack{s<\frac{k}{m} \\ i<m}}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \in \mathcal{I}.$$

We will write shortly "random k-orbit" when Σ and $\mathcal{I}$ are fixed.

The definition coincide with the one from [1] if restricted to the assumptions given there. Thus, in the definitions from [1] we drop the assumptions of completeness of the measurable space and completeness of the metric space. As a result instead of the notion of "almost all $\omega \in \Omega$" (understood as "all $\omega \in \Omega \setminus N$, where N has measure zero"), we introduce the notion of "all

$\omega \in \Omega \setminus N$, where $N \in \mathcal{I}$", where $\mathcal{I}$ is a fixed proper σ-ideal. That notion is crucial for both versions of definition, especially in condition (b). If we drop it there, it is usually easy to modify a concatenation of p identical m-orbits is such a way to fulfill the condition (b) (e.g. just change a value for one ω in one of the copies of m-orbit), and thus the condition (b) would be usually redundant. But the condition is needed, since we want to think of k as the minimal period of a k-orbit, as it is usually formulated in Sharkovsky Theorem generalizations and analogues.

For a random operator $\varphi\colon \Omega \times X \multimap X$ and $k, m \in \mathbb{N}$, such that $m \mid k$, we define the relation $\mathcal{O}_{\varphi,k,m} \subseteq \Omega \times X^k$ as follows:

$$\mathcal{O}_{\varphi,k,m}(\omega) = \left\{ (x_i)_{i=0}^{k-1} \in X^k \colon (x_i)_{i=0}^{m-1} \text{ is an } m\text{-orbit of } \varphi_\omega \wedge \forall_{\substack{s<\frac{k}{m} \\ i<m}} x_{sm+i} = x_i \right\}.$$

Measurability properties of that relation are the main focus in the proofs leading to the randomization results.

3 Randomization of Sharkovsky-type results

3.1 Characterizations of random operators with orbits

We shall say that Ω has a φ, k-partition if there exists the following partition of Ω:

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ for $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\Omega_0 \in \mathcal{I}$ and $\Omega_{i_j} \notin \mathcal{I}$ for $j < l$,
- $\mathrm{LCM}\{i_j \colon j < l\} = k$,
- φ_ω has an i_j-orbit for each $\omega \in \Omega_{i_j}$ for $j < l$.

We can formulate the following characterizations from the two articles:

Theorem 3.1 (A, Theorem 3.4)**.** *Assume that:*

- *X is a σ-compact Polish space,*
- *(Ω, Σ) is a measurable space,*
- *$\mathcal{I}$ is a proper σ-ideal on Ω,*

- $\varphi\colon \Omega \times X \multimap X$ *is a continuous random operator with compact values.*

Then φ *has a* $(\Sigma, \mathcal{I})$*-random* k*-orbit if and only if* Ω *has a* φ, k*-partition.*

Theorem 3.2 (B, Proposition 3.2)**.** *Assume that:*

- X *is a Suslin space,*
- (Ω, Σ) *is a measurable space and* Σ *is a Suslin family,*
- $\mathcal{I}$ *is a proper* σ*-ideal on* Ω*,*
- $\varphi\colon \Omega \times X \multimap X$ *is a random operator.*

Then φ *has a* $(\Sigma, \mathcal{I})$*-random* k*-orbit if and only if* Ω *has a* φ, k*-partition.*

These results have an analogous structure. In the artice A assumption on (Ω, Σ) is more general. In the article B the assumptions on X and on φ are more general (each Polish space is a Suslin space). The characterizations are analogous to Proposition 2 from [1], but the second one is its strict generalization, since each Polish space is a Suslin space and each complete measurable space is a Suslin family.

The result from article A is derived from the Kuratowski-Ryll-Nardzewski selection theorem (Proposition 2.2), the one from the article B – from the generalized Aumann-von Neumann selection theorem (Proposition 2.4). Thus for Theorem 3.1 to be proven the continuity assumption on the operator is used (the Berge's Maximum Theorem is actually involved), since the needed kind of measurability of relation $\mathcal{O}_{\varphi,k,m}$ is different.

In article A also a new kind of characterization result is introduced, namely a characterization of operators with orbits from an L^p-space.

Theorem 3.3 (A, Theorem 3.5)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$*,*
- $\Sigma = \mathcal{L}(\Omega)$*, where* $\Omega = \mathbb{R}^r$ *or* $\Omega = \mathbb{I}^r$ *for some* $r \in \mathbb{N}$*,*
- $\mathcal{I} = \mathcal{N}(\Omega)$*,*
- $\varphi\colon \Omega \times X \multimap X$ *is a random operator.*

Fix $p \in [1, +\infty]$*. Then* φ *has a* $(\Sigma, \mathcal{I})$*-random* k*-orbit* $(\xi_i)_{i=0}^{k-1} \in L^p(\Omega, X^k)$ *if and only if there is a function* $h \in L^p(\Omega, \mathbb{R})$ *and* Ω *has a* φ, k*-partition, such that* $\inf_{(x_i)_{i=0}^{k-1} \in \mathcal{O}_{\varphi,k,i_j}(\omega)} \left\| (x_i)_{i=0}^{k-1} \right\| \leq h(\omega)$ *for each* $\omega \in \Omega_{i_j}$*, where* $j < l$*.*

3.2 Randomized Sharkovsky theorems

Randomizations of Sharkovsky-like results can be proven using the characterizations from the previous section in combination with some generalizations of the classical Sharkovsky Theorem. In papers A and B a few randomizations are presented, especially the following.

Theorem 3.4 (A, Theorem 4.2)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$*,*
- (Ω, Σ) *is a measurable space,*
- $\mathcal{I}$ *is a proper* σ*-ideal on* Ω*,*
- $\varphi\colon \Omega \times X \multimap X$ *is a continuous random operator with compact and connected values.*

Then if φ *has a random* n*-orbit, then it has a random* k*-orbit for each* $k \lhd n$*.*

Theorem 3.5 (B, Theorem 4.1)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$*,*
- (Ω, Σ) *is a measurable space and* Σ *is a Suslin family,*
- $\mathcal{I}$ *is a proper* σ*-ideal on* Ω*,*
- $\varphi\colon \Omega \times X \multimap X$ *is a l.s.c. random operator with compact and connected values.*

Then if φ *has a random* n*-orbit, then it has a random* k*-orbit for each* $k \lhd n$*.*

Theorem 3.6 (B, Theorem 4.3, Corollary 4.4)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$*,*
- (Ω, Σ) *is a measurable space and* Σ *is a Suslin family and is nonatomic with respect to* $\mathcal{I}$*,*
- $\mathcal{I}$ *is a proper* σ*-ideal on* Ω*,*
- $\varphi\colon \Omega \times X \multimap X$ *is an u.s.c. random operator with compact and connected values.*

Then if φ has a random n-orbit, then it has a random k-orbit for each $k \lhd n$ with at most one exception.

In particular, suppose that φ has a random n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property. Then:

1. *If $q > 3$, then φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$.*
2. *If $q = 3$, then at least one of the following two cases occurs:*
   - *(i) φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$,*
   - *(ii) φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1}$.*
3. *If $q = 1$, then φ has a k-orbit for every $k \lhd n$.*

The first one is a result on continuous operators, the second – on l.s.c. operators, the third – on u.s.c. operators. The first is using Theorem 3.1, the second and third – Theorem 3.2.

Although both the first and the second result are using the l.s.c. Sharkovsky theorem (Proposition 2.6), the first one is restricted to continuous operators, because of restrictions in its corresponding characterization result.

The third result is using the u.s.c. Sharkovsky theorem (Proposition 2.7). Although the original "deterministic" theorem allows up to 2 exceptions in Sharkovsky ordering, in the randomized version it can be limited to 1 exception, because the random orbits can more easily be composed to have the wanted period by using different basic periods on different partitions of Ω.

While results from paper B are more general than the one from paper A with respect to continuity of the random operator, we gain more general assumptions on measurable space (Ω, Σ) in paper A. The first and the second result are generalizations of the Andres's result from [1] (Theorem 1).

Using Theorem 3.3 we also arrive at a Sharkovsky randomization result for orbits from an L^p-space.

Theorem 3.7 (A, Theorem 4.4)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$,
- $\Sigma = \mathcal{L}(\Omega)$, *where* $\Omega = \mathbb{R}^r$ *or* $\Omega = \mathbb{I}^r$ *for some* $r \in \mathbb{N}$,

- $\mathcal{I} = \mathcal{N}(\Omega)$,
- $f\colon \Omega \times X \to X$ *is a continuous random operator.*

Fix $p \in [1, +\infty]$*. Then if* f *has a random* n*-orbit belonging to the space* $L^p(\Omega, X^n)$*, then it has a random* k*-orbit belonging to the space* $L^p(\Omega, X^k)$ *for each* $k \lhd n$*.*

The randomized Sharkovsky theorems, when restricted to the case of multifunctions with one-element values (i.e. functions), usually can be reformulated to have a structure of the classical Sharkovsky Theorem, where the space $\mathbb{R}$ is substituted with the space of measurable functions and a continuous function with a continuous measurable operator. The following examples are corollaries of Theorem 3.4 and Theorem 3.7, respectively.

Corollary 3.8 (A, Corollary 4.3)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$,
- (Ω, Σ) *is a measurable space,*
- $\mathcal{I}$ *is a proper* σ*-ideal on* Ω*.*

Let the function $\mathfrak{f}\colon \mathfrak{M} \to \mathfrak{M}$ *be given by the formula* $\mathfrak{f}(\xi)(\omega) = f(\omega, \xi(\omega))$ *for every* $\omega \in \Omega$ *and for all* $[\xi] \in \mathfrak{M}$*, where* $f\colon \Omega \times X \to X$ *is a continuous random operator. Then if* $\mathfrak{f}$ *has an* n*-orbit, then it has a* k*-orbit for each* $k \lhd n$*.*

Corollary 3.9 (A, Corollary 4.5)**.** *Assume that:*

- $X = \mathbb{R}$ *or* $X = \mathbb{I}$,
- $\Sigma = \mathcal{L}(\Omega)$*, where* $\Omega = \mathbb{R}^r$ *or* $\Omega = \mathbb{I}^r$ *for some* $r \in \mathbb{N}$,
- $\mathcal{I} = \mathcal{N}(\Omega)$,

Fix $p \in [1, +\infty]$*. Let the function* $\mathfrak{f}\colon L^p(\Omega, X) \to L^p(\Omega, X)$ *be given by the formula* $\mathfrak{f}(\xi)(\omega) = f(\omega, \xi(\omega))$ *for every* $\omega \in \Omega$ *and for each* $[\xi] \in L^p(\Omega, X)$*, where* $f\colon \Omega \times X \to X$ *is a continuous random operator. Then if* $\mathfrak{f}$ *has an* n*-orbit, then it has a* k*-orbit for each* $k \lhd n$*.*

3.3 Random differential inclusions

Characterizations of random orbits can be a powerful tool even outside the classical Sharkovsky theorem randomization results. As an example the article B presents their application in the random differential inclusions.

The article is using a differential inclusion in the form:

$$x'(\omega, t) \in \varphi(\omega, t, x(\omega, t)), \tag{I_φ}$$

where $\varphi\colon \Omega \times [0, 1] \times \mathbb{R} \multimap \mathbb{R}$ is a random u-Carathéodory map on complete measure space (Ω, Σ, μ) with proper σ-ideal $\mathcal{I}$ and σ-algebra of Lebesgue measurable sets on $\mathbb{R}$ and $[0, 1]$ with σ-ideal of null sets, i.e.:

- $\varphi(\cdot, \cdot, x)\colon \Omega \times [0, 1] \multimap \mathbb{R}$ is measurable for all $x \in \mathbb{R}$,
- $\varphi(\omega, t, \cdot)\colon \mathbb{R} \multimap \mathbb{R}$ is u.s.c. for almost all $(\omega, t) \in \Omega \times [0, 1]$,
- there exists $a, b > 0$ such that $\sup\{|y|\colon y \in \varphi(\omega, t, x)\} \leq a + b|x|$ for almost all $(\omega, t) \in \Omega \times [0, 1]$ and all $x \in \mathbb{R}$.

φ is extended to $\Omega \times \mathbb{R} \times \mathbb{R}$ by: $\varphi(\omega, t+k, x) = \varphi(\omega, t, x)$ for $\omega \in \Omega$, $t \in [0, 1]$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$.

Function $x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a random solution of (I_φ) if $x(\omega, \cdot)$ is absolutely continuous for almost all $\omega \in \Omega$, $x(\cdot, t)$ is measurable for each $t \in \mathbb{R}$, and the condition (I_φ) is fulfilled (with differentiation over t) for almost all $(\omega, t) \in \Omega \times \mathbb{R}$. It is a random k-periodic subharmonic solution if $x(\omega, t) = x(\omega, t+k)$ for almost all $(\omega, t) \in \Omega \times \mathbb{R}$ and there is no $m \in \mathbb{N}$, $m < k$ such that $x(\omega, t) = x(\omega, t + m)$ for almost all $(\omega, t) \in \Omega \times \mathbb{R}$.

For such inclusions the following theorem is formulated:

Theorem 3.10 (B, Theorem 5.6)**.** *If (I_φ) has a random n-periodic subharmonic solution for some $n > 1$, then it has random k-periodic subharmonic solutions for any $k \in \mathbb{N}$.*

To leverage the characterization results the deterministic counterpart of the equation is introduced, namely the family indexed by $\omega \in \Omega$:

$$x'(t) \in \varphi(\omega, t, x(t)). \tag{I_{φ_ω}}$$

Function $x\colon \mathbb{R} \to \mathbb{R}$ is a solution of (I_{φ_ω}) if it is absolutely continuous and the condition (I_{φ_ω}) is fulfilled for almost all $t \in \mathbb{R}$. It is a k-periodic subharmonic solution if $x(t) = x(t + k)$ for almost all $t \in \mathbb{R}$ and there is no $m \in \mathbb{N}$, $m < k$ such that $x(t) = x(t + m)$ for almost all $t \in \mathbb{R}$.

With the given inclusion the Poincaré operator $P_k\colon \Omega \times \mathbb{R} \times \mathbb{R} \multimap \mathbb{R}$ defined by:

$$P_k(\omega, t_0, x_0) = \{x(t_0 + k)\colon x \text{ is a solution of } (I_{\varphi_\omega}), x(t_0) = x_0\}$$

is associated. The proof of the theorem is based on translating the hypothesis and the conclusion into statements on that Poincaré operator and applying the random operator characterization results from article B to it.

The results on inclusions even do not involve the Sharkovsky ordering – existence of any nontrivial orbit is equivalent to existence of all possible periods of orbits – thus the characterization results can be successfully applied outside of the context of that ordering.

4 Rings of Sharkovsky functions

4.1 Generated $\mathcal{A}$-rings

The article C introduces the notion of an $\mathcal{A}$-ring generated by a family $\mathcal{R} \subseteq \mathbb{R}^X$, where $X = \mathbb{R}$ or $X = \mathbb{I}$. The family

$$\mathcal{AR}(\mathcal{R}) = \bigcap \{\mathcal{Q} \subseteq \mathbb{R}^X\colon \mathcal{R} \subseteq \mathcal{Q}, \mathcal{Q} \text{ is an } \mathcal{A}\text{-ring}\}$$

is called an $\mathcal{A}$-ring generated by a family $\mathcal{R}$, and is the smallest (with respect to relation $\subseteq$) $\mathcal{A}$-ring containing $\mathcal{R}$. Thus the problem of the paper can be expressed as finding $\mathcal{AR}(\{f\}) \cap \mathcal{C}$ for $f \in \mathcal{D}$.

The article tackles firstly with finding $\mathcal{AR}(\{f\})$ before looking into continuous functions in that family. This also gives some insight into the algebraic properties of Darboux functions. The simplest case is when f is an identity function. For identity on $\mathbb{I}$ the result follows directly from Stone-Weierstrass Theorem, and for identity on $\mathbb{R}$ it is proven in C. Thus the results are the following:

Proposition 4.1 (C, Corollary 4.5)**.**

$$\mathcal{AR}(\{\mathrm{id}_\mathbb{I}\}) = \mathcal{C}$$

$$\mathcal{AR}(\{\mathrm{id}_\mathbb{R}\}) = \mathcal{C}_\mathcal{P}$$

When we take f as any Darboux function on $\mathbb{R}$ or $\mathbb{I}$ we arrive in article C at the following representation of the generated $\mathcal{A}$-ring:

Theorem 4.2 (C, Theorem 5.4)**.**

- *If $f \in \mathcal{D}$, then $\mathcal{AR}(\{f\}) = \mathcal{C}_\mathcal{P} \circ \{f\}$.*

- *If $f \in \mathcal{D} \cap \mathscr{B}$, then $\mathcal{AR}(\{f\}) = \mathcal{C} \circ \{f\}$.*
- *For $f \in \mathbb{I}^{\mathbb{I}}$, if $f \in \mathcal{D}$, then $\mathcal{AR}(\{f\}) = \mathcal{C} \circ \{f\}$.*

As a consequence the $\mathcal{A}$-ring generated by a Darboux function consists of only Darboux functions (C, Corollary 5.5).

4.2 Continuous functions in generated $\mathcal{A}$-rings

The set of preimages P_f of function f plays important role in finding the continuous function in $\mathcal{A}$-ring generated by f. Since P_f is a partition of the domain, some considerations around partitions apply here. In particular we make usage of relation $\prec$ between partitions and the closure of the preimage set $\mathrm{cl}(P_f)$. The proofs revolve around topological properties of P_f and $\mathrm{cl}(P_f)$, and the idea of separability of points by a function (which is just an ability to find a function having a different values on those points). The separability properties need a couple of non-trivial constructions to be performed to be proven, mainly levaraging topology of the real line or interval. Finally, we arrive at the following results:

Theorem 4.3 (C, Theorem 7.3)**.** *Assume that $f \in \mathcal{D}$.*

- *If $X = \mathbb{I}$, then*

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} = \{g \in \mathcal{C} \colon \ \mathrm{cl}(P_f) \prec P_g\}.$$

- *If $X = \mathbb{R}$ and $f(X)$ is closed, then*

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} = \{g \in \mathcal{C} \colon \ \mathrm{cl}(P_f) \prec P_g, \exists_{h \in \mathcal{P}} |g| \leq h \circ f\}.$$

The case when the base Darboux function has a real domain and not-closed image, i.e. an interval (a, b), $(a, b]$, or $[a, b)$ for some $a, b \in \overline{\mathbb{R}}$, is still open.

In the other cases we can answer the original Pawlak-Pawlak problem stated in [11] (see Problem 2 there). Theorem 4.3 gives the characterization of all continuous functions belonging to each $\mathcal{A}$-ring containing f, where $f \in \mathcal{D}$, which is even a more general result than the original formulation of the problem, where only $\mathcal{S}$-functions (a subfamily of Darboux functions) were considered. By solving the original problem we gain an insight to algebraic properties of $\mathcal{S}$-functions, a significant subfamily of Sharkovsky functions, i.e. functions fulfilling the assertion of the classical Sharkovsky Theorem. With being able to generalize the solution to all Darboux functions we can apply it to other subfamilies of Darboux functions which fulfill the assertion of the theorem, e.g. functions with connected G_δ graph (it is known they are Sharkovsky functions from [14]).

The case when the domain is a compact interval is the simpler one, the continuous functions in the generated $\mathcal{A}$-ring just need to fulfill natural condition on their preimage set, which cannot be more granular than the preimage set of the base function (actually its closure, a consequence of continuity of functions we are looking for).

