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## Letter of reference for Dr. Jacek Gulgowski

Jacek Gulgowski (University of Gdańsk, Department of Mathematics) is applying for being conferred with the degree of habilitation doctor (dr. hab.) in mathematical sciences. Being one of the members of the Habilitation Board, I was asked to provide a review on Gulgowski's scientific achievements which I am glad to supply in what follows.

First of all, let me point out that Gulgowski and I are not affiliated in the same University, have never strived for similar positions, have never applied for the same grant, and do not have joint publications. So there is no conflict of interest. On the other hand, I am quite familiar with Gulgowski's mathematical work, which includes, but is not restricted to, functions of bounded variation (in the classical or non-classical setting) and their applications to nonlinear analysis, integral equations, and boundary value problems. In what follows, I will give an overall assessment of his research achievements, his publications, and some other secondary features.

**1. Research achievements.** Jacek Gulgowski graduated from the Technical University in Gdańsk, where he obtained a Master degree in Informatics, and the University in Gdańsk, where he obtained a Master degree in Mathematics. He also received his PhD at the same Department in 2002. Subsequently, he held a position as Assistant Lecturer and, starting from 2002, as Assistant Professor at the Institute of Mathematics of the University of Gdańsk.

His research interests cover several fields: functional analysis and operator theory, functions of (classical or generalized) bounded variation, nonlinear operators, fixed point and bifurcation theory, integral equations and operators, boundary value problems, and many more application-oriented fields like approximation theory and algorithms. In what follows I will comment on those topics from this variety of interests[1] which I consider most remarkable, and which are related to the 6 articles presented for his cumulative habilitation. This is the main part of my review.

**1.1. Functions of bounded variation.** After Jordan's pioneering paper[2] [J], several notions of bounded variation in a more general sense have found many applications, the most important ones being *Wiener variation* [Wi], *Riesz variation* [Ri], *Young variation* [Y], and *Waterman variation* [Wt,Wt1,Wt2]. For instance, the Riesz variation is intimately related to Sobolev spaces [Ri1], while passing from the Wiener space $BV_p$ to the Young space $BV_\phi$ is done in the same way as passing from the Lebesgue space $L_p$ to the Orlicz space $L_\phi$. It is also interesting to note that the study of the Waterman space $\Lambda BV$ leads to "sharp" estimates for Fourier series and integrals [Wt3]: in case of the Waterman sequence $\Lambda = \Lambda_q = (n^q)_n$ for $0 < q < 1$, the Fourier series of a function $f \in \Lambda_q BV$ is $(C,\beta)$-bounded for $\beta = q-1$, and $(C,\alpha)$-summable for $\alpha > q-1$, and these values for $\alpha$ and $\beta$ are sharp.

There is another concept of bounded variation introduced by Terekhin [Te] in the 70th and called *q-integral p-variation.* The corresponding space $IBV_p^q$ has not found much attention in the literature, but seems to have important applications to approximation theory. Interestingly, in his paper [G] Gulgowski studies, apparently for the first time, this type of variation in connection with linear integral operators and nonlinear superposition operators, see below.

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[1] In the summary of professional accomplishments, the author choses a different partition of his results.
[2] The letters in square brackets refer to the list of references attached at the end of this review (pp. 8/9).

**1.2. Linear operator theory.** In the papers [BiCGS,BiGK,BiGK1,G] the author considers the linear integral operator

$$Kx(t) = \int_0^1 k(t,s)x(s)\,ds \qquad (0 \le t \le 1), \tag{1}$$

as well as its Volterra-type analogue

$$\tilde{K}x(t) = \int_0^t k(t,s)x(s)\,ds \qquad (0 \le t \le 1) \tag{2}$$

in various spaces of functions of bounded variation. To describe these results in some detail, let us assume throughout[3] that the function $k(t,\cdot) : [0,1] \to \mathbb{R}$ is (Lebesgue) measurable and integrable; otherwise the right-hand sides of (1) and (2) do not make sense. In addition, let us collect the following conditions for further reference.[4]

