Nancy, August 29, 2019

Report on the Phd thesis manuscript of Adam Dzedzej untitled:

« Multi-state two-dimensional number-conserving cellular automata »

To whom it may concern,

In this work, the author tackles the delicate problem to explore the properties of number conservation in cellular automata. The question of knowing under which conditions a cellular automaton rule can produce a global behaviour with a conserved quantity is a fundamental one, which has been explored since the 1970's (See Ref. [1] in the manuscript). It is of high importance because it is related to the question of understanding systems with interacting particles with a simple parallel discrete model where the components, here named the cells, are only allowed to change their own state according to the state of their neighbours. This study is also interesting *per se* as a way to develop new techniques to analyse the dynamics of discrete complex systems and the possibility to have emergent properties where the global behaviour is somehow "more" than the sum of the local reactions of their parts.

It is well known in this field that the complete exploration large spaces of rules (or behaviour) is a highly challenging task, most remarkably because the nontrivial properties of cellular automata are all subject to undecidability results (Rice-like theorems). This is why a great amount of research has been devoted to simple rules, generally with binary states and one-dimensional settings ; here, the author endeavours to go much beyond such limitations, tackling the two-dimensional case with more states.

The document, which is divided into eight chapters, is pleasant to read and have obviously been written with the greatest care. Let us briefly summarize its contents.

Chapter 1 introduces the problems and examines the main bibliographical references of the field. It first presents general questions and then focuses on the objectives of the thesis. The author has made real efforts to situate his own research in the more general landscape of cellular automata.

Chapter 2 gives a sound mathematical formalism to deal with the questions of the thesis. The basic definitions are given and connections are made with the existing literature. Chapter 3 presents the core of the work, namely the decomposition of a local cellular automaton rule with von Neumann neighbourhood into monomers and dimers, and the use of flow functions. It then gives useful theorems to characterize the forms of the local rules depending on what symmetry constraints they fulfil. It is interesting to notice that this gives us negative results as well as a means to considerably reduce the space of rules to explore. These first results involve original techniques, whose implications will be detailed in the next chapters.

In Chapter 4, the first systematic exploration of a rule space is made. This naturally begins with ternary rules: the full list of rules is obtained as a logical, but not trivial, consequence of the equations previously established. The author carefully distinguishes the examples that are "intrinsically" two-dimensional and analyses the dynamics of one such rule in detail. The use of flow functions gives a complementary information to analyse the results. The next chapter rises the difficulty by one step, tackling the four-state rules. Here, any direct exploration of the space of rules is strictly impossible and mathematical intermediate steps are needed. The author succeeds by making an ingenious use of the split-and-perturb method, introduced by Wolnik et al. (see Ref. [88] in the manuscript), in order to propose a theorem that describes the form of four-state number-conserving rules and thus allows the complete enumeration of such rules.

In Chapter 6, the question of number-conservation is crossed with another fundamental question in cellular automata: invertibility. The question as to know how to identify the invertible (or "reversible") rules, for which no loss of information occurs in their evolution, is a well-known question, which is known to be undecidable in all its generality. The authors shows that there are no three-state invertible number-conserving rule and explores the case of four states. An interesting representation is given, where states are represented on two layers, which allows one to get a more "semantical" view of the processes at play. Given its high importance, one may simply regret that this chapter has not been more developed.

The exploration of rule spaces ends at Chapter 7 with the analysis of the rules which hold the rotation symmetry. Here again, the author shows all his *maestria* and takes a full advantage of the tools and techniques he introduced in the previous chapters. Results are given by progressively increasing the number of states at play, and, surprisingly enough, the author succeeds in the exploration of this space of rules even for seven states, which to our knowledge has rarely been attained previously.

The manuscript ends with a brief concluding chapter where the steps taken are summarized. The author proposes conjectures and questions which can serve as starting point for future work.

To conclude, my appreciation of the PhD manuscript of Adam Dzedzej is that is a solid, serious and well-written contribution to the field of cellular automata and discrete complex systems. The text shows real efforts to present the methods and results in a pedagogical manner. The challenges which are tackled are completely in tune with the important questions currently studied in this field of research and the techniques presented are original and very likely to give rise to further developments. The quality of this research is furthermore attested by publications in recognised journals (*Journal of Physics A: Mathematical and Theoretical*, *Journal of cellular automata*, *Journal of Statistical Mechanics: Theory and Experiment)* and in high-level international conferences. **I thus strongly recommend to accept Adam Dzedzej as a Doctor of Philosophy of the University of Gdańsk and I support the attribution of honours to him without any reserve.**

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