In the real line domain case the continuous functions furthermore cannot be too distant from the base function, which means they need to be bounded by a polynomially bounded continuous function composed with the base function. This is actually strongly connected with the results described in the previous section, when the rings are represented through composition and polynomial boundedness appears only in the real line case. Polynomial boundedness naturally appears in the proofs as a result of the finite and algebraic character of the operations in the $\mathcal{A}$-ring definition, which cannot be broken by closure through uniform convergence.

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List of other articles of the author

The other results of the author outside of the scope of this dissertation are published in the following articles:

[D] Paweł Barbarski, Rafał Filipów, Nikodem Mrożek, and Piotr Szuca, *Uniform density u and $\mathcal{I}_u$-convergence on a big set*, Math. Commun. **16** (2011), no. 1, 125–130.

[E] Paweł Barbarski, Rafał Filipów, Nikodem Mrożek, and Piotr Szuca, *When does the Katětov order imply that one ideal extends the other?*, Colloq. Math. **130** (2013), no. 1, 91–102.

The results from papers D and E are a fruit of discussions of the authors, in which they all contributed to the proofs of the papers. The results belong to the area of ideal convergence. Ideal convergence is a generalization of classical notion of the convergence of a sequence on a real line known from mathematical analysis. In the classical definition the set of all points which are "far" from the limit of a sequence is finite, here instead of being finite a set need to belong to a fixed ideal on the set of natural numbers (a family closed under taking subsets and finite unions).

In article D one particular ideal, called ideal of uniform density zero, is considered. Multiple properties of the ideal are proven, e.g. properties inspired by the classical Bolzano-Weierstrass Theorem and topological properties. Some general theorems about convergence properties of ideals are proven, from which the statements on uniform density zero ideal follows.

In article E two relations between ideals are considered: Katětov order and containing an isomorphic ideal. In particular, conditions on ideals for which the two relations are equivalent are stated. Multiple examples of such ideals, their topological properties, and application of the results in the ideal convergence domain follows.

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The Sharkovskiĭ Theorem for spaces of measurable functions ☆

Paweł Barbarski

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

We formulate a generalization of the so-called Random Sharkovskiĭ Theorem from paper by Jan Andres, which is the randomized version of the classical Sharkovskiĭ Theorem. We use the method of transformation to deterministic case involving the Kuratowski–Ryll-Nardzewski selection theorem, which allows us to omit the assumption of completeness of the incorporated measurable space. Moreover, we formulate an analogue of the Sharkovskiĭ Theorem for spaces of measurable functions and for L^p-spaces.

1. Introduction

Oleksandr Sharkovskiĭ formulated his theorem classifying continuous real functions according to periods of periodic points which they possess in the paper [12] in the year 1964 (translated into English in [13]). This classification is based on the so-called “Sharkovskiĭ ordering” on natural numbers defined as follows:

$$\begin{array}{ccccccccc} 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \ldots \\ 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \ldots \\ 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \ldots \\ & & & & \vdots & & & & \\ \ldots & \rhd & 2^3 & \rhd & 2^2 & \rhd & 2 & \rhd & 1 \end{array}$$

The theorem can be generalized in numerous directions. The deterministic generalizations refer to the assumptions about i.a. function [15], space of points [11,8,14], or allow multifunctions instead of functions [3]. In this paper we state a random generalization of the theorem. There are few generalizations of this kind: [4,9,2].

The idea of investigating the existence of random fixed points in a deterministic way comes from F.S. DeBlasi, L. Górniewicz and G. Pianigiani (see [5, Chapter III.31]), and was developed in case of random orbits of a given period by J. Andres (see [2]). Our main theorem (Theorem 4.2) is a generalization of Theorem 1 from [2], actually we omit the assumption of measure space completeness in the random version of the Sharkovskiĭ Theorem. We also state a version of

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E-mail address: pbarbarski@mat.ug.edu.pl.

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the Random Sharkovskiĭ Theorem for orbits belonging to an L^p-space (Theorem 4.4). Moreover, we formulate an analogue of the Sharkovskiĭ Theorem for spaces of measurable functions (Corollary 4.3) and for L^p-spaces (Corollary 4.5). To obtain our results we use the method of transformation to deterministic case based on selection theorems for measurable multifunctions, but we incorporate the selection theorem from paper [10] by K. Kuratowski and C. Ryll-Nardzewski instead of the theorem of J. von Neumann and R. Aumann used in [2]. This forces us to use some more sophisticated facts about measurable multifunctions.

2. Preliminaries

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{I}$ we denote the set of natural numbers: $1, 2, 3, \ldots,$ the set of real numbers, and the interval $[0, 1]$, respectively. For a finite set $A \subset \mathbb{N}$, the symbol $\mathrm{LCM}(A)$ denotes the lowest common multiple of elements of A. The pair (Ω, Σ) always denotes a measurable space, i.e. a set Ω with a σ-algebra Σ, $\mathcal{I}$ always denotes a σ-ideal on a set Ω such that $\Omega \notin \mathcal{I}$, and the pair (X, d) always denotes a Polish space, i.e. complete separable metric space. We say that a measurable space (Ω, Σ) is complete when there exists a complete measure on it. For $Y \subset X$ by $\overline{Y}$ we mean the closure of Y in X. For $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$, the symbol $\|\cdot\|$ denotes the Euclidean norm on X, and the symbol d denotes the Euclidean metric. For a closed set $B \subset X$ we use the standard notation: $d(x, B) = \inf\{d(x, y)\colon\ y \in B\}$.

For σ-algebras Σ_1 and Σ_2 by $\Sigma_1 \otimes \Sigma_2$ we mean the product σ-algebra. The symbol $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets in X, and symbols $\mathcal{L}(\mathbb{R}^k)$, $\mathcal{L}(\mathbb{I}^k)$, $\mathcal{N}(\mathbb{R}^k)$, and $\mathcal{N}(\mathbb{I}^k)$ denote the σ-algebras of Lebesgue measurable sets and the σ-ideals of Lebesgue measure zero sets in a given space, respectively.

Let $\xi \sim \zeta$ hold for measurable $\xi, \zeta\colon \Omega \to X$ if and only if $\{\omega \in \Omega\colon\ \xi(\omega) \neq \zeta(\omega)\} \in \mathcal{I}$. We denote the set of equivalence classes of this relation by:

$$\mathfrak{M}_{(\Omega,\Sigma),\mathcal{I},(X,d)} = \{\xi : \Omega \to X\colon\ \xi \text{ is measurable}\}/\!\sim.$$

We will write $\mathfrak{M}$ instead of $\mathfrak{M}_{(\Omega,\Sigma),\mathcal{I},(X,d)}$ when Ω, Σ, $\mathcal{I}$, X, and d are clear from the context. When $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$, $\Sigma = \mathcal{L}(\Omega)$, $\mathcal{I} = \mathcal{N}(\Omega)$, and $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$, we also use the standard notation:

$$L^p(\Omega, X) = \left\{\xi : \Omega \to X\colon\ \xi \text{ is measurable} \wedge \int_\Omega \left\|\xi(\omega)\right\|^p d\omega < +\infty\right\}/\!\sim$$

for $p \in [1, +\infty)$ and

$$L^{+\infty}(\Omega, X) = \{\xi : \Omega \to X\colon\ \xi \text{ is measurable and bounded}\}/\!\sim.$$

The symbol $[\xi]$ stands for the class of equivalence of ξ.

We naturally identify a relation $\varphi \subset A \times B$ with a function $\varphi : A \to \mathcal{P}(B)$. Furthermore, if $\varphi \subset (A \times B) \times C$, i.e. $\varphi : A \times B \to \mathcal{P}(C)$, then for any $x \in A$ we define the relation $\varphi_x = \varphi(x, \cdot)$. A multifunction $\varphi : A \multimap B$ is a relation with nonempty values, i.e. $\varphi : A \to \mathcal{P}(B) \setminus \{\emptyset\}$, we identify a function $f : A \to B$ with the multifunction with one-element values fulfilling the condition $\varphi(x) = \{f(x)\}$ for every $x \in A$.

By a superposition of a function $f : B \to C$ with a relation $\varphi \subset A \times B$ we mean the relation $f \circ \varphi \subset A \times C$ defined by $f \circ \varphi(a) = f(\varphi(a))$ for $a \in A$. By a product of relations $F_i \subset A \times B$ for $i \in I$ we mean the relation $\prod_{i\in I} F_i$ defined by $(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a)$ for $a \in A$.

The notion of measurability is naturally expanded on relations. We denote by

$$\varphi^-(B) = \bigl\{\omega \in \Omega\colon\ \varphi(\omega) \cap B \neq \emptyset\bigr\}$$

the large preimage of the set $B \subset X$ under the relation $\varphi \subset \Omega \times X$.

Definition 2.1. A relation $\varphi \subset \Omega \times X$ is called measurable if $\varphi^-(F) \in \Sigma$ for every closed $F \subset X$. It is called weakly measurable if $\varphi^-(G) \in \Sigma$ for every open $G \subset X$.

The paper [6] contains a thorough analysis of the notion of a measurable relation. We shall state here a few facts from this paper. Every measurable relation is weakly measurable (see Proposition 2.1). The product of at most countably many weakly measurable relations is weakly measurable (see Proposition 2.3(ii)). For a weakly measurable multifunction $\varphi : X \multimap \mathbb{R}$ the function defined by $f(x) = \inf \varphi(x)$ for $x \in X$ is measurable (see Theorem 5.8). The superposition of a continuous function with a weakly measurable relation is weakly measurable (it is clear from the definition).

Another useful fact is a simple modification of Theorem 6.4 in [6] and its proof is analogical to the original proof.

Proposition 2.2. *Assume that X is σ-compact, Y is a metric space, and $f : \Omega \times X \to Y$ is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$ and such that f_ω is continuous for each $\omega \in \Omega$. Then for every closed set $B \subset Y$ the relation $F \subset \Omega \times X$ defined by $F(\omega) = f_\omega^{-1}(B)$ is measurable.*

Proof. For $n \in \mathbb{N}$ put $U_n = \{y \in Y\colon\ d(y, B) < \frac{1}{n}\}$, $B_n = \{y \in Y\colon\ d(y, B) \leqslant \frac{1}{n}\}$ and define the relations $F_n \subset \Omega \times X$ by $F_n(\omega) = f_\omega^{-1}(U_n)$. Clearly $B = \bigcap_n U_n$ and $B = \bigcap_n B_n$. Since U_n is open for each n, the relation F_n is weakly measurable for each n by [6, Theorem 6.2], and by [6, Proposition 2.6] so is the relation $\overline{F_n}$, defined by $\overline{F_n}(\omega) = \overline{F_n(\omega)}$. Since $F_n(\omega) \subset \overline{F_n}(\omega) \subset f_\omega^{-1}(\overline{U_n}) \subset f_\omega^{-1}(B_n)$ for all $\omega \in \Omega$, $F = \bigcap_n \overline{F_n}$, and therefore the relation F is measurable by [6, Corollary 4.2]. $\square$

To formulate the selection theorems we use the standard notion of a selector of a multifunction.

Definition 2.3. We call a function $f : \Omega \to X$ a selector of a multifunction $\varphi : \Omega \multimap X$ and write $f \subset \varphi$ if $f(\omega) \in \varphi(\omega)$ for each $\omega \in \Omega$.

We formulate the selection theorem of Kuratowski and Ryll-Nardzewski from [10] as follows:

Theorem 2.4 *(Kuratowski and Ryll-Nardzewski). If a multifunction $\varphi : \Omega \multimap X$ is weakly measurable and has closed values, then it has a measurable selector $f \subset \varphi$.*

A selection theorem concerning selectors from an L^p-space follows strictly from Lemma 2.3.2 in [7]:

Theorem 2.5. *Suppose that $p \in [1, +\infty]$, $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$, $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$. If a multifunction $\varphi : \Omega \multimap X$ is such that $\varphi \in \mathcal{L}(\Omega) \otimes \mathcal{B}(X)$ and there is $h \in L^p(\Omega, \mathbb{R})$ such that $\inf_{x \in \varphi(\omega)} \|x\| \leqslant h(\omega)$ for each $\omega \in \Omega$, then φ has a measurable selector $f \subset \varphi$ such that $f \in L^p(\Omega, X)$.*

We recall that the notion of continuity can be expanded on relations in the following manner:

Definition 2.6. Let Y be a metric space. A relation $\varphi \subset X \times Y$ is called upper semicontinuous (u.s.c.) if $\varphi^-(F)$ is closed for every closed $F \subset X$. It is called lower semicontinuous (l.s.c.) if $\varphi^-(G)$ is open for every open $G \subset X$. If φ is both l.s.c. and u.s.c., then it is called continuous.

In case of relations with one-element values (i.e. functions) the notions of a lower and upper continuity coincide. An extended analysis of the notion of continuity of relations can be found in Chapter 1 of the book [7]. We shall state here some useful facts from this book. The product of l.s.c. multifunctions is l.s.c. (see Proposition 2.57) and the product of u.s.c. multifunctions with compact values is u.s.c. (see Proposition 2.58). For a continuous multifunction $\varphi : X \multimap X$ with compact values and a continuous function $u : X \times X \to \mathbb{R}$ the function defined by $v(x) = \inf\{u(x, y)\colon\ y \in \varphi(x)\}$ for $x \in X$ is continuous. The latter is a trivial consequence of Berge's Maximum Theorem, since $\inf\{u(x, y)\colon\ y \in \varphi(x)\} = -\sup\{-u(x, y)\colon\ y \in \varphi(x)\}$ (see Theorem 3.4 in [7, Chapter 1]).

We introduce the notion of a measurable operator as a generalization of the notion of a random operator.

Definition 2.7. A multifunction $\varphi : \Omega \times X \multimap X$ with closed values is called a measurable operator if it is weakly measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$. If (Ω, Σ) is complete, then φ is called a random operator. A measurable operator φ is called u.s.c. if φ_ω is u.s.c. for each $\omega \in \Omega$, it is called l.s.c. if φ_ω is l.s.c. for each $\omega \in \Omega$, and it is called continuous if φ_ω is continuous for each $\omega \in \Omega$.

In description of discrete dynamics of multifunctions it is convenient to use the notion of a periodic orbit instead of the notion of a periodic point.

Definition 2.8. Assume that $\varphi : A \multimap A$. A sequence $(x_i)_{i=1}^k \in A^k$ is called a k-orbit of the multifunction φ, if $x_{i+1} \in \varphi(x_i)$ for $i \leqslant k-1$, $x_1 \in \varphi(x_k)$, and there is no $m < k$ such that $m \mid k$ and $x_{(s-1)m+i} = x_i$ for $i \leqslant m$ and $s \leqslant \frac{k}{m}$.

A theorem analogical to the classical Sharkovskiĭ Theorem holds for the l.s.c. multifunctions. Here we state it in a form which is a special case of Theorem 6 in [3].

Theorem 2.9. *Suppose that $X = \mathbb{R}$ or $X = \mathbb{I}$, and $\varphi : X \multimap X$ is a l.s.c. multifunction with compact connected values. Then if φ has an n-orbit, then it has a k-orbit for each $k \lhd n$.*

We shall also need the following theorem which is a simple modification of the classical Sharkovskiĭ Theorem (see Remark 2.1.3 in [1]).

Theorem 2.10. *Assume that $X = \mathbb{R}$ or $X = \mathbb{I}$, $f : X \to X$ is a continuous function, $k, n \in \mathbb{N}$, and $k \lhd n$. If $(x_i)_{i=1}^n$ is an n-orbit of f, then f has a k-orbit $(x_i')_{i=1}^k \subset [\min_{i \leqslant n} x_i, \max_{i \leqslant n} x_i]$.*

We introduce a notion of a measurable orbit, which is a generalization of a notion of a random orbit for measurable operators.

Definition 2.11. Let $\varphi : \Omega \times X \multimap X$ be a measurable operator. A sequence of measurable functions $(\xi_i)_{i=1}^k$ where $\xi_i : \Omega \to X$ for $i \leqslant k$ is called a $(\Sigma, \mathcal{I})$-measurable k-orbit of the operator φ, if

$$\Omega \setminus \bigl\{\omega \in \Omega\colon \forall_{i \leqslant k-1} \xi_{i+1}(\omega) \in \varphi\bigl(\omega, \xi_i(\omega)\bigr) \wedge \xi_1(\omega) \in \varphi\bigl(\omega, \xi_k(\omega)\bigr)\bigr\} \in \mathcal{I},$$

and there is no $m < k$ such that $m \mid k$ and

$$\Omega \setminus \bigl\{\omega \in \Omega\colon \forall_{s \leqslant \frac{k}{m}} \forall_{i \leqslant m} \xi_{(s-1)m+i}(\omega) = \xi_i(\omega)\bigr\} \in \mathcal{I}.$$

We will write shortly "measurable k-orbit" when Σ and $\mathcal{I}$ are fixed. If φ is a random operator and $\mathcal{I}$ is the σ-ideal of null sets, then $(\xi_i)_{i=1}^k$ is called a random k-orbit.

3. Transformation to the deterministic case

The method of transformation to the deterministic case is based on theorems characterizing measurable operators possesing measurable orbits of a given period. An example of such a theorem in case of random operators is given in Proposition 2 in [2]. We give here a theorem for measurable operators mapping into a σ-compact space X.

Firstly, we introduce some useful notation. For a measurable operator $\varphi : \Omega \times X \multimap X$, $k, m \in \mathbb{N}$ such that $m \mid k$, $S = \{p < m\colon\ p \mid m\}$, and $s = |S|$ we define the function $d_{\varphi,k,m} : \Omega \times X^k \to [0, +\infty)$ as follows:

$$d_{\varphi,k,m}\bigl(\omega, (x_i)_{i=1}^k\bigr) = d\Bigl((x_i)_{i=1}^k, \varphi(\omega, x_m) \times \varphi(\omega, x_1) \times \cdots \times \varphi(\omega, x_{m-1})\bigl(\{x_1\} \times \{x_2\} \times \cdots \times \{x_m\}\bigr)^{\frac{k}{m}-1}\Bigr),$$

and the function $D_{\varphi,k,m} : \Omega \times X^k \to [0, +\infty)^{1+s}$ as follows:

$$D_{\varphi,k,m} = \bigl(d_{\varphi,k,m}, (d_{\varphi,k,p})_{p \in S}\bigr);$$

moreover for $\omega \in \Omega$ we denote $d_{\varphi,k,m,\omega} = d_{\varphi,k,m}(\omega, \cdot)$ and $D_{\varphi,k,m,\omega} = D_{\varphi,k,m}(\omega, \cdot) = (d_{\varphi,k,m,\omega}, (d_{\varphi,k,p,\omega})_{p \in S})$. Furthermore, we define the relation $\mathcal{O}_{\varphi,k,m} \subset \Omega \times X^k$ as follows:

$$\mathcal{O}_{\varphi,k,m}(\omega) = \Bigl\{(x_i)_{i=1}^k \in X^k\colon\ (x_i)_{i=1}^m \text{ is an } m\text{-orbit of } \varphi_\omega \wedge \forall_{\substack{s \leqslant \frac{k}{m} \\ i \leqslant m}} x_{(s-1)m+i} = x_i\Bigr\}.$$

Now we state some simple lemmas about those functions and relations.