$$\exists m \in L_1\, \forall s \in [0,1] : \ var(k(\cdot,s)) \le m(s). \tag{C1}$$

$$\exists m \in L_1\, \forall s \in [0,1] : \ var_\Lambda(k(\cdot,s)) \le m(s). \tag{C2}$$

$$\exists M > 0 : \ \sup_\xi var\left(\int_0^\xi k(\cdot,s)\,ds\right) \le M. \tag{C3}$$

$$\exists M > 0 : \ \sup_\xi var_p\left(\int_0^\xi k(\cdot,s)\,ds\right) \le M. \tag{C4}$$

$$\exists M > 0 : \ \sup_\xi var_\Lambda\left(\int_0^\xi k(\cdot,s)\,ds\right) \le M. \tag{C5}$$

$$\exists m_0 \in L_{q/(q-1)} : \ \|k(\cdot,s)\| \le m_0(s). \tag{C6}$$

$$\exists m_1 \in L_{q/(q-1)} : \ ivar_1^q(k(\cdot,s)) \le m_1(s). \tag{C7}$$

Here $var(k(\cdot,s))$ denotes the Jordan variation, $var_p(k(\cdot,s))$ the Wiener $p$-variation, $var_\Lambda(k(\cdot,s))$ the Waterman $\Lambda$-variation, and $ivar_p^q(k(\cdot,s))$ the Terekhin $q$-integral $p$-variation of the map $t \mapsto k(t,s)$. Clearly, (C1) implies (C3), and (C2) implies (C5) (with $M := \|m\|_{L_1}$). The converse is not true, as the kernel function $k(t,s) := t/(t^2+s^2)$ with $k(0,0) := 0$ shows.

In the above mentioned papers, the author obtains, in part with coauthors, the following results for the integral operator (1):

**Theorem 1** [BiGK]. *Under condition* (C1), *the operator* (1) *maps* $BV_p$ *into* $BV$ *and is compact.*

**Theorem 2** [BiCGS]. *Under condition* (C2), *the operator* (1) *maps* $\Lambda BV$ *into itself and is compact.*

**Theorem 3** [BiGK1]. *Under condition* (C3), *the operator* (1) *maps* $BV$ *into itslf and is bounded.*

**Theorem 4** [BiGK1]. *Under condition* (C4), *the operator* (1) *maps* $BV$ *into* $BV_p$ *and is bounded.*

**Theorem 5** [BiCGS]. *Under condition* (C5), *the operator* (1) *maps* $BV$ *into* $\Lambda BV$ *and is bounded.*

**Theorem 6** [G]. *Under conditions* (C6) *and* (C7), *the operator* (1) *maps* $L_q$ *into* $IBV_1^q$ *and is bounded.*

[3] Sometimes a weaker assumption suffices, e.g., the integrability of the function $k(0,\cdot)$.

[4] In condition (C4), $p \in (1,\infty)$ is fixed; in conditions (C6) and (C7), $q \in (1,\infty)$ is fixed.

Although the conditions (C6) and (C7) look somewhat artificial, it is interesting to note that they are satisfied by the fractional Riemann-Liouville operator

(3)
$$K_\tau x(t) := \frac{1}{\Gamma(\tau)} \int_0^t \frac{x(s)}{(t-s)^{1-\tau}}\, ds$$

in case $1 - 1/q < \tau < 1$. This is a consequence of the fact that the function $u_\alpha(t) := t^{-\alpha}$ belongs, for $q > 1$ and $0 < \alpha < 1/q$, to the space $IBV_1^q$, but, being unbounded, of course to none of the other spaces of functions of bounded variation.

More precisely, Gulgowski shows in [G] that condition (C6) holds with

$$m_0(s) := \frac{1}{\Gamma(\tau)} \frac{(1-s)^{1/q-1+\tau}}{(1-(1-\tau))q^{1/q}}$$

and condition (C7) holds with $m_1(s)$ involving $ivar_1^q(u_{1-\tau})$. As a consequence, the weakly singular operator (3) does not only map $L_q$ into itself (which is a classical result), but also $L_q$ into $IBV_1^q$, by Theorem 6. I consider this one of the strongest results in Gulgowski's mathematical work.