Lemma 3.1. *Let $\varphi : \Omega \times X \multimap X$ be a measurable operator and let $k, m \in \mathbb{N}$ be such that $m \mid k$. Then*

$$\mathcal{O}_{\varphi,k,m}(\omega) = \Bigl\{(x_i)_{i=1}^k \in X^k\colon\ d_{\varphi,k,m,\omega}\bigl((x_i)_{i=1}^k\bigr) = 0 \wedge \forall_{\substack{p<m \\ p \mid m}} d_{\varphi,k,p,\omega}\bigl((x_i)_{i=1}^k\bigr) > 0\Bigr\}$$

for all $\omega \in \Omega$.

Proof. For every $p \leqslant m$, such that $p \mid m$, $d_{\varphi,k,p,\omega}((x_i)_{i=1}^k) = 0$ if and only if $x_{i+1} \in \varphi(\omega, x_i)$ for $i \leqslant p-1$, $x_1 \in \varphi(\omega, x_p)$, and $x_{(s-1)p+i} = x_i$ for $i \leqslant p$ and $s \leqslant \frac{k}{p}$, since the set $\varphi(\omega, x_p) \times \varphi(\omega, x_1) \times \cdots \times \varphi(\omega, x_{p-1}) \times (\{x_1\} \times \{x_2\} \times \cdots \times \{x_p\})^{\frac{k}{p}-1}$ is closed. The lemma is an immediate consequence of this remark. $\square$

Lemma 3.2. *Let $\varphi : \Omega \times X \multimap X$ be a measurable operator. Then for every $k, m \in \mathbb{N}$ such that $m \mid k$ the function $d_{\varphi,k,m}$ (and as a result $D_{\varphi,k,m}$) is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$. If φ is a continuous measurable operator with compact values, then $d_{\varphi,k,m,\omega}$ (and as a result $D_{\varphi,k,m,\omega}$) is continuous for each $\omega \in \Omega$.*

Proof. *Measurability*. For each $i \leqslant m$ the relation φ^i defined by $\varphi^i(\omega, (x_i)_{i=1}^k) = \varphi(\omega, x_i)$ is clearly weakly measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$, since φ is weakly measurable. The relation P^i defined by $P^i(\omega, (x_i)_{i=1}^k) = \{x_i\}$ is weakly measurable for each $i \leqslant k$. Therefore the relation $\varphi^{k,m} = \prod_{i=1}^k P^i \times \varphi^m \times \varphi^1 \times \cdots \times \varphi^{m-1} \times (\prod_{i=1}^m P^i)^{\frac{k}{m}-1}$ is weakly measurable. Since $d_{\varphi,k,m}(\omega, (x_i)_{i=1}^k) = \inf d \circ \varphi^{k,m}(\omega, (x_i)_{i=1}^k)$ and d is continuous, $d_{\varphi,k,m}$ is measurable.

Continuity. For each $i \leqslant m$ the relation φ_ω^i is clearly l.s.c., since φ_ω is l.s.c. Also P_ω^i is l.s.c., and therefore the relation $\varphi^{\omega,k,m} = \varphi_\omega^m \times \varphi_\omega^1 \times \cdots \times \varphi_\omega^{m-1} \times (\prod_{i=1}^m P_\omega^i)^{\frac{k}{m}-1}$ is l.s.c. By an analogical reasoning $\varphi^{\omega,k,m}$ is also u.s.c. Thus it is continuous. Since $d_{\varphi,k,m,\omega}((x_i)_{i=1}^k) = \inf\{d((x_i)_{i=1}^k, (y_i)_{i=1}^k)\colon\ (y_i)_{i=1}^k \in \varphi^{\omega,k,m}((x_i)_{i=1}^k)\}$, $d_{\varphi,k,m,\omega}$ is continuous. $\square$

We shall need also the following lemma:

Lemma 3.3. *Suppose that $\varphi : \Omega \times X \multimap X$ is a measurable operator, and $k \in \mathbb{N}$. If φ has a measurable k-orbit $\xi = (\xi_i)_{i=1}^k$, then the following splitting of Ω exists*:

$$\Omega = \Omega_0 \cup \bigcup_{j=1}^{l} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_1, i_2, \ldots, i_l$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j \leqslant l$,
- $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = k$,
- φ_ω *has an* i_j*-orbit for each* $\omega \in \Omega_{i_j}$, *where* $j \leqslant l$,
- $\xi(\omega) \in \mathcal{O}_{\varphi,k,i_j}(\omega)$ *for each* $\omega \in \Omega_{i_j}$, *where* $j \leqslant l$.

Proof. The proof is a simple modification of the "Only if" part of the proof of Proposition 2 in [2]. The only difference is that we use measurable orbits instead of random orbits and we write $\Omega_m \in \mathcal{I}$ and $\Omega_m \notin \mathcal{I}$ instead of $\mu(\Omega_m) = 0$ and $\mu(\Omega_m) > 0$ respectively. (In fact $\mu(\Omega_m) = 0$ is equivalent to $\Omega_m \in \mathcal{N}(\mathbb{R}^k)$ or $\Omega_m \in \mathcal{N}(\mathbb{I}^k)$.) □

Now we are ready to formulate the characterization.

Theorem 3.4. *Suppose that X is σ-compact, $\varphi : \Omega \times X \multimap X$ is a continuous measurable operator with compact values, and $k \in \mathbb{N}$. φ has a measurable k-orbit if and only if the following splitting of Ω exists*:

$$\Omega = \Omega_0 \cup \bigcup_{j=1}^{l} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_1, i_2, \ldots, i_l$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j \leqslant l$,
- $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = k$,
- φ_ω *has an* i_j*-orbit for each* $\omega \in \Omega_{i_j}$, *where* $j \leqslant l$.

Proof. By Lemma 3.3 it is enough to show the implication to the left. Fix $j \leqslant l$ and $t \in \mathbb{N}$, and denote $m = i_j$, $S = \{p < m\colon p \mid m\}$, and $s = |S|$. We define the relation $\mathcal{O}_{\varphi,k,m,t} : \Omega \multimap X^k$ as follows:

$$\mathcal{O}_{\varphi,k,m,t}(\omega) = D^{-1}_{\varphi,k,m,\omega}\left(\{0\} \times \left[\frac{1}{t}, +\infty\right)^s\right).$$

Since the set $\{0\} \times [\frac{1}{t}, +\infty)^s$ is closed in the metric space $Y = [0, +\infty)^{1+s}$, and by virtue of Lemma 3.2 and Proposition 2.2, the relation $\mathcal{O}_{\varphi,k,m,t}$ is measurable.

By Lemma 3.1 $\mathcal{O}_{\varphi,k,m} = \bigcup_t \mathcal{O}_{\varphi,k,m,t}$. Since φ_ω has m-orbit for each $\omega \in \Omega_m$, $\mathcal{O}_{\varphi,k,m}(\omega)$ is nonempty for each $\omega \in \Omega_m$. Hence for each $\omega \in \Omega_m$ there is t such that $\mathcal{O}_{\varphi,k,m,t}(\omega)$ is nonempty. Let t_ω be the least such t. We denote $\Omega_{m,t} = \{\omega \in \Omega_m\colon\ t_\omega = t\}$. The family $\{\Omega_{m,t}\colon\ t \in \mathbb{N}\}$ is disjoint and $\Omega_m = \bigcup_t \Omega_{m,t}$. Moreover,

$$\begin{aligned}\Omega_{m,t} &= \left\{\omega \in \Omega_m\colon\ \mathcal{O}_{\varphi,k,m,t}(\omega) \neq \emptyset \wedge \forall_{t'<t}\mathcal{O}_{\varphi,k,m,t'}(\omega) = \emptyset\right\} \\ &= \mathcal{O}^-_{\varphi,k,m,t}\left(X^k\right) \setminus \bigcup_{t'<t} \mathcal{O}^-_{\varphi,k,m,t'}\left(X^k\right) \in \Sigma.\end{aligned}$$

We define the multifunction $\mathcal{O}_{\varphi,k} : \Omega \multimap X^k$ as follows:

$$\mathcal{O}_{\varphi,k}(\omega) = \begin{cases} X^k & \text{for } \omega \in \Omega_0, \\ \mathcal{O}_{\varphi,k,i_j,t}(\omega) & \text{for } \omega \in \Omega_{i_j,t} \text{ for some } j \leqslant l \text{ and } t \in \mathbb{N}. \end{cases}$$

The relation $\mathcal{O}_{\varphi,k}$ is clearly measurable, and consequently also weakly measurable. Furthermore, it has closed values. Hence by virtue of Theorem 2.4 it has a measurable selector $(\xi_i)_{i=1}^k : \Omega \to X^k$.

For every $j \leqslant l$ and $\omega \in \Omega_{i_j}$ there is $t \in \mathbb{N}$ such that $\omega \in \Omega_{i_j,t}$, and consequently $(\xi_i(\omega))_{i=1}^k \in \mathcal{O}_{\varphi,k,i_j}(\omega)$. Hence $(\xi_i(\omega))_{i=1}^{i_j}$ is an i_j-orbit of φ_ω. Since $\Omega_0 \in \mathcal{I}$, $\Omega_{i_j} \notin \mathcal{I}$ for $j \leqslant l$, and $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = k$, $(\xi_i)_{i=1}^k$ is a measurable k-orbit of the operator φ. □

We may also formulate an analogous theorem for orbits from an L^p-space.

Theorem 3.5. *Suppose that $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$ for some $r \in \mathbb{N}$, $\Sigma = \mathcal{L}(\Omega)$, $\mathcal{I} = \mathcal{N}(\Omega)$, $X = \mathbb{R}$ or $X = \mathbb{I}$, $\varphi : \Omega \times X \multimap X$ is a measurable operator, $k \in \mathbb{N}$, and $p \in [1, +\infty]$. φ has a measurable k-orbit $\xi = (\xi_i)_{i=1}^k \in L^p(\Omega, X^k)$ if and only of there is a function $h \in L^p(\Omega, \mathbb{R})$ and the following splitting of Ω:*

$$\Omega = \Omega_0 \cup \bigcup_{j=1}^{l} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_1, i_2, \ldots, i_l$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j \leqslant l$,
- $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = k$,
- φ_ω *has an* i_j*-orbit for each* $\omega \in \Omega_{i_j}$, *where* $j \leqslant l$,
- $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{\varphi,k,i_j}(\omega)} \|(x_i)_{i=1}^k\| \leqslant h(\omega)$ *for each* $\omega \in \Omega_{i_j}$, *where* $j \leqslant l$.

Proof. "$\Leftarrow$". This is a slight modification of the "If" part of the proof of [2, Proposition 2]. It is clear that $\mathcal{O}_{\varphi,k,i_j}(\omega) \neq \emptyset$ for each $\omega \in \Omega_{i_j}$. We define the multifunction $\mathcal{O}_{\varphi,k} : \Omega \multimap X^k$ as follows:

$$\mathcal{O}_{\varphi,k}(\omega) = \begin{cases} X^k & \text{for } \omega \in \Omega_0, \\ \mathcal{O}_{\varphi,k,i_j}(\omega) & \text{for } \omega \in \Omega_{i_j} \text{ for some } j \leqslant l. \end{cases}$$

Fix $j \leqslant l$ and denote $m = i_j$, $S = \{t < m\colon\ t \mid m\}$, and $s = |S|$. Since $\mathcal{O}_{\varphi,k,m} = D^{-1}_{\varphi,k,m}(\{0\} \times (0, +\infty)^s)$, by virtue of Lemma 3.2 $\mathcal{O}_{\varphi,k,m} \in \mathcal{L}(\Omega) \otimes \mathcal{B}(X^k)$. Moreover $\Omega_0 \times X^k \in \mathcal{L}(\Omega) \otimes \mathcal{B}(X^k)$, and consequently $\mathcal{O}_{\varphi,k} \in \mathcal{L}(\Omega) \otimes \mathcal{B}(X^k)$. Hence by Theorem 2.5 the multifunction $\mathcal{O}_{\varphi,k}$ has a measurable selector $(\xi_i)_{i=1}^k : \Omega \to X^k$ such that $(\xi_i)_{i=1}^k \in L^p(\Omega, X^k)$.

For every $j \leqslant l$ and $\omega \in \Omega_{i_j}$, $(\xi_i(\omega))_{i=1}^k \in \mathcal{O}_{\varphi,k,i_j}(\omega)$. Hence $(\xi_i(\omega))_{i=1}^{i_j}$ is an i_j-orbit of φ_ω. Since $\Omega_0 \in \mathcal{I}$, $\Omega_{i_j} \notin \mathcal{I}$ for $j \leqslant l$, and $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = k$, $(\xi_i)_{i=1}^k$ is a measurable k-orbit of the operator φ.

"$\Rightarrow$". By Lemma 3.3 it is enough to show that there is a function $h \in L^p(\Omega, \mathbb{R})$ such that $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{\varphi,k,i_j}(\omega)} \|(x_i)_{i=1}^k\| \leqslant h(\omega)$ for each $\omega \in \Omega_{i_j}$.

We define the function h by equality $h(\omega) = \|\xi(\omega)\|$. $\xi \in L^p(\Omega, X^k)$, thus $h \in L^p(\Omega, \mathbb{R})$. Fix $j \leqslant l$ and $\omega \in \Omega_{i_j}$. Then $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{\varphi,k,i_j}(\omega)} \|(x_i)_{i=1}^k\| \leqslant h(\omega)$, since $\xi(\omega) \in \mathcal{O}_{\varphi,k,i_j}(\omega)$. □

4. The Sharkovskiĭ Theorem

In this section we give the main results of the paper. We shall use the theorems from the previous section and a version of the Sharkovskiĭ Theorem stated in form of Theorem 2.9.

Before we state the main theorems, we state a simple lemma.

Lemma 4.1. *If* $\mathrm{LCM}\{i_j\colon\ j \leqslant l\} = n$ *and* $k \lhd n$, *then there is* $j' \leqslant l$ *such that* $k \lhd i_{j'}$.

Proof. If $n = 2^q$ for some $q \in \mathbb{N}$, then there is $j' \leqslant l$ such that $i_{j'} = n$, thus $k \lhd i_{j'}$.

If $n = p \cdot 2^q$ for some $q \in \mathbb{N}$ and odd $p > 1$, then for each $j \leqslant l$ there is $q_j \leqslant q$ and odd $p_j \leqslant p$ such that $i_j = p_j \cdot 2^{q_j}$. Clearly there is $j' \leqslant l$ such that $p_{j'} > 1$. Then $k \lhd n \lhd i_{j'}$. □

We state here the theorem which we may call the Measurable Sharkovskiĭ Theorem for continuous measurable operators. It is a generalization of the Random Sharkovskiĭ Theorem (Theorem 1 from [2]). Here we omit the assumption of completeness of the measurable space (Ω, Σ), which is present in Andres's version.

Theorem 4.2 *(Measurable Sharkovskiĭ Theorem). Assume that (Ω, Σ) is a measurable space, $\mathcal{I}$ is a σ-ideal on Ω such that $\Omega \notin \mathcal{I}$, $X = \mathbb{R}$ or $X = \mathbb{I}$, and $\varphi : \Omega \times X \multimap X$ is a continuous measurable operator with compact and connected values. Then if φ has a measurable n-orbit, then it has a measurable k-orbit for each $k \lhd n$.*

Proof. The proof is analogical to that of Theorem 1 in [2], but we paraphrase it for completeness. Suppose that φ has a measurable n-orbit. There is a splitting of Ω as in Theorem 3.4. Fix $k \lhd n$. By Lemma 4.1 there is $j' \leqslant l$ such that $k \lhd i_{j'}$. Moreover, $1 \lhd i_j$ for each $j \leqslant l$ and $j \neq j'$. Since φ_ω has an i_j-orbit for each $\omega \in \Omega_{i_j}$ and every $j \leqslant l$, by Theorem 2.9 φ_ω has a k-orbit for each $\omega \in \Omega_{i_{j'}}$ and a 1-orbit for each $\omega \in \Omega_{i_j}$ and every $j \leqslant l$ and $j \neq j'$. Put $\Omega'_k = \Omega_{i_{j'}}$, $\Omega'_1 = \bigcup\{\Omega_{i_j}\colon\ j \leqslant$

$l \wedge j \neq j'\}$, and $\Omega'_0 = \Omega_0$. As a result we get the following splitting of Ω: $\Omega = \Omega'_0 \cup \Omega'_k$ if $l = 1$, or $\Omega = \Omega'_0 \cup \Omega'_k \cup \Omega'_1$ if $l > 1$. By Theorem 3.4 φ has a measurable k-orbit. □

The Measurable Sharkovskiĭ Theorem in the case of multifunctions with one-element values (i.e. functions) can be reformulated so that it will have a structure analogical to the structure of the classical Sharkovskiĭ Theorem where in place of the space $\mathbb{R}$ or $\mathbb{I}$ we put a space of measurable functions and in place of continuous function we put a continuous measurable operator.

Corollary 4.3. *Assume that (Ω, Σ) is a measurable space, $\mathcal{I}$ is a σ-ideal on Ω such that $\Omega \notin \mathcal{I}$, and $X = \mathbb{R}$ or $X = \mathbb{I}$. Let the function $\mathfrak{f} : \mathfrak{M} \to \mathfrak{M}$ be given by the formula $\mathfrak{f}(\xi)(\omega) = f(\omega, \xi(\omega))$ for every $\omega \in \Omega$ and for all $[\xi] \in \mathfrak{M}$, where $f : \Omega \times X \to X$ is a continuous measurable operator. Then if $\mathfrak{f}$ has an n-orbit, then it has a k-orbit for each $k \lhd n$.*

We also state a version of the Random Sharkovskiĭ Theorem for orbits from an L^p-space.

Theorem 4.4. *Assume that $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$ for some $r \in \mathbb{N}$, $\Sigma = \mathcal{L}(\Omega)$, $\mathcal{I} = \mathcal{N}(\Omega)$, $X = \mathbb{R}$ or $X = \mathbb{I}$, $f : \Omega \times X \to X$ is a continuous measurable operator, and $p \in [1, +\infty]$. Then if f has a measurable n-orbit belonging to the space $L^p(\Omega, X^n)$, then it has a measurable k-orbit belonging to the space $L^p(\Omega, X^k)$ for each $k \lhd n$.*

Proof. Suppose that f has a measurable n-orbit belonging to the space $L^p(\Omega, X^n)$. There is a function $h \in L^p(\Omega, \mathbb{R})$ and a splitting of Ω as in Theorem 3.5. Fix $k \lhd n$. By Lemma 4.1 there is $j' \leqslant l$ such that $k \lhd i_{j'}$. We get a splitting of Ω as in the proof of Theorem 4.2 $\Omega = \Omega'_0 \cup \Omega'_k$ if $l = 1$, or $\Omega = \Omega'_0 \cup \Omega'_k \cup \Omega'_1$ if $l > 1$.

Let $h' : \Omega \to \mathbb{R}$ be such that $h'(\omega) = \sqrt{k} \cdot h(\omega)$ for each $\omega \in \Omega$. Then $h' \in L^p(\Omega, \mathbb{R})$. Fix $\omega \in \Omega'_k$. Then $\omega \in \Omega_{i_{j'}}$. Hence $\inf_{(x_i)_{i=1}^n \in \mathcal{O}_{f,n,i_{j'}}(\omega)} \|(x_i)_{i=1}^n\| \leqslant h(\omega)$. Fix $\varepsilon > 0$. Let $(x'_i)_{i=1}^n \in \mathcal{O}_{f,n,i_{j'}}(\omega)$ be such that $\|(x'_i)_{i=1}^n\| < h(\omega) + \frac{\varepsilon}{\sqrt{k}}$. Then $(x'_i)_{i=1}^{i_{j'}}$ is an $i_{j'}$-orbit of f_ω. Since $k \lhd i_{j'}$, by Theorem 2.10 there is a k-orbit $(x''_i)_{i=1}^k$ of f_ω such that $(x''_i)_{i=1}^k \subset [\min_{i \leqslant i_{j'}} x'_i, \max_{i \leqslant i_{j'}} x'_i]$. Then $(x''_i)_{i=1}^k \in \mathcal{O}_{f,k,k}(\omega)$. Hence $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{f,k,k}(\omega)} \|(x_i)_{i=1}^k\| \leqslant \|(x''_i)_{i=1}^k\| \leqslant \sqrt{k} \cdot \max_{i \leqslant i_{j'}} \|x'_i\| \leqslant \sqrt{k} \cdot \|(x'_i)_{i=1}^n\| < \sqrt{k} \cdot h(\omega) + \varepsilon$, and consequently $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{f,k,k}(\omega)} \|(x_i)_{i=1}^k\| \leqslant \sqrt{k} \cdot h(\omega) = h'(\omega)$. Analogically for $\omega \in \Omega'_1$ we have $\inf_{(x_i)_{i=1}^k \in \mathcal{O}_{f,k,1}(\omega)} \|(x_i)_{i=1}^k\| \leqslant h'(\omega)$.

By Theorem 3.5 f has a measurable k-orbit belonging to the space $L^p(\Omega, X^k)$. □

This version can be also reformulated so that it will have a form of Sharkovskiĭ Theorem for L^p-spaces.

Corollary 4.5. *Assume that $\Omega = \mathbb{R}^r$ or $\Omega = \mathbb{I}^r$ for some $r \in \mathbb{N}$, $\Sigma = \mathcal{L}(\Omega)$, $\mathcal{I} = \mathcal{N}(\Omega)$, $X = \mathbb{R}$ or $X = \mathbb{I}$, and $p \in [1, +\infty]$. Let the function $\mathfrak{f} : L^p(\Omega, X) \to L^p(\Omega, X)$ be given by the formula $\mathfrak{f}(\xi)(\omega) = f(\omega, \xi(\omega))$ for every $\omega \in \Omega$ and for each $[\xi] \in L^p(\Omega, X)$, where $f : \Omega \times X \to X$ is a continuous measurable operator. Then if $\mathfrak{f}$ has an n-orbit, then has a k-orbit for each $k \lhd n$.*

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RANDOMIZED SHARKOVSKY-TYPE RESULTS AND RANDOM SUBHARMONIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS

JAN ANDRES AND PAWEŁ BARBARSKI

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ABSTRACT. Two multivalued deterministic versions of the celebrated Sharkovsky cycle coexistence theorem are randomized in terms of very general random periodic orbits. It is also shown that nontrivial subharmonics of scalar random upper-Carathéodory differential inclusions imply the coexistence of random subharmonics of all orders.

1. INTRODUCTION

The aim of the present paper is to randomize: (i) two multivalued deterministic Sharkovsky-type theorems (cf. [20]) obtained by the authors in [3, 7] and (ii) the deterministic theorem for scalar differential equations and inclusions, saying that the existence of a pure (nontrivial) subharmonic solution implies the coexistence of subharmonic solutions of all orders, whose various proofs can be found in [2, 4, 18, 19, 21].

The first goal is related to the existence of random periodic orbits rather than random periodic points (cf. [9, 10]) and there are quite rare results in this field (see [1, 6, 8, 16, 23]). The main technique, Proposition 3.2 below, will be developed for a very general class of multivalued random operators in a Suslin space. In this way, all our earlier results in [1, 6, 8] will be generalized both as a method in Section 3 as well as its application to Sharkovsky-type theorems in Section 4.

As concerns the second goal, the forcing property for the subharmonic solutions of random differential inclusions was already studied by the authors in [4] and [6], but the proofs were either only indicated in [4] or incomplete in [6]. Here, full proofs of the related Theorem 5.6 below will be given in Section 5. Although random periodic solutions of differential equations have been investigated as random fixed points of the associated random operators (see e.g. [11, 12, 22]), as far as we know, there are no further results for random subharmonic solutions considered as random periodic orbits of the associated random operators.

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Although the proofs of the statements might seem to be rather technical (some preliminaries are therefore recalled in Section 2), the main randomized theorems are formulated in an extremely simple and transparent way. Moreover, since the developed randomization technique is quite universal and powerful, it can be easily applied elsewhere.

2. Preliminaries

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{I}$, we denote the set of natural numbers, the set of real numbers, and the interval $[0,1]$, respectively. For finite $I \subset \mathbb{N}$, the symbol $\mathrm{LCM}\, I$ denotes the lowest common multiple of elements of I. We denote $D(n) = \{p < n\colon p|n\}$ and $d(n) = |D(n)|$, for $n \in \mathbb{N}$. We also fix the following notation: (Ω, Σ) is a measurable space, $\mathcal{I}$ is a proper σ-ideal on Ω (i.e. $\Omega \notin \mathcal{I}$), and μ (if it exists) is a complete σ-finite nontrivial measure on (Ω, Σ). We say that Σ is *complete* if such a measure exists. We say that Σ is *nonatomic* with respect to $\mathcal{I}$ if for every $A \in \Sigma$ such that $A \notin \mathcal{I}$, there is a splitting $A = B \cup C$ such that $B, C \in \Sigma$ and $B, C \notin \mathcal{I}$.

We recall the notion of a Suslin family. Let S be the set of all finite sequences of natural numbers. For a family $\{A_s\colon s \in S\}$ we put $\mathcal{A}\{A_s\colon s \in S\} = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=0}^{\infty} A_{\sigma\restriction n}$. We say that a family $\mathcal{F}$ is a *Suslin family* if $\mathcal{AF} = \mathcal{F}$, where $\mathcal{AF} = \{\mathcal{A}\{A_s\colon s \in S\}\colon A_s \in \mathcal{F} \text{ for } s \in S\}$.

By (X, d), we always understand a metric space. A metric space which is a continuous image of a Polish space is called a *Suslin space*. Each Suslin space is separable. For $Y \subset X$, by $\overline{Y}$ we mean the closure of Y in X. For $X = \mathbb{R}^k$ or $X = \mathbb{I}^k$, the symbol d denotes the Euclidean metric. For a closed set $B \subset X$, we use the standard notation: $d(x, B) = \inf\{d(x, y)\colon y \in B\}$.

For σ-algebras Σ_1 and Σ_2, by $\Sigma_1 \otimes \Sigma_2$ we mean their product σ-algebra. The symbol $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets in X. If X_1, X_2 are separable, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$. For a complete measure space (Ω, Σ, μ), we denote $\mathcal{N}(\Omega) = \{A \subset \Omega\colon \mu(A) = 0\}$. For $\Omega = \mathbb{R}^k$ or $\Omega = \mathbb{I}^k$, the symbol $\mathcal{L}(\Omega)$ denotes the σ-algebra of Lebesgue measurable sets on Ω.

We naturally identify a relation $\varphi \subset A \times B$ with a map $\varphi\colon A \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ stands for all subsets of B. If we want to emphasize the properties of φ as a subset of $A \times B$, we use the notion of a graph Γ_φ, where $\Gamma_\varphi = \{(a, b) \in A \times B\colon b \in \varphi(a)\}$. Furthermore, if $\Gamma_\varphi \subset (A \times B) \times C$, i.e. $\varphi\colon A \times B \to \mathcal{P}(C)$, then for any $x \in A$, we define the relation $\varphi_x = \varphi(x, \cdot)$. A multivalued map $\varphi\colon A \multimap B$ is a relation with nonempty values, i.e., $\varphi\colon A \to \mathcal{P}(B) \setminus \{\emptyset\}$. For single valued maps, we identify a map $f\colon A \to B$ with the multivalued map with one-element values fulfilling the condition $\varphi(x) = \{f(x)\}$, for every $x \in A$.

By a superposition of a map $f\colon B \to C$ with a relation $\varphi \subset A \times B$, we mean the relation $f \circ \varphi \subset A \times C$, defined by $f \circ \varphi(a) = f(\varphi(a))$, for $a \in A$. By a product of relations $F_i \subset A \times B$ for $i \in I$, we mean the relation $\prod_{i\in I} F_i$ defined by $(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a)$, for $a \in A$.

We denote by

$$\varphi^-(B) = \{\omega \in \Omega\colon \varphi(\omega) \cap B \neq \emptyset\}$$

the large preimage of the set $B \subset X$ under the relation $\varphi \subset \Omega \times X$. A relation $\varphi \subset \Omega \times X$ is called *measurable* if $\varphi^-(F) \in \Sigma$, for every closed $F \subset X$. It is called *weakly measurable* if $\varphi^-(G) \in \Sigma$, for every open $G \subset X$. The paper [14] contains a thorough analysis of the notion of a measurable relation. We shall state several facts from that paper here. Every measurable relation is weakly measurable (see [14, Proposition 2.1]). If Σ is complete and X is a Suslin space, the notions of measurability, weak measurability and a measurable graph coincide for a relation with closed values (see [14, Theorem 3.5]). The countable sum of measurable relations is measurable (see [14, Proposition 2.3(i)]). If a codomain is separable, then the product of at most countably many weakly measurable relations is weakly measurable (see [14, Proposition 2.3(ii)]). For a weakly measurable multivalued map $\varphi\colon \Omega \multimap \mathbb{R}$, the map defined by $f(\omega) = \inf \varphi(\omega)$, for $\omega \in \Omega$, is measurable (see [14, Theorem 5.8]). The superposition of a continuous map with a weakly measurable relation is weakly measurable (which is clear from the definition).

Assume that X and Y are metric spaces. A relation $\varphi \subset X \times Y$ is called *upper semicontinuous* (u.s.c.) if $\varphi^{-}(F)$ is closed in X, for every closed $F \subset Y$. It is called *lower semicontinuous* (l.s.c.) if $\varphi^{-}(G)$ is open in X, for every open $G \subset Y$. If φ is both l.s.c. and u.s.c., then it is called *continuous*.

We call a map $f\colon \Omega \to X$ a *selection* of a multivalued map $\varphi\colon \Omega \multimap X$ and write $f \subset \varphi$ if $f(\omega) \in \varphi(\omega)$, for each $\omega \in \Omega$. We shall use a generalization of the Aumann-von Neumann selection theorem which is a simple consequence of Corollary to Theorem 7 in the article [17] by Leese. We state it as follows:

Lemma 2.1. *Assume that Σ is a Suslin family and X is a Suslin space. Let $\varphi\colon \Omega \multimap X$ have a measurable graph, i.e. $\Gamma_\varphi \in \Sigma \otimes \mathcal{B}(X)$. Then it has a measurable selection $f \subset \varphi$.*

Another fact is a simple consequence of [17, Lemma 3].

Lemma 2.2. *Assume that Σ is a Suslin family and X is a Suslin space. Then every $\varphi\colon \Omega \multimap X$ with a measurable graph is measurable.*

A sequence $(x_i)_{i=0}^{k-1} \in A^k$ is called a *k-orbit* of the multivalued map $\varphi\colon A \multimap A$ if $x_{i+1} \in \varphi(x_i)$, for $i < k-1$, $x_0 \in \varphi(x_{k-1})$, and there is no $m < k$ such that $m|k$ and $x_{sm+i} = x_i$, for $i < m$ and $s < \frac{k}{m}$.

We recall the Sharkovsky ordering of natural numbers [20]:

$$\begin{array}{ccccccccc}
3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \dots \\
2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \dots \\
2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \dots \\
 & & & & \vdots & & & & \\
\dots & \rhd & 2^3 & \rhd & 2^2 & \rhd & 2 & \rhd & 1.
\end{array}$$

We state here a simple fact about this ordering (cf. [8, Lemma 4.1]):

Lemma 2.3. *If $\mathrm{LCM}\{i_j\colon j < l\} = n$ and $k \lhd n$, then there is $j' < l$ such that $k \lhd i_{j'}$.*

We also recall here some generalizations of Sharkovsky's Theorem which are simple corollaries of Theorem 6 from [3] and Theorem 2 form [7], respectively.

Proposition 2.4. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be an l.s.c. multivalued map with compact and convex values. If φ has an n-orbit, then φ has a k-orbit, for every $k \lhd n$.*

Proposition 2.5. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a u.s.c. multivalued map with compact and convex values. Suppose that φ has an n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property.*

(1) *If $q > 3$, then φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$.*

(2) *If $q = 3$, then at least one of the following two cases occurs:*
   (i) *φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1} \cdot 3, 2^{m+2}$,*
   (ii) *φ has a k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1}$.*

(3) *If $q = 1$, then φ has a k-orbit for every $k \lhd n$.*

We recall here the definition of a random operator after [1].

Definition 2.6. Let $\varphi\colon \Omega \times X \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is weakly measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$. We call it an *l.s.c. (u.s.c.) random operator* if φ_ω, i.e., $\varphi(\omega, \cdot)$, is l.s.c. (u.s.c., respectively), for almost all $\omega \in \Omega$.

Remark 2.7. For the definition of a random operator, it is usually still required that φ be compact-valued (cf. [13]), and $\varphi(\omega, \cdot)\colon X \multimap X$ to be u.s.c. (cf. again [13]) or Hausdorff-continuous (cf. [15, Chapter 5.6]), for almost all $\omega \in \Omega$. We omit these additional assumptions and introduce here a more general approach.

For a random operator $\varphi\colon \Omega \times X \multimap X$ and $k, m \in \mathbb{N}$, we define the function $d_{\varphi,k,m}\colon \Omega \times X^k \to [0, +\infty)$ as follows (cf. [1]):

$$d_{\varphi,k,m}\left(\omega, (x_i)_{i=0}^{k-1}\right) = d\Big((x_i)_{i=0}^{k-1}, \varphi(\omega, x_{m-1}) \times \varphi(\omega, x_0) \times \ldots \times \varphi(\omega, x_{m-2}) \\ \times \left(\{x_0\} \times \{x_1\} \times \ldots \times \{x_{m-1}\}\right)^{\frac{k}{m}-1}\Big),$$

and the function $D_{\varphi,k,m}\colon \Omega \times X^k \to [0, +\infty)^{1+d(m)}$ as follows:

$$D_{\varphi,k,m} = \left(d_{\varphi,k,m}, (d_{\varphi,k,p})_{p \in D(m)}\right).$$

Moreover, for $\omega \in \Omega$, we denote $d_{\varphi,k,m,\omega} = d_{\varphi,k,m}(\omega, \cdot)$ and

$$D_{\varphi,k,m,\omega} = D_{\varphi,k,m}(\omega, \cdot) = \left(d_{\varphi,k,m,\omega}, (d_{\varphi,k,p,\omega})_{p \in D(m)}\right).$$

We also define the relation $\mathcal{O}_{\varphi,k,m} \subset \Omega \times X^k$ as follows:

$$\mathcal{O}_{\varphi,k,m}(\omega) = \left\{(x_i)_{i=0}^{k-1} \in X^k \colon (x_i)_{i=0}^{m-1} \text{ is an } m\text{-orbit of } \varphi_\omega \wedge \forall_{\substack{s < \frac{k}{m} \\ i < m}} x_{sm+i} = x_i\right\},$$

and set $\Delta_{\varphi,k} = \{\omega \in \Omega \colon \varphi_\omega \text{ has a } k\text{-orbit}\}$.

The following simple facts from the paper (cf. [8, Lemmas 3.1 and 3.2]) will be useful.

Lemma 2.8. *Assume that X is separable. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. Then, for every $k, m \in \mathbb{N}$ such that $m|k$, the function $d_{\varphi,k,m}$ (and as a result $D_{\varphi,k,m}$) is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$.*

Lemma 2.9. *Let $\varphi\colon \Omega \times X \multimap X$ be a random operator and let $k, m \in \mathbb{N}$ be such that $m|k$. Then*

$$\mathcal{O}_{\varphi,k,m}(\omega) = \left\{(x_i)_{i=0}^{k-1} \in X^k \colon d_{\varphi,k,m,\omega}\left((x_i)_{i=0}^{k-1}\right) = 0 \wedge \forall_{\substack{p<m \\ p|m}} d_{\varphi,k,p,\omega}\left((x_i)_{i=0}^{k-1}\right) > 0\right\},$$

for all $\omega \in \Omega$.

Remark 2.10. Equivalently we can write:

$$\mathcal{O}_{\varphi,k,m}(\omega) = D_{\varphi,k,m,\omega}^{-1}\left(\{0\} \times (0, +\infty)^{d(m)}\right),$$

or simply:

$$\Gamma_{\mathcal{O}_{\varphi,k,m}} = D_{\varphi,k,m}^{-1}\left(\{0\} \times (0, +\infty)^{d(m)}\right).$$

As a result:

$$\Gamma_{\mathcal{O}_{\varphi,k,m}} \in \Sigma \otimes \mathcal{B}(X^k).$$

Thus, if Σ is a Suslin family and X is a Suslin space, then, by Lemma 2.2, $\mathcal{O}_{\varphi,k,m}$ is measurable, and its domain $\mathcal{O}^{-}_{\varphi,k,m}(\mathbb{R}^k) \in \Sigma$. In particular, $\Delta_{\varphi,k} = \mathcal{O}^{-}_{\varphi,k,k}(\mathbb{R}^k) \in \Sigma$.

Now, we can state the crucial definition of a random orbit.

Definition 2.11. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to X$, for $i = 0, \dots, k-1$, is called a *random k-orbit* of the operator φ if

(a) $\Omega \setminus \{\omega \in \Omega\colon \forall_{i<k-1}\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega)) \wedge \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\} \in \mathcal{I}$,

(b) there is no $m < k$ such that $m|k$ and

$$\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \in \mathcal{I}.$$

Remark 2.12. If we assume that (Ω, Σ, μ) is a complete measure space, $\mathcal{I} = \mathcal{N}(\Omega)$, and (X, d) is a Polish space, then the definition of a random orbit coincides with the definition given in [1], i.e.:

Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. A sequence of measurable functions $(\xi_i)_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to X$, for $i = 0, \dots, k-1$, is called a *random k-orbit* of the operator φ if

(a) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for $i = 0, \dots, k-2$ and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,

(b) the sequence $(\xi_i)_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements (i.e. it is not a concatenation), where $mp = k$ (for almost all $\omega \in \Omega$).

Equivalently:

(a) $\mu\left(\Omega \setminus \{\omega \in \Omega\colon \forall_{i<k-1}\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega)) \wedge \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\}\right) = 0$,

(b) there is no $m < k$ such that $m|k$ and

$$\mu\Big(\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\}\Big) = 0.$$

3. Characterization of operators with random orbits

Lemma 3.1. *Assume that X is separable. Let $\varphi\colon \Omega \times X \multimap X$ be a random operator. If φ has a random k-orbit, then there exists a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \dots, i_{l-1}$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j = 0, \dots, l-1$,
- $\mathrm{LCM}\{i_j\colon j = 0, \dots, l-1\} = k$,
- φ_ω *has an i_j-orbit for each* $\omega \in \Omega_{i_j}$ *for* $j < l$.

Proof. Let:

$$\Omega_m = \{\omega \in \Omega\colon (\xi_i(\omega))_{i=0}^{k-1} \in \mathcal{O}_{\varphi,k,m}(\omega)\},$$

for any $m|k$. Let $\{i_j\colon j < l\} = \{m|k\colon \Omega_m \notin \mathcal{I}\}$, and:

$$\Omega_0 = \Omega \setminus \bigcup_{j=0}^{l-1} \Omega_{i_j}.$$

The sets Ω_m for $m = 0, i_0, i_1, \dots, i_{l-1}$ are disjoint.