**1.3. Nonlinear operator theory.** Given a function[5] $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, the *superposition operator* (also called *Nemytskij operator* in the Russian literature) $S_f$ generated by $f$ is given by

(4)
$$S_f x(t) = f(t, x(t)) \qquad (0 \leq t \leq 1).$$

In the special case $f = f(u)$, the corresponding operator

(5)
$$C_f x(t) = f(x(t))$$

is often called *composition operator.*[6]

The operators (4) and (5) occur virtually in every field of nonlinear analysis, but in spite of their simple form they exhibit many surprising features. The problem of characterizing all functions $f : \mathbb{R} \to \mathbb{R}$ such that $C_f x \in BV$ for all $x \in BV$ was solved by Josephy [Jo] in 1981: local Lipschitz continuity of $f$ is a necessary and sufficient condition. Moreover, in this case the operator $C_f$ is automatically bounded.[7]

On the other hand, the corresponding problem in the nonautomous case $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ was open for many years. In the Russian paper [L] the author claims to give a sufficient condition on $f$ for $S_f(BV) \subseteq BV$ in this case; unfortunately, this is false. Bugajewska in her paper [Ba] was apparently the first mathematician in the nonlinear community who pointed out this fact, and a specific counterexample was given subsequently by Maćkowiak [Mc] a few years later.

It turns out that the following conditions[8] are useful for the solution of this problem:

(C8)
$$\begin{cases} \forall r > 0\, \exists M_r > 0\, \forall \{t_0, \ldots, t_m\} \in \mathcal{P}[0,1]\, \forall u_0, \ldots, u_m \in [-r, r] : \sum\limits_{j=1}^m |u_j - u_{j-1}| \leq r \\ \Rightarrow \sum\limits_{j=1}^m |f(t_j, u_j) - f(t_{j-1}, u_j)| \leq M_r \text{ and } \sum\limits_{j=1}^m |f(t_{j-1}, u_j) - f(t_{j-1}, u_{j-1})| \leq M_r. \end{cases}$$

(C9)
$$f \in AC(I) \text{ and } f' \in L_{p/(p-1)}(I) \quad (I \subset \mathbb{R} \text{ compact interval}).$$

[5] The domain of definition $[0,1]$ of the first argument of $f$ is chosen without loss of generality; passing from $[0,1]$ to another interval or even a higher-dimensional domain in most cases is straightforward.

[6] Gulgowski calls (4) *nonautonomous superposition operator* and (5) *autonomous superposition operator* and denotes both by $F$.

[7] Recall that boundedness and continuity are, in contrast to linear operators, independent properties for a nonlinear operator.

[8] Condition (C9) means that $f$ belongs to the space $RBV_{p/(p-1)}(I)$ of bounded Riesz variation which the author does not take into account. This space is closely related to Sobolev spaces

**Theorem 7** [BaBiKMc]. *The superposition operator* (4) *maps $BV$ into itself and is bounded if and only if condition* (C8) *holds.*

Let us remark that the boundedness requirement is essential in Theorem 7. In fact, in [BaBiKMc] the authors give an example of an operator (4) which maps $BV$ into itself, but is not bounded. So the superposition operator $S_f$ behaves quite differently than the composition operator $C_f$ in the space $BV$, where one gets "automatic boundedness".