For all $m = i_0, i_1, \dots, i_{l-1}$, we have:

$$\Omega_m = \left\{ \omega \in \Omega \colon d_{\varphi,k,m}\left(\omega, (\xi_i(\omega))_{i=0}^{k-1}\right) = 0 \wedge \forall_{\substack{p<m \\ p|m}} d_{\varphi,k,p}\left(\omega, (\xi_i(\omega))_{i=0}^{k-1}\right) > 0 \right\}.$$

Let $\xi' = \left(id_\Omega, (\xi_i)_{i=0}^{k-1}\right)$. Then:

$$\Omega_m = \left(D_{\varphi,k,m} \circ \xi'\right)^{-1} \left(\{0\} \times (0, +\infty)^{d(m)}\right).$$

The function $D_{\varphi,k,m}$ is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$. Hence, $D^{-1}_{\varphi,k,m}(\{0\} \times (0, +\infty)^{d(m)}) \in \Sigma \otimes \mathcal{B}(X^k)$. Since X is separable, the σ-algebra $\Sigma \otimes \mathcal{B}(X^k)$ is generated by the sets $M \times \prod_{i=0}^{k-1} B_i$, where $M \in \Sigma$ and $B_i \in \mathcal{B}(X)$ for $i < k$. Fix such a set $M \times \prod_{i=0}^{k-1} B_i$. Then:

$$\xi'^{-1}\left(M \times \prod_{i=0}^{k-1} B_i\right) = M \cap \bigcap_{i=0}^{k-1} \xi_i^{-1}(B_i) \in \Sigma,$$

because functions ξ_i, for $i < k$, are measurable. Thus, $\xi'^{-1}(B) \in \Sigma$, for any $B \in \Sigma \otimes \mathcal{B}(X^k)$, and subsequently:

$$\Omega_m = \xi'^{-1}\left(D^{-1}_{\varphi,k,m}(\{0\} \times (0, +\infty)^{d(m)})\right) \in \Sigma,$$

for any $m = i_0, i_1, \dots, i_{l-1}$. Therefore, $\Omega_0 \in \Sigma$. Clearly, $\Omega_{i_j} \notin \mathcal{I}$, for any $j < l$. By the condition (a), we have $\Omega \setminus \bigcup_{m|k} \Omega_m \in \mathcal{I}$, and since

$$\Omega_0 = \left(\Omega \setminus \bigcup_{m|k} \Omega_m\right) \cup \bigcup \{\Omega_m \colon \Omega_m \in \mathcal{I} \wedge m|k\},$$

we get $\Omega_0 \in \mathcal{I}$.

Suppose that $k' = \mathrm{LCM}\{i_j \colon j < l\} < k$. Obviously, $k'|k$. Then $\xi_{sm+i}(\omega) = \xi_i(\omega)$, for any $s < \frac{k}{k'}$ and $i < k'$, and $\omega \in \Omega \setminus \Omega_0$, which contradicts the condition (b). Hence, $\mathrm{LCM}\{i_j \colon j < l\} = k$. $\square$

Proposition 3.2. *Assume that Σ is a Suslin family and X is a Suslin space. Let $\varphi \colon \Omega \times X \multimap X$ be a random operator. Then φ has a random k-orbit iff there exists a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \dots, i_{l-1}$,
- $\Omega_0 \in \mathcal{I}$ *and* $\Omega_{i_j} \notin \mathcal{I}$ *for* $j = 0, \dots, l-1$,
- $\mathrm{LCM}\{i_j \colon j = 0, \dots, l-1\} = k$,
- φ_ω *has an* i_j*-orbit for each* $\omega \in \Omega_{i_j}$ *for* $j < l$.

Proof. By Lemma 3.1, it is enough to show the implication to the left. We define the multivalued map $\mathcal{O}_{\varphi,k} \colon \Omega \to X^k$ as follows:

$$\mathcal{O}_{\varphi,k}(\omega) = \begin{cases} X^k & \text{for } \omega \in \Omega_0, \\ \mathcal{O}_{\varphi,k,i_j}(\omega) & \text{for } \omega \in \Omega_{i_j} \text{ for } j < l \text{ and } t \in \mathbb{N}. \end{cases}$$

By Remark 2.10, we get $\Gamma_{\mathcal{O}_{\varphi,k}} \in \Sigma \otimes \mathcal{B}(X^k)$. By Lemma 2.1, we receive a measurable function $\xi\colon \Omega \to X^k$ such that $\xi \subset \mathcal{O}_{\varphi,k}$.

The sequence $(\xi_i)_{i=0}^{k-1}$ is a random k-orbit of φ. To see this, first observe that each ξ_i is measurable, because ξ is measurable and $\mathcal{B}(X)^k = \mathcal{B}(X^k)$. For each $\omega \in \Omega_{i_j}$, where $j < l$, we have $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for $i < k-1$, and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$. Thus, the condition (a) is fulfilled. Suppose that there is $m < k$ such that $m|k$ and

$$\Omega \setminus \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \in \mathcal{I}.$$

Fix any $j < l$. Then $\Omega_{i_j} \cap \{\omega \in \Omega\colon \forall_{s<\frac{k}{m}} \forall_{i<m}\, \xi_{sm+i}(\omega) = \xi_i(\omega)\} \neq \emptyset$, because $\Omega_{i_j} \notin \mathcal{I}$. Thus, $i_j|m$, and $\mathrm{LCM}\{i_j : j < l\} \leq m < k$, a contradiction. Consequently, the condition (b) is fulfilled. $\square$

Remark 3.3. The above proposition is a generalization of Proposition 2 in [1].

4. Randomized Sharkovsky-type results

Now we can state two randomized versions of the Sharkovsky Theorem, whose proofs are based on the above Proposition 3.2.

Theorem 4.1. *Assume that Σ is a Suslin family, $\mathcal{I}$ is a proper σ-ideal, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is an l.s.c. random operator with compact and connected values. If φ has a random n-orbit, then it has a random k-orbit, for each $k \lhd n$.*

Proof. Suppose that φ has a random n-orbit. There is a splitting of Ω as in Proposition 3.2. Fix $k \lhd n$. By Lemma 2.3, there is $j' < l$ such that $k \lhd i_{j'}$. Moreover, $1 \lhd i_j$ for each $j < l$ and $j \neq j'$. Since φ_ω has an i_j-orbit, for each $\omega \in \Omega_{i_j}$ and every $j < l$, by Proposition 2.4, φ_ω has a k-orbit, for each $\omega \in \Omega_{i_{j'}}$ and a 1-orbit, for each $\omega \in \Omega_{i_j}$ and every $j < l$ and $j \neq j'$. Put $\Omega'_k = \Omega_{i_{j'}}$, $\Omega'_1 = \bigcup\{\Omega_{i_j} : j \leq l \wedge j \neq j'\}$, and $\Omega'_0 = \Omega_0$. As a consequence, we get the following splitting of Ω: $\Omega = \Omega'_0 \cup \Omega'_k$ if $l = 1$, or $\Omega = \Omega'_0 \cup \Omega'_k \cup \Omega'_1$ if $l > 1$. By Proposition 3.2, φ has a random k-orbit. $\square$

As a special single-valued case of Theorem 4.1, we obtain the following randomization of the classical Sharkovsky theorem in [20].

Corollary 4.2. *Assume that Σ is a Suslin family, $\mathcal{I}$ is a proper σ-ideal, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous random operator. If f has a random n-orbit, then it has a random k-orbit, for each $k \lhd n$.*

Theorem 4.3. *Assume that Σ is a Suslin family which is nonatomic with respect to a proper σ-ideal $\mathcal{I}$, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is a u.s.c. random operator with compact and connected values. Suppose that φ has a random n-orbit ($n = 2^m q$, where q is odd), and n is the maximal number in the Sharkovsky ordering with that property.*

(1) *If $q > 3$, then φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$.*
(2) *If $q = 3$, then at least one of the following two cases occurs:*
   (i) *φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+2}$,*
   (ii) *φ has a random k-orbit for every $k \lhd n$, except possibly for $k = 2^{m+1}$.*
(3) *If $q = 1$, then φ has a k-orbit for every $k \lhd n$.*

Proof. There is a splitting of Ω as in Proposition 3.2.

Recall that $\Delta_{\varphi,k} \in \Sigma$, for any k.

First, fix $k \rhd n$, $k \neq n$. Suppose that $\Delta_{\varphi,k} \notin \mathcal{I}$. Then we can take $\Omega_0' = \Omega_0$, $\Omega_k' = \Delta_{\varphi,k} \setminus \Omega_0$, and $\Omega_1' = \Omega \setminus (\Omega_k' \cup \Omega_0')$, and by Proposition 3.2, we get a random k-orbit for φ which contradicts the maximality of n. Thus, $\Delta_{\varphi,k} \in \mathcal{I}$.

Let $\Omega_0'' = \bigcup_{k \rhd n} \Delta_{\varphi,k} \cup \Omega_0$. Then $\Omega_0'' \in \Sigma$ and $\Omega_0'' \in \mathcal{I}$. Moreover, φ_ω has no k-orbit with $k \rhd n$ and $k \neq n$, for every $\omega \in \Omega \setminus \Omega_0''$. Fix $j = 0, \ldots, l-1$. Since $\Omega_{i_j} \setminus \Omega_0'' \neq \emptyset$, $i_j \lhd n$. Furthermore, $i_j | n$. Hence, $i_j = n$ or $i_j = 2^{m_j}$ (where $m_j \leq m$), and there is $j' < l$ such that $i_{j'} = n$. Let $\Omega_n'' = \Omega_n \setminus \Omega_0''$. Thus, $\Omega_n'' \in \Sigma$, $\Omega_n'' \notin \mathcal{I}$, and for every $\omega \in \Omega_n''$, φ_ω has n-orbit, and n is the maximal number in the Sharkovsky ordering with that property.

Suppose that $q > 3$. Fix $k \lhd n$, $k \neq 2^{m+2}$. By Proposition 2.5, for every $\omega \in \Omega_n$, φ_ω has a k-orbit. Now, take $\Omega_k' = \Omega_n''$, $\Omega_0' = \Omega_0''$, $\Omega_1' = \Omega \setminus (\Omega_k' \cup \Omega_0')$. Then by Proposition 3.2, φ has a random k-orbit.

Suppose that $q = 3$. For $k \lhd n$, $k \neq 2^{m+1} \cdot 3$, $k \neq 2^{m+2}$, $k \neq 2^{m+1}$, we proceed analogously to the previous paragraph. By Proposition 2.5, $\Omega_n'' \subset \Delta_{\varphi,2^{m+1}} \cup \big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big)$. Then $\Delta_{\varphi,2^{m+1}} \cap \Omega_n'' \notin \mathcal{I}$ or $\big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big) \cap \Omega_n'' \notin \mathcal{I}$.

Suppose that $\Delta_{\varphi,2^{m+1}} \cap \Omega_n'' \notin \mathcal{I}$. Let $k = 2^{m+1}$. Now, take $\Omega_0' = \Omega_0''$, $\Omega_k' = \big(\Delta_{\varphi,2^{m+1}} \cap \Omega_n''\big) \setminus \Omega_0'$, $\Omega_1' = \Omega \setminus (\Omega_k' \cup \Omega_0')$. Then, by Proposition 3.2, φ has a random k-orbit. Let $k = 2^{m+1} \cdot 3$. For every $\omega \in \Omega_n''$, φ_ω has an n-orbit, i.e., $2^m \cdot 3$-orbit. There is a splitting $\Delta_{2^{m+1}} \cap \Omega_n'' = \Omega' \cup \Omega''$ such that $\Omega', \Omega'' \in \Sigma$, $\Omega', \Omega'' \notin \mathcal{I}$. Take $\Omega_0' = \Omega_0$, $\Omega_{2^{m+1}}' = \Omega'$, $\Omega_{2^m \cdot 3}' = \Omega''$, and $\Omega_1' = \Omega \setminus (\Omega_{2^{m+1}}' \cup \Omega_{2^m\cdot 3}' \cup \Omega_0')$. Since $\mathrm{LCM}\{2^{m+1}, 2^m \cdot 3, 1\} = 2^{m+1} \cdot 3 = k$, by Proposition 3.2, φ has a random k-orbit.

Suppose that $\big(\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Delta_{\varphi,2^{m+2}}\big) \cap \Omega_n'' \notin \mathcal{I}$. Then $\Delta_{\varphi,2^{m+1}\cdot 3} \cap \Omega_n'' \notin \mathcal{I}$, and $\Delta_{\varphi,2^{m+2}} \cap \Omega_n'' \notin \mathcal{I}$. For $k = 2^{m+1} \cdot 3$ and $k = 2^{m+2}$, we proceed quite analogously as for $k = 2^{m+1}$ in the previous paragraph.

The case when $q = 1$ is analogous to the case, when $q > 3$. □

Corollary 4.4. *Assume that Σ is a Suslin family which is nonatomic with respect to a proper σ-ideal $\mathcal{I}$, and $\varphi\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ is a u.s.c. random operator with compact and connected values. If φ has a random n-orbit, then it has a random k-orbit for each $k \lhd n$ with at most one exception.*

Remark 4.5. Observe that in the case (2)(i), unlike in the deterministic Proposition 2.5, we have a larger area for manipulation in the randomized Theorem 4.3. We can namely divide Ω into two parts (by nonatomicity) and take the period 2^{m+1} on one side and $2^m \cdot 3$ on the other one. Thus, the period $2^{m+1} \cdot 3$ occurs by their combination via Proposition 3.2, and subsequently $2^{m+1} \cdot 3$ is no longer an exception. In particular, the maximal number of exceptional cases reduces to one, as stated in Corollary 4.4.

5. Subharmonics of random differential inclusions

Assume that (Ω, Σ, μ) is a complete measure space and $\mathcal{I}$ is a proper σ-ideal on Ω. On $\mathbb{R}$ and $[0,1]$, we use the σ-algebra of Lebesgue measurable sets and the σ-ideal of null sets. Assume that $\varphi\colon \Omega \times [0,1] \times \mathbb{R} \multimap \mathbb{R}$ is a *random u-Carathéodory map*, i.e.:

- $\varphi(\cdot,\cdot,x)\colon \Omega \times [0,1] \multimap \mathbb{R}$ is measurable, for all $x \in \mathbb{R}$,
- $\varphi(\omega,t,\cdot)\colon \mathbb{R} \multimap \mathbb{R}$ is u.s.c., for almost all $(\omega,t) \in \Omega \times [0,1]$,
- there exists $a, b > 0$ such that $\sup\{|y|\colon y \in \varphi(\omega,t,x)\} \leq a + b|x|$, for almost all $(\omega,t) \in \Omega \times [0,1]$ and all $x \in \mathbb{R}$.

We extend φ to $\Omega \times \mathbb{R} \times \mathbb{R}$ in the following manner: $\varphi(\omega, t+k, x) = \varphi(\omega, t, x)$, for $\omega \in \Omega$, $t \in [0,1]$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$. Let $\varphi_\omega(t,x) = \varphi(\omega, t, x)$. We shall consider the random differential inclusion:

$$(I_\varphi) \qquad x'(\omega,t) \in \varphi(\omega,t,x(\omega,t)) \quad [\equiv \varphi(\omega,t+1,x(\omega,t))]$$

and a one-parameter family of deterministic differential inclusions:

$$(I_{\varphi_\omega}) \qquad x'(t) \in \varphi_\omega(t,x(t)) \quad [\equiv \varphi_\omega(t+1,x(t))]\,.$$

We say that $x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a *random solution* of (I_φ) if $x(\omega,\cdot)$ is absolutely continuous, for almost all $\omega \in \Omega$, $x(\cdot,t)$ is measurable, for each $t \in \mathbb{R}$, and the condition (I_φ) is fulfilled (where the differentiation is over t), for almost all $(\omega,t) \in \Omega \times \mathbb{R}$. For $k \in \mathbb{N}$, we say that a solution is a *random k-periodic subharmonic solution* if $x(\omega,t) = x(\omega,t+k)$, for almost all $(\omega,t) \in \Omega \times \mathbb{R}$ and there is no $m \in \mathbb{N}$, $m < k$ such that $x(\omega,t) = x(\omega,t+m)$, for almost all $(\omega,t) \in \Omega \times \mathbb{R}$.

We say that $x\colon \mathbb{R} \to \mathbb{R}$ is a *solution* of (I_{φ_ω}) if, for a fixed $\omega \in \Omega$, it is absolutely continuous and the condition (I_{φ_ω}) is fulfilled, for almost all $t \in \mathbb{R}$. For $k \in \mathbb{N}$, we say that a solution is a *k-periodic subharmonic solution* if $x(t) = x(t+k)$, for almost all $t \in \mathbb{R}$, and there is no $m \in \mathbb{N}$, $m < k$ such that $x(t) = x(t+m)$, for almost all $t \in \mathbb{R}$. For $k = 1$, we also speak about (*random*) *harmonic* solutions. Thus, for $k > 1$, we can speak about *pure* or *nontrivial* (*random*) *subharmonics.*

We define the (*random*, cf. [6, Proposition 4.3]) *Poincaré operator* $P_k\colon \Omega \times \mathbb{R} \times \mathbb{R} \multimap \mathbb{R}$ associated with (I_φ) as follows:

$$P_k(\omega,t_0,x_0) = \{x(t_0+k)\colon x \text{ is a solution of } (I_{\varphi_\omega}), x(t_0) = x_0\}.$$

It is known that $P_k = P_1^k$. For more details and properties of the random Poincaré operators, see [5, Chapter III,4,E] and [6]. We shall also use the following notation: $P_{k,\omega,t_0}(x_0) = P_k(\omega,t_0,x_0)$ and $P_{k,t_0}(\omega,x_0) = P_k(\omega,t_0,x_0)$.

We can now draw a diagram of implications between certain sentences. In that diagram, we assume that $n, k > 1$. By "$\exists \bigcup_i^m \Omega_i = \Omega\ \forall_\omega \Phi$", we mean that there exists a partition of Ω such that

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ for $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ and $\mu(\Omega_{i_j}) > 0$ for $j = 0, \ldots, l-1$,
- $\mathrm{LCM}\{i_j\colon j = 0, \ldots, l-1\} = m$,
- for $i = i_j$ and for all $\omega \in \Omega_i$, Φ.

The implications in the diagram are proved or cited below. Theorem 2 from [2] will be stated here in the form of the following proposition.

Proposition 5.1. *Let $\varphi\colon \mathbb{R} \multimap \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins, i.e., $\varphi^*(x) = \sup\{y\colon y \in \varphi(x)\}$ and $\varphi_*(x) = \inf\{y\colon y \in \varphi(x)\}$, are nondecreasing. If φ has an n-orbit with $n > 1$, then it also has a k-orbit, for every $k \in \mathbb{N}$.*

Remark 5.2. We know (see [2] and [4]) that the Poincaré operators associated with (I_{φ_ω}) are u.s.c. maps with compact, connected values and their margins are nondecreasing. In particular, they satisfy the assumptions of Proposition 5.1.

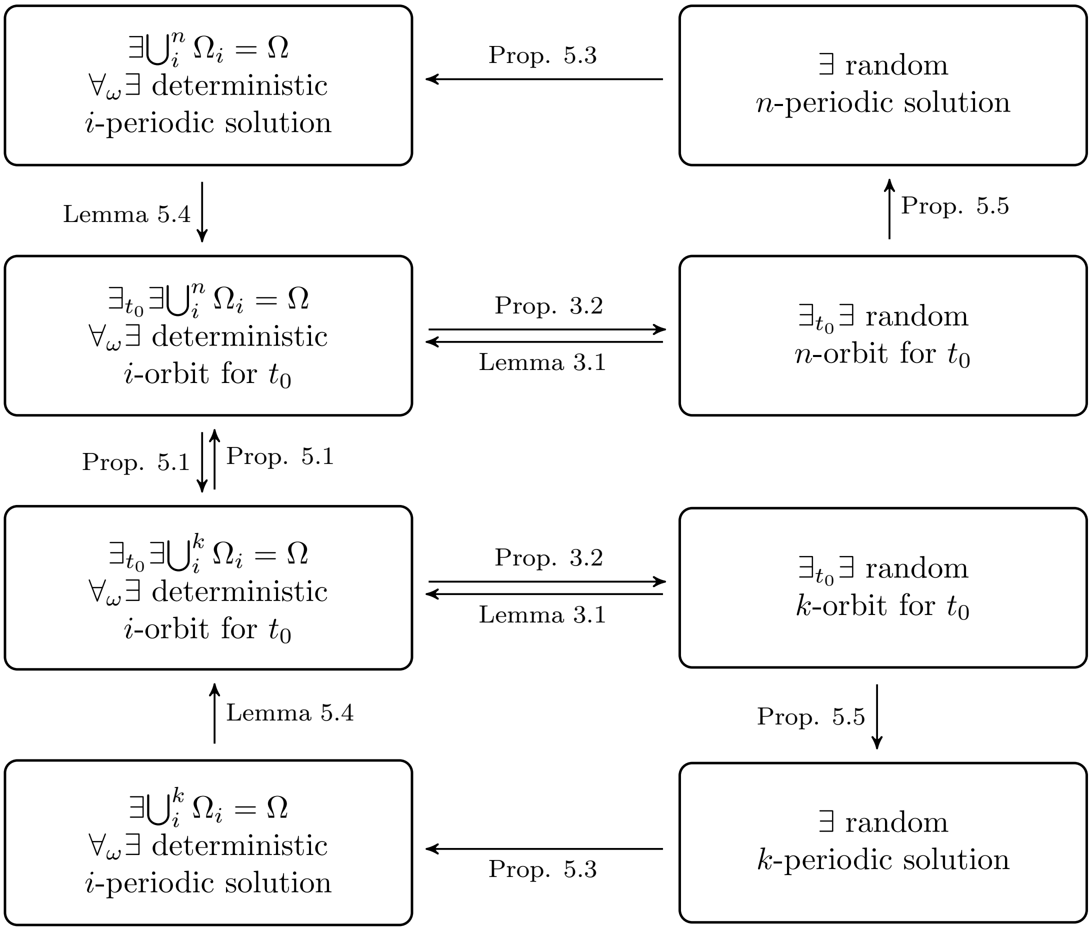

FIGURE 1. Diagram of implications among various sentences about the given inclusion (I_φ)

Proposition 5.3. *If the inclusion (I_φ) has a random n-periodic subharmonic solution (where $n \in \mathbb{N}$, $n > 1$), then there is a partition of Ω such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ *and* $\mu(\Omega_{i_j}) > 0$ *for* $j < l$,
- $\mathrm{LCM}\{i_j : j < l\} = n$,
- (I_{φ_ω}) *has an i_j-periodic solution for each* $\omega \in \Omega_{i_j}$, $j < l$.

Proof. Let x be a random n-periodic subharmonic solution of (I_φ). Let

$$\Omega_i = \{\omega \in \Omega\colon x_\omega \text{ is } i\text{-periodic subharmonic solution of } (I_{\varphi_\omega})\},$$

for $i|n$, $i \in \mathbb{N}$. Then

$$\begin{aligned}\Omega_i &= \{\omega \in \Omega \colon \forall_t x(\omega,t) = x(\omega,t+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \{\omega \in \Omega \colon \forall_t x(\omega,t) = x(\omega,t+j)\}\\ &= \{\omega \in \Omega \colon \forall_{q\in\mathbb{Q}} x(\omega,q) = x(\omega,q+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \{\omega \in \Omega \colon \forall_{q\in\mathbb{Q}} x(\omega,q) = x(\omega,q+j)\}\\ &= \bigcap_{q\in\mathbb{Q}} \{\omega \in \Omega \colon x(\omega,q) = x(\omega,q+i)\} \setminus \bigcup_{\substack{j|n\\ j<i}} \bigcap_{q\in\mathbb{Q}} \{\omega \in \Omega \colon x(\omega,q) = x(\omega,q+j)\}.\end{aligned}$$

Hence, $\Omega_i \in \Sigma$, for any $i|n$, $i \in \mathbb{N}$. Furthermore, $\mu(\Omega \setminus \bigcup_{i|n} \Omega_i) = 0$. Let $\{i_j : j < l\} = \{i|n \colon \mu(\Omega_i) > 0\}$ and $\Omega_0 = \Omega \setminus \bigcup_{j<l} \Omega_{i_j}$. Clearly, $\Omega_0 \in \Sigma$ and $\mu(\Omega_0) = 0$. At the same time, $\mathrm{LCM}\{i_j : j < l\} = n$, because otherwise $\mathrm{LCM}\{i_j : j < l\} = k < n$, and for almost all $\omega \in \Omega$, x_ω has some period $i|k$, by which x has some period $i|k$, a contradiction. □

Lemma 5.4. *If there exists a partition of* Ω *such that*

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j},$$

where:

- $\Omega_m \in \Sigma$ *for* $m = 0, i_0, i_1, \ldots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ *and* $\mu(\Omega_{i_j}) > 0$ *for* $j < l$,
- $\mathrm{LCM}\{i_j : j < l\} = n$,
- (I_{φ_ω}) *has an* i_j*-periodic solution for each* $\omega \in \Omega_{i_j}$, $j < l$,

then there is $t_0 \in [0,n)$ *and a partition of* Ω *such that*

$$\Omega = \Omega_0' \cup \Omega_1' \cup \Omega_n',$$

where:

- $\Omega_m' \in \Sigma$ *for* $m = 0, 1, n$,
- $\mu(\Omega_0') = 0$ *and* $\mu(\Omega_n') > 0$,
- P_{1,ω,t_0} *has an* n*-orbit for each* $\omega \in \Omega_n'$.

Proof. Since $n > 1$, there is $j_0 < l$ such that $i_{j_0} > 1$. Fix $\omega \in \Omega_{j_{i_0}}$. (I_{φ_ω}) has a j_{i_0}-periodic solution $x_\omega \colon \mathbb{R} \to \mathbb{R}$. The set $A_\omega = \{t \colon x_\omega(t) \neq x_\omega(t+1)\}$ is open and nonempty, because otherwise $n = 1$. Take $q_\omega \in \mathbb{Q} \cap A_\omega$. Then $(x_\omega(q_\omega + s))_{s=0}^{i_0-1}$ is an orbit of P_{1,ω,q_ω} or a concatenation of identical orbits of period larger than 1. Thus, P_{1,ω,q_ω} has an l-orbit for some $l > 1$. Let

$$\Omega_q'' = \{\omega \in \Omega \colon P_{1,\omega,q} \text{ has an } l\text{-orbit for some } l > 1\},$$

for $q \in \mathbb{Q} \cap [0,n)$.

$$\Omega_q'' = \bigcup_{p=2}^{\infty} \mathcal{O}^-_{P_{1,q},p,p}(\mathbb{R}^n).$$

The relation $\mathcal{O}_{P_{1,q},p,p}$ is measurable. Hence, $\Omega_q'' \in \Sigma$. Since $\bigcup_{q\in\mathbb{Q}\cap[0,n)} \Omega_q'' \supset \Omega_{i_{j'}}$, there is $q_0 \in \mathbb{Q} \cap [0,n)$ such that $\mu(\Omega_{q_0}'') > 0$. Now, take $t_0 = q_0$, $\Omega_0' = \Omega_0$, $\Omega_n' = \Omega_{q_0}''$, $\Omega_1' = \Omega \setminus (\Omega_0' \cup \Omega_n')$. Then P_{1,ω,t_0} has an n-orbit, for each $\omega \in \Omega_n'$, by Proposition 5.1. □

In Proposition 5.6 in [4], we can find the proof of the following fact.

Proposition 5.5. *(I_φ) has a random n-periodic subharmonic solution, provided P_{1,t_0} has a random n-orbit, for some $t_0 \in [0,1]$.*

As a result of commutativity of the diagram, we can state the following main theorem of this section.

Theorem 5.6. *If (I_φ) has a random n-periodic subharmonic solution, for some $n > 1$, then it has random k-periodic subharmonic solutions, for all $k \in \mathbb{N}$.*

Remark 5.7. Although the diagram in Figure 1 does not explicitly contain all implications indicated there by the arrows, it is completely commutative. In other words, the missing arrows can be completed.

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Department of Mathematical Analysis and Application of Mathematics, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: jan.andres@upol.cz

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: barbarski.pawel@gmail.com

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Paweł Barbarski, Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland. email: barbarski.pawel@gmail.com

CONTINUOUS FUNCTIONS IN RINGS GENERATED BY A SINGLE DARBOUX FUNCTION

Abstract

The problem of characterizing continuous functions belonging to each $\mathcal{A}$-ring containing a fixed Darboux function is fully solved for a compact interval domain and solved with some minor exception for a real line domain. Many properties of generated $\mathcal{A}$-rings and their continuous functions are stated.

1 Introduction

Continuous functions are widely researched in the context of dynamical systems (see [1]). The Sharkovsky Theorem on the coexistence of periodical orbits of continuous functions (see [17]) is the fundamental result of the discrete dynamics. It has many generalizations (e.g. [11], [16], [4], [2], [6], [3]). The Darboux-like functions as a superset of continuous functions which still preserve many interesting properties of continuous functions are the natural candidate to consider in generalizing the discrete-dynamics results (see e.g. [18], [8]). On the other hand we can equip the sets of functions with algebraic structures (see [9] and its citations). Algebraic considerations can also expose some properties of Darboux-like functions (see e.g. [7], [10], [12], [14]).

An example of Darboux-like functions which can be considered in algebraic structures and have some interesting dynamics are the first return limiting

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notions functions. In [13] authors consider the relationship between special kind of them, called $\mathcal{S}$-functions, and Sharkovsky Theorem. In particular they state a theorem that every $\mathcal{S}$-function is a Sharkovsky function ([13, Th. 3.3]). Moreover, they prove some facts about special kind of rings of $\mathcal{S}$-function in this context, but also formulate some problems. We shall discuss one of them, namely: Characterize continuous functions belonging to each $\mathcal{A}$-ring of $\mathcal{S}$-functions containing some $f \in \mathcal{S}$ (see [13, Problem 2]). Actually, we shall consider its generalization to all Darboux functions.

After some preliminaries we consider what are the generated $\mathcal{A}$-rings in general. We also reformulate the considered problem in terms of $\mathcal{A}$-rings generated by a single function. We generalize the problem for a Darboux function instead of $\mathcal{S}$-function. We state the problem both for functions with domain and codomain which is a compact interval and for the real line case. Afterwards we discuss the properties of $\mathcal{A}$-rings generated by identity and by any single Darboux function. Then we can look into the continuous functions in those rings. In the last section we state the main results of the paper, namely the solution of the problem. The solution is full in case of compact interval domain of the function. In case of real line domain we state the solution only for function with image which is a closed interval (in particular the real line), thus leaving the remaining cases of open interval and one-side-open interval as an open problem.

2 Preliminaries

By $\mathbb{N}$, $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{I}$ we denote the set of natural numbers, the set of real numbers, the set $\mathbb{R} \cup \{-\infty, +\infty\}$, and the closed unit interval in $\mathbb{R}$, respectively. By ω_1 we denote the first uncountable ordinal number. For $A \subseteq \mathbb{R}$, by $\mathrm{cl}(A)$ we denote a closure of set A. We use the following notation:

$$\{0,1\}^{<\omega} = \{s \colon \{1,\dots,n\} \to \{0,1\} \colon n \in \mathbb{N}\},$$
$$\{0,1\}^{<n} = \{s \colon \{1,\dots,k\} \to \{0,1\} \colon k < n\}$$

for $n \in \mathbb{N}$. For $s \colon \{1,\dots,n\} \to \{0,1\}$ we define $\mathrm{len}(s) = n$.

We consider functions $f \colon X \to \mathbb{R}$, where X is $\mathbb{R}$ or $\mathbb{I}$ (or equivalently any other compact non-degenerated interval). The family of all such functions we denote by $\mathbb{R}^X$. Let $\mathscr{C}(X)$, $\mathcal{P}(X)$, $\mathcal{C}(X)$, $\mathcal{B}(X)$, $\mathcal{D}(X)$, $\mathcal{B}_1(X)$ denote its subfamilies of all constant functions, all polynomials, all continuous functions, all bounded functions, all Darboux functions, and all Baire class one functions, respectively. Let $\mathcal{DB}_1(X) = \mathcal{D}(X) \cap \mathcal{B}_1(X)$. For families of functions $\mathcal{R}_1 \subseteq \mathbb{R}^{\mathbb{R}}$, $\mathcal{R}_2 \subseteq \mathbb{R}^X$ we shall denote $\mathcal{R}_1 \circ \mathcal{R}_2 = \{f \circ g \colon f \in \mathcal{R}_1, g \in \mathcal{R}_2\}$. By $\mathrm{id}_X \colon X \to \mathbb{R}$

we denote identity function; i.e. $\mathrm{id}(x) = x$. We shall omit X if it is clear from the context.

In [13] authors consider the family of all $\mathcal{S}(\mathbb{R})$-functions (a kind of a return limiting notion). By Theorem 2.2 and Proposition 2.3 from [13] it is clear that:

$$\mathcal{C}(\mathbb{R}) \subseteq \mathcal{DB}_1(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq \mathcal{D}(\mathbb{R}).$$

We omit the definition of a $\mathcal{S}$-function here, since we will generalize the mentioned problem to all Darboux functions.

We also adapt the following definitions from the mentioned paper. Let $\mathcal{R}$ denote some family of functions from X to $\mathbb{R}$. We consider the following conditions:

1. $\mathscr{C} \subseteq \mathcal{R}$,
2. if $f, g \in \mathcal{R}$, then $f + g, fg \in \mathcal{R}$,
3. if $(f_n)_n \subseteq \mathcal{R}$ and $f_n \rightrightarrows f$, then $f \in \mathcal{R}$,
4. if $f, g \in \mathcal{R}$, then $\max(f, g), \min(f, g) \in \mathcal{R}$,

$\mathcal{R}$ is called *an $\mathcal{A}$-ring* (*Aumann ring*, notion inspired by [5]) if it fulfills conditions (1) – (4).

Remark. Assume condition (1). Condition (2) is equivalent to $\mathcal{R}$ being a ring and to $\mathcal{R}$ being an algebra. Condition (3) is equivalent to $\mathcal{R}$ being uniformly closed. In that contex an $\mathcal{A}$-ring is a uniformly closed algebra closed under max and min operations.

The original problem from [13] is to characterize continuous functions belonging to each $\mathcal{A}$-ring of $\mathcal{S}(\mathbb{R})$-functions containing some fixed $f \in \mathcal{S}(\mathbb{R})$. We shall reformulate the problem and generalize it to Darboux functions in the following section.

3 Generated $\mathcal{A}$-Rings

Consider the following fact:

Proposition 3.1. *Let X be $\mathbb{R}$ or a compact interval. Let $\mathcal{R} \subseteq \mathbb{R}^X$. The family*

$$\mathcal{AR}(\mathcal{R}) = \bigcap \{\mathcal{Q} \subseteq \mathbb{R}^X : \mathcal{R} \subseteq \mathcal{Q}, \mathcal{Q} \textit{ is an } \mathcal{A}\textit{-ring of functions}\}$$

is the smallest $\mathcal{A}$-ring containing $\mathcal{R}$.

We shall call the set from the proposition *the $\mathcal{A}$-ring generated by $\mathcal{R}$*. For every $f \in \mathcal{S}(\mathbb{R})$, we have $\mathcal{AR}(\{f\}) \subseteq \mathcal{S}(\mathbb{R})$ by [13, Theorem 3.3]. Now the problem can be reformulated and generalized in the following form:

Problem 1. Let X be $\mathbb{R}$ or $\mathbb{I}$ and $f \in \mathcal{D}$. Find $\mathcal{AR}(\{f\}) \cap \mathcal{C}$.

Since $\mathcal{C}$ is an $\mathcal{A}$-ring, it is clear that $\mathcal{AR}(\mathcal{C}) = \mathcal{C}$ and $\mathcal{AR}(\mathcal{P}) = \mathcal{AR}(\{\mathrm{id}\}) \subseteq \mathcal{C}$. Let $\mathcal{R} \subseteq \mathbb{R}^X$. Denote

$$\begin{aligned}
\mathrm{cl}(\mathcal{R}) &= \{f\colon f_n \rightrightarrows f \text{ for some } (f_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}\},\\
\mathcal{A}(\mathcal{R}) &= \{f+g, fg, \min(f,g), \max(f,g)\colon f,g \in \mathcal{R}\} \cup \mathrm{cl}(\mathcal{R}),\\
\mathcal{A}_0(\mathcal{R}) &= \mathcal{R} \cup \mathscr{C},\\
\mathcal{A}_{\alpha+1}(\mathcal{R}) &= \mathcal{A}(\mathcal{A}_\alpha(\mathcal{R})) \text{ for ordinal } \alpha \in \omega_1,\\
\mathcal{A}_\alpha(\mathcal{R}) &= \bigcup_{\beta<\alpha} \mathcal{A}_\beta(\mathcal{R}) \text{ for limit ordinal } \alpha \in \omega_1 \cup \{\omega_1\}.
\end{aligned}$$

We formulate a useful characterization of generated $\mathcal{A}$-rings.

Lemma 3.2. *Let $\mathcal{R} \subseteq \mathbb{R}^X$, then*

$$\mathcal{AR}(\mathcal{R}) = \mathcal{A}_{\omega_1}(\mathcal{R}).$$

Proof. Firstly, $\mathcal{A}_{\omega_1}(\mathcal{R})$ is an $\mathcal{A}$-ring containing $\mathcal{R}$. Clearly $\mathcal{R}, \mathscr{C} \subseteq \mathcal{A}_0(\mathcal{R}) \subseteq \mathcal{A}_{\omega_1}(\mathcal{R})$ and the sequence $(\mathcal{A}_\alpha(\mathcal{R}))_\alpha$ is increasing. To see conditions (2) and (4), take $f, g \in \mathcal{A}_{\omega_1}(\mathcal{R})$. There are α, β such that $f \in \mathcal{A}_\alpha(\mathcal{R})$ and $g \in \mathcal{A}_\beta(\mathcal{R})$, thus $f, g \in \mathcal{A}_{\max(\alpha,\beta)}(\mathcal{R})$, and $f+g, fg, \max(f,g), \min(f,g) \in \mathcal{A}_{\max(\alpha,\beta)+1}(\mathcal{R})$. To see condition (3), take $(f_n)_n \subseteq \mathcal{A}_{\omega_1}(\mathcal{R})$ with $f_n \rightrightarrows f$. Since there is no countable set of ordinals cofinal with ω_1, there is some $\alpha < \omega_1$ such that $\{f_n\}_n \subseteq \mathcal{A}_\alpha(\mathcal{R})$. Hence $f \in \mathcal{A}_{\alpha+1}(\mathcal{R})$.

Secondly, $\mathcal{A}_{\omega_1}(\mathcal{R})$ is the smallest such $\mathcal{A}$-ring. Take an $\mathcal{A}$-ring $\mathcal{R}' \supseteq \mathcal{R}$ Certainly, $\mathcal{R}' \supseteq \mathcal{A}_1(\mathcal{R})$ by conditions (1) – (4) for $\mathcal{R}'$. Analogically, when $\mathcal{A}_\alpha(\mathcal{R}) \subseteq \mathcal{R}'$, then $\mathcal{A}_{\alpha+1}(\mathcal{R}) \subseteq \mathcal{R}'$. Thus, inductively we can see that $\mathcal{A}_{\omega_1}(\mathcal{R}) \subseteq \mathcal{R}'$. □

4 $\mathcal{A}$-Rings Generated by Identity

By the Stone-Weierstrass Theorem it is clear that $\mathcal{AR}(\{\mathrm{id}_\mathbb{I}\}) = \mathcal{C}(\mathbb{I})$. Let us find the set $\mathcal{AR}(\{\mathrm{id}_\mathbb{R}\})$. We will consider all polynomially bounded continuous functions; i.e.

$$\mathcal{C}_\mathcal{P} = \{f \in \mathcal{C}(\mathbb{R})\colon \exists_{P\in\mathcal{P}(\mathbb{R})} |f| \leq P\}.$$

Proposition 4.1. *$\mathcal{C}_\mathcal{P}$ is an $\mathcal{A}$-ring.*

Proof. Clearly $\mathscr{C}(\mathbb{R}) \subseteq \mathcal{C}_{\mathcal{P}}$. Take f, g such that $|f| < P_1$ and $|g| < P_2$ for $P_1, P_2 \in \mathcal{P}(\mathbb{R})$. Then

$$|f+g| \leq |f| + |g| < P_1 + P_2,$$

$$|fg| = |f||g| < P_1 P_2,$$

$$|\min(f,g)|, |\max(f,g)| \leq \max(|f|,|g|) < P_1 + P_2.$$

Take $f_n \rightrightarrows f$ with $(f_n)_n \subseteq \mathcal{C}_{\mathcal{P}}$. There is n such that $|f_n - f| \leq 1$, and there is $P \in \mathcal{P}$ such that $|f_n| < P$. Thus we have $|f| \leq |f_n| + 1 \leq P + 1$. $\square$

Let X be $\mathbb{R}$ or a compact interval. For $h \colon X \to \mathbb{R}$ and $a, b \in \overline{\mathbb{R}}$ such that $(a,b) \subseteq X$ define $h_{a,b} \colon \mathbb{R} \to \mathbb{R}$ as

$$h_{a,b}(x) = \begin{cases} h(a) & \text{for } x \leq a \\ h(x) & \text{for } x \in (a,b) \\ h(b) & \text{for } x \geq b \end{cases} .$$

Lemma 4.2. *If $h \in \mathcal{AR}(\{\mathrm{id}_X\})$, then $h_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$.*

Proof. It is sufficient to prove that the family $\{h \colon h_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})\}$ is an $\mathcal{A}$-ring containing id_X. Firstly, for $c \in \mathscr{C}(X)$ we have $c_{a,b} \in \mathscr{C}(\mathbb{R}) \subseteq \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$ and $(\mathrm{id}_X)_{a,b} = \max(\min(\mathrm{id}_{\mathbb{R}}, b), a) \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$. Secondly, for $f_{a,b}, g_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$, we have $(f+g)_{a,b} = f_{a,b} + g_{a,b}$, $(fg)_{a,b} = f_{a,b} g_{a,b}$, $(\min(f,g))_{a,b} = \min(f_{a,b}, g_{a,b})$, $(\max(f,g))_{a,b} = \max(f_{a,b}, g_{a,b})$, and all of those functions belong to $\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$. Finally, for $f_n \rightrightarrows f$, where $(f_n)_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$, we have $(f_n)_{a,b} \rightrightarrows f_{a,b}$, thus $f_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$. $\square$

Lemma 4.3. *Assume that $a, b \in \mathbb{R}$, $a < b$. If $h \in \mathcal{C}(\mathbb{R})$, then $h_{a,b} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$.*

Proof. Clearly, $h \restriction [a,b] \in \mathcal{C}([a,b]) = \mathcal{AR}\left(\{\mathrm{id}_{[a,b]}\}\right)$. We get the conclusion by Lemma 4.2, since $h_{a,b} = (h \restriction [a,b])_{a,b}$. $\square$

Proposition 4.4.

$$\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) = \mathcal{C}_{\mathcal{P}}.$$

Proof. By Proposition 4.1 it is sufficient to prove the "$\supseteq$" inclusion.

Suppose that $|f| \leq P$ for $P \in \mathcal{P}$. Define $g(x) = \frac{f(x)}{1+x^2 P(x)}$. Then $g \in \mathcal{C}(\mathbb{R})$. Thus $g_{-m,m} \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$ for any $m \in \mathbb{N}$ by Lemma 4.3. Furthermore, $g_{-m,m} \rightrightarrows g$, since $g(x) \to 0$ for $x \to \infty$ or $x \to -\infty$. Since $f(x) = g(x)(1 + x^2 P(x))$, we have $f \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$. $\square$

Corollary 4.5.

$$\mathcal{AR}(\{\mathrm{id}_{\mathbb{I}}\}) = \mathcal{C}(\mathbb{I})$$

$$\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) = \mathcal{C}_{\mathcal{P}}$$

5 $\mathcal{A}$-Rings Generated by a Darboux Function

Let X be $\mathbb{R}$ or $\mathbb{I}$. Suppose $f \in \mathcal{D}$. Then there are $m_f, M_f \in \overline{\mathbb{R}}$ such that $(m_f, M_f) \subseteq f(X) \subseteq [m_f, M_f]$.

Proposition 5.1. *Suppose that $f \in \mathcal{D}$. Then*

$$\mathcal{AR}(\{f\}) = \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) \circ \{f\}$$

Proof. "$\subseteq$". Let $\mathcal{Q} = \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) \circ \{f\} = \{h \circ f \colon h \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})\}$. Then $\{f\} \subseteq \mathcal{Q}$, since $f = \mathrm{id}_{\mathbb{R}} \circ f \in \mathcal{Q}$.

We shall show that $\mathcal{Q}$ is an $\mathcal{A}$-ring. Since $c \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$ for every $c \in \mathscr{C}$ and $c = c \circ f$, the condition (1) is fulfilled. Since $h_1 \circ f + h_2 \circ f = (h_1 + h_2) \circ f$, $(h_1 \circ f)(h_2 \circ f) = h_1 h_2 \circ f$, $\max(h_1 \circ f, h_2 \circ f) = \max(h_1, h_2) \circ f$, $\min(h_1 \circ f, h_2 \circ f) = \min(h_1, h_2) \circ f$ and $\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$ is an $\mathcal{A}$-ring, the conditions (2) and (4) are also fulfilled.

For (3) take $h_n \circ f \rightrightarrows g$, where $\{h_n\}_n \subseteq \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$. Thus for any $\varepsilon > 0$ there is N such that for $n, m > N$

$$\sup_{x \in X} |h_n \circ f(x) - h_m \circ f(x)| < \varepsilon,$$

and

$$\sup_{y \in f(X)} |h_n(y) - h_m(y)| < \varepsilon,$$

hence

$$\sup_{y \in [m_f, M_f]} |h_n(y) - h_m(y)| \leq \varepsilon,$$

and as a result

$$\sup_{y \in \mathbb{R}} |(h_n)_{m_f, M_f}(y) - (h_m)_{m_f, M_f}(y)| \leq \varepsilon.$$

Consequently there is h such that $(h_n)_{m_f, M_f} \rightrightarrows h$, and $h \in \mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\})$ by Lemma 4.2. Thus for any $\varepsilon > 0$ there is N such that for $n > N$

$$\sup_{y \in \mathbb{R}} |(h_n)_{m_f, M_f}(y) - h(y)| \leq \varepsilon,$$

and

$$\sup_{x \in X} |(h_n)_{m_f, M_f} \circ f(x) - h \circ f(x)| \leq \varepsilon.$$

Finally, $h_n \circ f = (h_n)_{m_f, M_f} \circ f \rightrightarrows h \circ f$, and $g = h \circ f \in \mathcal{Q}$.

"$\supseteq$". Clearly, $\mathrm{id} \circ f = f \in \mathcal{AR}(\{f\})$, $\mathscr{C} \circ f = \mathscr{C} \subseteq \mathcal{AR}(\{f\})$. By Lemma 3.2, $\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) = \mathcal{A}_{\omega_1}(\{\mathrm{id}_{\mathbb{R}}\})$. Thus we can proceed inductively. Assume that $\mathcal{A}_\alpha(\{\mathrm{id}_{\mathbb{R}}\}) \circ \{f\} \subseteq \mathcal{AR}(\{f\})$. We shall prove that $\mathcal{A}_{\alpha+1}(\{\mathrm{id}_{\mathbb{R}}\}) \circ \{f\} \subseteq \mathcal{AR}(\{f\})$. Since operations $+, \cdot, \max, \min$ commutate with operation $\circ$ (as was shown in the previous part of the proof) it is sufficient to discuss only the uniform convergence.

Take $h \in \mathcal{A}_{\alpha+1}(\{\mathrm{id}_{\mathbb{R}}\})$ and $h_n \rightrightarrows h$, where $h_n \in \mathcal{A}_\alpha(\{\mathrm{id}_{\mathbb{R}}\})$. Thus for any $\varepsilon > 0$ there is N such that for $n > N$

$$\sup_{y \in \mathbb{R}} |h_n(y) - h(y)| < \varepsilon,$$

hence

$$\sup_{x \in X} |h_n(f(x)) - h(f(x))| < \varepsilon,$$

thus $h_n \circ f \rightrightarrows h \circ f$. Since $h_n \circ f \in \mathcal{AR}(\{f\})$, $h \circ f \in \mathcal{AR}(\{f\})$. $\square$

Lemma 5.2.

$$\mathcal{AR}(\{f\}) \supseteq \{h \circ \max(y_1, \min(y_2, f)) \colon y_1, y_2 \in \mathbb{R}, h \colon [y_1, y_2] \to \mathbb{R}, h \in \mathcal{C}\}$$

Proof. Clearly $g = \max(y_1, \min(y_2, f)) \in \mathcal{AR}(\{f\})$. By Theorem 7.26 in [15] there is a sequence of polynomials $(P_n)_n$ such that $P_n \restriction [y_1, y_2] \rightrightarrows h$. Thus $\sup_{y \in [y_1, y_2]} |P_n(y) - h(y)| \to 0$, hence $\sup_{x \in X} |P_n(g(x)) - h(g(x))| \to 0$, since $g(X) \subseteq [y_1, y_2]$. Therefore $P_n \circ g \rightrightarrows h \circ g$. Let $P_n(y) = \sum_{i=0}^{k_n} c_{n,i} y^i$ for every $y \in [y_1, y_2]$. Then $P_n \circ g = \sum_{i=0}^{k_n} c_{n,i} g^i$, hence $P_n \circ g \in \mathcal{AR}(\{f\})$, and consequently $h \circ g \in \mathcal{AR}(\{f\})$. $\square$

Lemma 5.3. *Suppose that $f \in \mathcal{B} \cap \mathcal{D}$. Then*

$$\mathcal{AR}(\{f\}) = \{h \circ f \colon h \in \mathcal{C}, h \colon [m_f, M_f] \to \mathbb{R}\}.$$

Proof. Since $f(X) \subseteq [m_f, M_f]$ and $\mathcal{AR}(\{\mathrm{id}_{\mathbb{R}}\}) \subseteq \mathcal{C}$, the proof of the inclusion "$\subseteq$" is trivial by Proposition 5.1. By Lemma 5.2 we get the inclusion "$\supseteq$". $\square$

Theorem 5.4. *Assume that X is $\mathbb{I}$ or $\mathbb{R}$.*

- *If $f \in \mathcal{D}$, then $\mathcal{AR}(\{f\}) = \mathcal{C}_{\mathcal{P}} \circ \{f\}$.*
- *If $f \in \mathcal{D} \cap \mathcal{B}$, then $\mathcal{AR}(\{f\}) = \mathcal{C}(\mathbb{R}) \circ \{f\}$.*

In particular if $f\colon \mathbb{I} \to \mathbb{I}$ *and* $f \in \mathcal{D}$, *then* $\mathcal{AR}(\{f\}) = \mathcal{C}(\mathbb{I}) \circ \{f\}$.

PROOF. The first case follows from Proposition 5.1 with Corollary 4.5, the second one from Lemma 5.3. $\square$

Corollary 5.5. *Assume that* $f \in \mathcal{D}$. *Then* $\mathcal{AR}(\{f\}) \subseteq \mathcal{D}$.

PROOF. It is clear from Theorem 5.4, since the composition of continuous function (actually even a Darboux function) with a Darboux function is a Darboux function. $\square$

6 Continuous Functions in $\mathcal{A}$-Rings

For a Darboux function $f\colon X \to \mathbb{R}$ (where X is $\mathbb{I}$ or $\mathbb{R}$) denote

$$P_f = \{f^{-1}(\{y\})\colon y \in \mathbb{R}\} \setminus \{\emptyset\}.$$

P_f is a partition of X. If $f \in \mathcal{C}$, then every element of P_f is closed.

Proposition 6.1. *Let* P *be a partition of* X *and* $\mathcal{R}$ *be a family of functions. Suppose that* $f \restriction A$ *is constant for every* $f \in \mathcal{R}$ *and every* $A \in P$. *Then* $f \restriction A$ *is constant for every* $f \in \mathcal{AR}(\mathcal{R})$ *and every* $A \in P$.

PROOF. Suppose that $A \in P$ and $f \restriction A$, $g \restriction A$ are constant. Then $(f+g) \restriction A, (fg) \restriction A, \min(f,g) \restriction A, \max(f,g) \restriction A$ are constant. Suppose that $f_n \restriction A$ are constant and $f_n \rightrightarrows f$. Then $f \restriction A$ is constant. By induction using Lemma 3.2 we get the assertion. $\square$

Corollary 6.2.

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} \subseteq \{g \in \mathcal{C}\colon g \restriction A \textit{ is constant for every } A \in P_f\}.$$

For partitions P_1, P_2 we shall write $P_1 \prec P_2$ when for every $A \in P_1$ there is $B \in P_2$ such that $A \subseteq B$. We shall say that a partition is closed if it has only closed elements.

For a partition P and $A \in P$ let

$$\mathrm{cl}_P(A) = \bigcap \{A' \supseteq A\colon A' \in P' \text{ for some closed } P' \succ P\}.$$

and

$$\mathrm{cl}(P) = \{\mathrm{cl}_P(A)\colon A \in P\}.$$

Proposition 6.3. *Let* P *be a partition of* X. *Then* $\mathrm{cl}(P)$ *is the smallest (in the sense of the ordering* $\prec$*) closed partition which is bigger than* P.

Proof. It is clear that the elements of $\mathrm{cl}(P)$ are closed. Suppose $\mathrm{cl}_P(A) \neq \mathrm{cl}_P(B)$ for some $A, B \in P$. Then there are $A', B' \in P'$ for some closed $P' \succ P$ such that $A \subseteq A'$, $B \subseteq B'$, and $A' \neq B'$. Thus $A' \cap B' = \emptyset$, hence $A \cap B = \emptyset$. Consequently, $\mathrm{cl}(P)$ is a closed partition.

Clearly $\mathrm{cl}_P(A) \supseteq A$ for each $A \in P$, thus $P \prec \mathrm{cl}(P)$.

Fix a closed partition $P' \succ P$ and fix $A \in P$. There is $A' \in P'$ such that $A' \supseteq A$. Then $\mathrm{cl}_P(A) \subseteq A'$. Hence $\mathrm{cl}(P) \prec P'$. □

Corollary 6.4.

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} \subseteq \{g \in \mathcal{C} \colon g \restriction A \textit{ is constant for every } A \in \mathrm{cl}(P_f)\}.$$

Definition 6.5. Let P be a partition of X. We say that a family of functions $\mathcal{R}$ separates the elements of the partition P, if for each $A, B \in P$ and $A \neq B$ and for every $x_1 \in A$ and $x_2 \in B$ there is a function $f \in \mathcal{R}$ such that $f(x_1) \neq f(x_2)$.

Lemma 6.6. *Let P be a partition of X. Suppose that $\mathcal{R}$ is an algebra of functions which separates the elements of the partition P and $\mathscr{C} \subseteq \mathcal{R}$. Let $A, B \in P$, $A \neq B$, $x_1 \in A$, $x_2 \in B$, and $c_1, c_2 \in \mathbb{R}$. Then there is a function $f \in \mathcal{R}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. There is a function $g \in \mathcal{R}$ such that $g(x_1) \neq g(x_2)$. Let

$$f(x) = \frac{c_1(g(x) - g(x_2)) - c_2(g(x) - g(x_1))}{g(x_1) - g(x_2)}.$$

Then $f \in \mathcal{R}$, $f(x_1) = c_1$, and $f(x_2) = c_2$. □

Proposition 6.7. *Assume that P is a partition of $\mathbb{I}$ and $\mathcal{R} \subseteq \mathcal{C}(\mathbb{I})$ separates the elements of the partition P. Then*

$$\mathcal{AR}(\mathcal{R}) \supseteq \{f \in \mathcal{C}(\mathbb{I}) \colon f \restriction A \textit{ is constant for each } A \in P\}.$$

Proof. We procede analogically as in Steps 3 and 4 of the proof of Theorem 7.32 from [15].

Let $\mathcal{Q} = \mathcal{AR}(\mathcal{R})$. Let f be continuous such that $f \restriction A$ is constant for each $A \in P$. We shall define a family $\{g_{x,y} \colon x, y \in \mathbb{I}\}$ of functions from $\mathcal{Q}$ in the following way.

Fix $x \in \mathbb{I}$. There is $A_x \in P$ such that $x \in A_x$. $\mathcal{R} \subseteq \mathcal{Q}$, thus $\mathcal{Q}$ is an algebra which separates the elements of the partition P and $\mathscr{C}(\mathbb{I}) \subseteq \mathcal{Q} \subseteq \mathcal{C}(\mathbb{I})$. Let $y \in \mathbb{I}$.

If $y \in \mathbb{I} \setminus A_x$, then by Lemma 6.6 there is a function $g_{x,y} \in \mathcal{Q}$ such that $g_{x,y}(x) = f(x)$ and $g_{x,y}(y) = f(y)$.

If $y \in A_x$, then $f(x) = f(y)$, and we put $g_{x,y}(t) = f(x)$ for each $t \in \mathbb{I}$. Obviously, $g_{x,y} \in \mathcal{Q}$.

Clearly, $g_{x,y} \in \mathcal{C}(\mathbb{I})$ for each $y \in \mathbb{I}$, thus for each $\varepsilon > 0$ there exists an open set $J_{x,y}$ containing y such that $g_{x,y}(t) > f(t) - \varepsilon$ for each $t \in J_{x,y}$. By the compactness of $\mathbb{I}$ there exist $y_1, \ldots, y_n$ such that $\mathbb{I} \subseteq J_{x,y_1} \cup \ldots \cup J_{x,y_n}$. Put $g_x = \max(g_{x,y_1}, \ldots, g_{x,y_n})$. Thus $g_x \in \mathcal{Q}$. It is clear that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \varepsilon$ for each $t \in \mathbb{I}$. Since $g_x \in \mathcal{C}(\mathbb{I})$, there exists an open set I_x containing x such that $g_x(t) < f(t) + \varepsilon$ for each $t \in I_x$. By the compactness of $\mathbb{I}$ there exist $x_1, \ldots, x_m$ such that $\mathbb{I} \subseteq I_{x_1} \cup \ldots \cup I_{x_m}$. Put $g = \min(g_{x_1}, \ldots, g_{x_m})$. Thus $g \in \mathcal{Q}$ and $|g(t) - f(t)| < \varepsilon$ for each $t \in \mathbb{I}$. Finally, $f \in \mathcal{Q}$, because $\mathcal{Q}$ is a uniformly closed algebra. $\square$

Lemma 6.8. *Assume that $f \in \mathcal{D}$. For each $A \in \mathrm{cl}(P_f)$ there are $a, b \in \mathbb{R}$ such that $a \leq b$ and one of the following occurs:*

1. $f(A) = [a, b]$,
2. $f(A) = [a, b)$ *and* $b \notin f(X)$,
3. $f(A) = (a, b]$ *and* $a \notin f(X)$,
4. $f(A) = (a, b)$ *and* $a, b \notin f(X)$.

Proof. Let $A \in \mathrm{cl}(P_f)$. It is enough to prove that $f(A)$ is connected and closed in $f(X)$. Suppose that there is $y \notin f(A)$ such that $f(A) \cap (-\infty, y) \neq \emptyset$ and $f(A) \cap (y, +\infty) \neq \emptyset$. Let $A_1 = f^{-1}(f(A) \cap (-\infty, y))$ and $A_2 = f^{-1}(f(A) \cap (y, +\infty))$. Then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. We shall show that A_1 and A_2 are closed. Take a sequence $x_n \to x$ such that $x_n \in A_1$. Clearly $x \in A$, since A is closed. Suppose $x \in A_2$. Then for each n we have $f(x_n) < y < f(x)$. Thus there is x'_n between x_n and x such that $f(x'_n) = y$. Hence $x'_n \to x$ and $y \notin f(A)$, so there is $A' \in \mathrm{cl}(P_f)$ such that $A' \neq A$ and $x'_n \in A'$. Therefore $x \in A'$, and consequently $x \notin A$, which is a contradiction. Thus $x \in A_1$ and A_1 is closed. Analogically we can prove that A_2 is closed. Therefore $P' = (\mathrm{cl}(P_f) \setminus \{A\}) \cup \{A_1, A_2\}$ is a partition with closed elements and $P' \prec \mathrm{cl}(P_f)$, which is a contradiction. Hence $f(A)$ is connected. Therefore there are $a, b \in \mathbb{R}$ such that $a \leq b$ and $(a, b) \subseteq f(A) \subseteq [a, b]$.

Suppose that $b \in f(X)$ and $b \notin f(A)$. Let $B \in \mathrm{cl}(P_f)$ be such that $b \in f(B)$. Since $f(B)$ is connected, $f(B) \subseteq [b, +\infty)$. Take $x_1 \in A$ and $x_2 \in B$. Without loss of generality we can assume that $x_1 < x_2$. Let $x_3 = \sup A \cap (-\infty, x_2]$. Then $x_3 \in A$ and $x_3 < x_2$. Since $f(x_3) \in f(A)$ and $f(x_2) \in f(B)$, $f(x_3) < b$ and $f(x_2) \geq b$. Let $y = \frac{f(x_3)+b}{2}$. Then $y \in f(A)$ and $f(x_3) < y < f(x_2)$. Hence there is x such that $x_3 < x < x_2$ and $f(x) = y$, and therefore $x \in A$, which is a contradiction. Thus if $b \in f(X)$, then $b \in f(A)$. The proof of the fact that $a \in f(X)$ implies $a \in f(A)$ is analogical. $\square$

Lemma 6.9. *Assume that $f \in \mathcal{D}$. There is a family $\mathcal{R}$ which separates the elements of partition $\mathrm{cl}(P_f)$ such that $\mathcal{R} \subseteq \mathcal{AR}(\{f\}) \cap \mathcal{C}$.*

Proof. Fix $A, B \in \mathrm{cl}(P_f)$, $A \neq B$, $x_1 \in A$ and $x_2 \in B$. Without loss of generality we can assume that $f(x_1) < f(x_2)$. By Lemma 6.8 $f(A) \cap [f(x_1), f(x_2)] = [f(x_1), a_0]$ and $f(B) \cap [f(x_1), f(x_2)] = [b_0, f(x_2)]$ for some a_0, b_0 such that $f(x_1) \leq a_0 < b_0 \leq f(x_2)$. Furthermore for any $C \in \mathrm{cl}(P_f)$ other than A or B if $f(C) \cap (a_0, b_0) \neq \emptyset$, then $f(C) \subseteq (a_0, b_0)$. Let

$$I_f = \bigcup\{f(C) \subseteq (a_0, b_0) \colon C \in \mathrm{cl}(P_f) \wedge |f(C)| > 1\}.$$

Cleary $I_f \subseteq (a_0, b_0)$. We shall consider two cases:

1. I_f is not dense in (a_0, b_0),
2. I_f is dense in (a_0, b_0).

Case 1. There exist y_1, y_2 such that $a_0 < y_1 < y_2 < b_0$ and $y \notin I_f$ for each $y \in (y_1, y_2)$. Fix such y. Certainly $y \in f(X)$, since $f \in \mathcal{D}$, and there is $C \in \mathrm{cl}(P_f)$ such that $y \in f(C)$. Clearly $f(C)$ has exactly one element. Hence $f(C) = \{y\}$. Since $C \in \mathrm{cl}(P_f)$, we have $f^{-1}(\{y\}) = C \in \mathrm{cl}(P_f)$, and therefore $f^{-1}(\{y\})$ is closed for each $y \in (y_1, y_2)$. Put $g_{x_1,x_2} = \max(y_1, \min(y_2, f))$. Then $g_{x_1,x_2} \in \mathcal{AR}(\{f\})$ and $g_{x_1,x_2}(x_1) = y_1 \neq y_2 = g_{x_1,x_2}(x_2)$.

Put $g = g_{x_1,x_2}$. We shall show that $g \in \mathcal{C}$. Suppose there is a sequence $(t_n)_n \subseteq X$ such that $t_n \to t$ and $g(t_n) \nrightarrow g(t)$. There is $\varepsilon > 0$ and a subsequence $(t_{k_n})_n$ such that $g(t_{k_n}) > g(t) + \varepsilon$ or $g(t_{k_n}) < g(t) - \varepsilon$. Without loss of generality we can assume the former. For each n there is u_n between t_{k_n} and t such that $g(u_n) = g(t) + \frac{\varepsilon}{2}$. Thus $u_n \to t$. Moreover, $g^{-1}(\{g(t) + \frac{\varepsilon}{2}\}) = f^{-1}(\{g(t) + \frac{\varepsilon}{2}\})$ is closed, since $g(t) + \frac{\varepsilon}{2} \in (y_1, y_2)$. Consequently, $g(t) = g(t) + \frac{\varepsilon}{2}$, which is a contradiction. Thus $g \in \mathcal{C}$.

Case 2. By Lemma 6.8 and definition I_f is a sum of disjoint closed nondegenerate intervals. By the density of I_f, number of those intervals is infinite. By the separability of $\mathbb{R}$ number of those intervals is countable. Let enumerate them as $([a_n, b_n])_n$. We shall inductively enumerate that family of intervals by finite sequences of zeros and ones: $\{[c_s, d_s] \colon s \in \{0,1\}^{<\omega}\}$. Let $[c_\emptyset, d_\emptyset] = [a_1, b_1]$. Suppose that intervals $[c_s, d_s]$ for $s \in \{0,1\}^{<n}$ are given. Let

$$c'_s = \max\left(\left\{d_r \colon r \in \{0,1\}^{<n} \wedge d_r < c_s\right\} \cup \{a_0\}\right),$$

and let $[c_{s^\frown 0}, d_{s^\frown 0}]$ be the first interval from the sequence $([a_n, b_n])_n$ laying between c'_s and c_s. Let

$$d'_s = \min\left(\left\{c_r\colon r \in \{0,1\}^{<n} \cup \left(\{0,1\}^{<n} \times \{0\}\right) \wedge c_r > d_s\right\} \cup \{b_0\}\right),$$

and let $[c_{s^\frown 1}, d_{s^\frown 1}]$ be the first interval from the sequence $([a_n, b_n])_n$ laying between d_s and d'_s. It is clear that $\{[a_n, b_n]\colon n \in \mathbb{N}\} = \{[c_s, d_s]\colon s \in \{0,1\}^{<\omega}\}$. We shall define a sequence of functions $(h_n\colon [a_0, b_0] \to [a_0, b_0])_n$. Let

$$y_s = \frac{1}{2^{\mathrm{len}(s)+1}} + \sum_{k=1}^{\mathrm{len}(s)} s(k) \cdot \frac{1}{2^k}$$

and put $h_n(x) = y_s$ for $x \in [c_s, d_s]$ for $s \in \{0,1\}^{<n}$, $h_n(a_0) = 0$, $h_n(b_0) = 1$, and let h_n be linear between neighboring intervals $[c_s, d_s]$, between point a_0 and its neighboring interval, and between point b_0 and its neighboring interval.

It is clear that each h_n is non-decreasing and continuous. We shall show that $(h_n)_n$ is uniformly convergent. Fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\frac{1}{2^N} < \varepsilon$. For each $n, m > N$ $h_n \restriction [c_s, d_s] = h_m \restriction [c_s, d_s]$ for each $s \in \{0,1\}^{<N}$. Hence $\sup_{x\in\mathbb{I}} |h_n(x) - h_m(x)| \le \frac{1}{2^N} < \varepsilon$. Thus there is $h \in \mathcal{C}$ such that $h_n \rightrightarrows h$. It is clear that $h(x) = y_s$ if and only if $x \in [c_s, d_s]$. Put $g_{x_1,x_2} = h \circ \max(a_0, \min(b_0, f))$. By Lemma 5.2 $g_{x_1,x_2} \in \mathcal{AR}(\{f\})$. Moreover, $g_{x_1,x_2}(x_1) = h(a_0) = 0 \neq 1 = h(b_0) = g_{x_1,x_2}(x_2)$.

Put $g = g_{x_1,x_2}$. Clearly $g \in \mathcal{D}$ by Corollary 5.5. We shall prove that $g \in \mathcal{C}$. As in Case 1 suppose there is a sequence $(t_n)_n \subseteq \mathbb{I}$ such that $t_n \to t$ and $g(t_n) \not\to g(t)$. There is $\varepsilon > 0$ and a subsequence $(t_{k_n})_n$ such that $g(t_{k_n}) > g(t) + \varepsilon$ or $g(t_{k_n}) < g(t) - \varepsilon$. Without loss of generality we can assume the former. There is $s \in \{0,1\}^{<\omega}$ such that $y_s \in (g(t), g(t)+\varepsilon)$. For each n there is u_n between t_{k_n} and t such that $g(u_n) = y_s$. Thus $u_n \to t$ and $f(u_n) \in [c_s, d_s]$. There is $A_s \in \mathrm{cl}(P_f)$ such that $f(A_s) = [c_s, d_s]$. Consequently, $u_n \in A_s$ and by the closeness of A_s we have that $t \in A_s$. Thus $g(t) = y_s > g(t)$, which is a contradiction. Hence $g \in \mathcal{C}$.

Let $\mathcal{R} = \{g_{x_1,x_2}\colon x_1 \in A, x_2 \in B \text{ for some } A, B \in \mathrm{cl}(P_f), A \neq B\}$. Then $\mathcal{R} \subseteq \mathcal{AR}(\{f\}) \cap \mathcal{C}$ and $\mathcal{R}$ separates the elements of the partition $\mathrm{cl}(P_f)$. $\square$

Theorem 6.10. *Assume that $X = \mathbb{I}$ and $f \in \mathcal{D}$. Then*

$$\mathcal{AR}(\{f\}) \cap \mathcal{C}(\mathbb{I}) = \{g \in \mathcal{C}(\mathbb{I})\colon g \restriction A \textit{ is constant for every } A \in \mathrm{cl}(P_f)\}.$$

Proof. By Corollary 6.4 we only have to prove the inclusion "$\supseteq$". By Lemma 6.9 there is a family $\mathcal{R}$ which separates the elements of $\mathrm{cl}(P_f)$ such that $\mathcal{R} \subseteq \mathcal{AR}(\{f\}) \cap \mathcal{C}(\mathbb{I})$. By Proposition 6.7

$$\mathcal{AR}(\mathcal{R}) \supseteq \{g \in \mathcal{C}(\mathbb{I})\colon g \restriction A \text{ is constant for each } A \in \mathrm{cl}(P_f)\}.$$

Furthermore, $\mathcal{AR}(\mathcal{R}) \subseteq \mathcal{AR}(\{f\}) \cap \mathcal{C}(\mathbb{I})$. $\square$

7 Final Results

Assume that X is $\mathbb{R}$ or $\mathbb{I}$ and $f\colon X \to \mathbb{R}$.

Lemma 7.1. *Assume that $f \in \mathcal{D}$. Then $f(U)$ is open in $f(X)$ for every open U such that for every $A \in \mathrm{cl}(P_f)$, either $A \cap U = \emptyset$, or $A \subseteq U$.*

Proof. When $f \in \mathscr{C}$, it is trivial, hence we can assume that f is not constant.

Take U fulfilling the assumption. Since U is open, it is an at most countable sum of pairwise disjoint open intervals U_n for $n \in \mathbb{N}$. Thus $f(U_n)$ are intervals. We shall prove that none of them can be degenerate.

On the contrary, assume that $f(U_n)$ has only one element for some n. Then f is constant on U_n, and there is $B \in P_f$ such that $U_n \subseteq B$. Since $P_f \prec \mathrm{cl}(P_f)$, there exists $A \in \mathrm{cl}(P_f)$ such that $B \subseteq A$. Thus $U_n \subseteq A$, hence $\mathrm{cl}(U_n) \subseteq A$, since A is closed. Consequently, $A \cap U \neq \emptyset$, since $U_n \subseteq U$. Then by the assumption we reason that $A \subseteq U$, thus $\mathrm{cl}(U_n) \subseteq U$. Since the sets U_n are pairwise disjoint, we conclude that $U_n = X$. But then f is a constant function, which contradicts the assumption from the first paragraph of the proof.

Suppose that $f(U)$ is not open in $f(X)$. There exists $y_0 \in f(U)$ which is not an internal element of $f(U)$ in $f(X)$, and it cannot be an internal element of any $f(U_n)$ (with respect to $f(X)$). Thus it can be only an endpoint of some of intervals $f(U_n)$ (and not an endpoint of $f(X)$), and always an upper endpoint or always a lower endpoint. Without loss of generality assume that it is an upper endpoint. Take any x_0 such that $f(x_0) = y_0$. There is some interval $U_n \ni x_0$. Take any $y_1 > y_0$ such that $y_1 \in f(X) \setminus f(U)$ and x_1 such that $f(x_1) = y_1$, without loss of generality we can assume that $x_0 < x_1$. Let $s = \inf\{x > x_0 \colon f(x) > y_0\}$. Since $\sup f(U_n) = y_0$ and $x_0 \in U_n$, $s > x_0$.

Suppose that $f(s) > y_0$. Then there should exist x such that $x_0 < x < s$ and $f(x) = \frac{y_0+f(s)}{2} > y_0$, a contradiction with the definition of s.

Suppose that $f(s) = y_0$. Then $s \in U$. Let $U_m \ni s$. But $y_0 = \sup f(U_m)$, thus $f(x) \leq y_0$ for every $x \in U_m$, a contradiction with the definition of s.

Suppose that $f(s) < y_0$. By the definition of s there should be an infinite descending sequence $(s_n)_n$ such that $s_n \to s$ and $f(s_n) > y_0$. Thus there also should be a descending sequence $(x_n)_n$ such that $x_n \to s$ and $f(x_n) = y_0$. Thus $\mathrm{cl}\left(\{x_n\}_n\right) \subseteq U$, and $s \in U$, and we conclude with a contradiction as in the previous case.

Consequently, $f(U)$ is open. $\square$

Proposition 7.2. *For any $f \in \mathcal{D}(X)$ and any $g \in \mathcal{C}(X)$ such that $\mathrm{cl}(P_f) \prec P_g$, there exists $h \in \mathcal{C}(f(X))$ such that $g = h \circ f$.*

Proof. For each $y \in f(X)$ define $h(y) = g(x)$, where $x \in f^{-1}(\{y\})$. Function h is properly defined on $f(X)$, since by assumption $\mathrm{cl}(P_f) \prec P_g$ there is only one element of the set $g(f^{-1}(\{y\})) = \{g(x) \colon x \in f^{-1}(\{y\})\}$.

Let U be an open set. Then $h^{-1}(U) = f(g^{-1}(U))$, since $\mathrm{cl}(P_f) \prec P_g$. Moreover, $g^{-1}(U)$ is open. By the assumption for every $A \in \mathrm{cl}(P_f)$, $A \subseteq B$ for some $B \in P_g$, thus either $A \cap g^{-1}(U) = \emptyset$, or $A \subseteq g^{-1}(U)$. By Lemma 7.1 $f(g^{-1}(U))$ is open in $f(X)$. □

Theorem 7.3. *Assume that $f \in \mathcal{D}$.*

- *If $X = \mathbb{I}$, then*

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} = \{g \in \mathcal{C} \colon \mathrm{cl}(P_f) \prec P_g\}.$$

- *If $X = \mathbb{R}$ and $f(X)$ is closed, then*

$$\mathcal{AR}(\{f\}) \cap \mathcal{C} = \{g \in \mathcal{C} \colon \mathrm{cl}(P_f) \prec P_g, \exists_{P \in \mathcal{P}} |g| \le P \circ f\}.$$

Proof. The first case follows from Theorem 6.10.

For the second case, by Theorem 5.4 it is sufficient to prove that

$$(\mathcal{C}_{\mathcal{P}} \circ \{f\}) \cap \mathcal{C} = \{g \in \mathcal{C} \colon \mathrm{cl}(P_f) \prec P_g, \exists_{P \in \mathcal{P}} |g| \le P \circ f\}.$$

Take g belonging to the set on the left side. Then $g = h \circ f$, where $h \in \mathcal{C}_{\mathcal{P}}$. It is clear that $P_f \prec P_{h \circ f}$, and since $g \in \mathcal{C}$, also $\mathrm{cl}(P_f) \prec P_{h \circ f}$. There is $P \in \mathcal{P}$ such that $|h| \le P$. Hence $|g(x)| = |h(f(x))| \le P(f(x))$.

Take g belonging to the set on the right side. By Proposition 7.2 there exists $h \in \mathcal{C}(f(X))$ such that $g = h \circ f$. Clearly $f(X) = [a, b]$ for some $a, b \in \overline{\mathbb{R}}$. Then $g = h_{a,b} \circ f$ and $|h_{a,b}(f(x))| \le P(f(x))$ for any $x \in X$. Thus $|h_{a,b}(y)| \le P(y)$ for any $y \in f(X)$, and $|h_{a,b}(y)| \le P(y) + |h(a)| + |h(b)|$ for any $y \in \mathbb{R}$ (for finite a, b; for infinite — we can skip $h(a)$ and $h(b)$, respectively, in the inequality). Hence $h_{a,b} \in \mathcal{C}_{\mathcal{P}}$. □

Problem 2. Let $f \colon \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{D}$ and $f(\mathbb{R})$ is not closed (i.e. $f(\mathbb{R})$ is one of (a, b), $(a, b]$, $[a, b)$ for some $a, b \in \overline{\mathbb{R}}$). Characterize continuous functions belonging to each $\mathcal{A}$-ring containing $f \colon \mathbb{R} \to \mathbb{R}$, i.e. find $\mathcal{AR}(\{f\}) \cap \mathcal{C}$.

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