To decide whether or not the operator $C_f$ is also continuous in the norm of $BV$ if $C_f(BV) \subseteq BV$ was a surprisingly difficult open problem for more than 50 years. Some sufficient conditions have been given in the literature. Thus, in [BaBiKMc,BiGK1] it is shown that $C_f$ is continuous if $f$ is a sum of power series centered at 0 with infinite radius of convergence. In the same spirit Bugajewski and Gulgowski have proved the following

**Theorem 8** [BiGK]. *The composition operator* (5) *maps $BV$ into itself and is continuous if $f$ is continuously differentiable.*

The problem of deciding whether or not the operator $C_f$ is always continuous whenever it maps $BV$ into itself is now positively solved:

**Theorem 9.** *If $f \in Lip_{loc}(\mathbb{R})$ the operator* (5) *is automatically continuous in $BV$.*

This result shows that the requirement $f \in C^1(\mathbb{R})$ in Theorem 8 is too strong. We mention that Theorem 9 has an interesting history. In the paper [Mo] from 1937, Morse claims to prove that the local Lipschitz continuity of $f$ guarantees the continuity of $C_f$ in the $BV$ norm. However, the proof of this claim is about 30 pages long, it uses cryptic tools from other fields, and its correctness is at least doubtful. In the recent paper [Mc1], the author gives a more straightforward and elegant proof of this fact. An even shorter (and almost elementary) proof of Theorem 9 was given quite recently by Reinwand in [Rw].

We do not go into details; instead, we mention another result by Bugajewski, Gulgowski, and Kasprzak which gives a sufficient condition for the continuity of the composition between $BV$ and the (larger) Wiener space $BV_p$.

**Theorem 10** [BiGK]. *The composition operator* (5) *maps $BV$ into $BV_p$ and is continuous if condition* (C9) *holds.*

Concerning more general notions of variation, several other conditions for the inclusions $C_f(BV_p) \subseteq BV_p$, $C_f(BV_\phi) \subseteq BV_\phi$, and $C_f(\Lambda BV) \subseteq \Lambda BV$ may be found in the monograph [ABM]. Typically, local Lipschitz continuity of $f$ in the autonomous case is sufficient here, and often close to being necessary. Moreover, boundedness of $C_f$ in these spaces is often "for free", as in the space $BV$.

The continuity problem for $C_f$ (let alone for $S_f$) is again quite subtle. To conclude we briefly give two results in this spirit by Gulgowski and coauthors for the Waterman space $\Lambda BV$. The first result is parallel to Theorem 8, the scond result shows that one gets continuity of $C_f$ "for free" if one enlarges the target space.

**Theorem 11** [BiCGS]. *The composition operator* (5) *maps $\Lambda BV$ into itself and is continuous if $f$ is continuously differentiable.*

**Theorem 12** [BiCGS]. *Suppose that $f$ is continuous and $\Lambda = (\lambda_n)_n$ is some Waterman sequence. Then there exists another Waterman sequence $\Gamma = (\gamma_n)_n$ such that the composition operator* (5) *maps $\Lambda BV$ into $\Gamma BV$ and is continuous.*

We point out that in general one cannot choose $\Gamma = \Lambda$ in Theorem 12. Even worse, it is shown in [BiCGS] that, if[9] $\gamma_n = o(\lambda_n)$ as $n \to \infty$, and $C_f$ maps $\Lambda BV$ into $\Gamma BV$, then $f$ must be constant.

Analogous results have been given by Gulgowski in [G,G2] also for more general spaces, e.g., those of Young or Terekhin type. We do not go into details, since the corresponding hypotheses are very technical. We just confine ourselves to the following surprising result which holds for Borel functions $f : \mathbb{R} \to \mathbb{R}$:

[9]This condition implies that $\Gamma BV$ is a strict subspace of $\Lambda BV$. So requiring composition operators to map a large Waterman space into a smaller Waterman space leads to an extremely strong degeneracy.

**Theorem 13** [BiGK]. *The composition operator* (5) *maps* $IBV_1^q$ *into itself if and only if* $f \in Lip(\mathbb{R})$.

The "if" part of the proof of this theorem is trivial. The "only if" part is highly nontrivial; it is proved by showing first that $C_f(IBV_1^q) \subseteq IBV_1^q$ implies that $f$ is locally bounded, then locally Lipschitz, and then globally Lipschitz. Theorem 13 is surprising inasmuch *global* Lipschitz continuity is not needed for acting conditions in all other spaces, but *local* Lipschitz continuity suffices. Thus, the map $f(u) = u^2$ generates a composition operator $C_f$ which maps any of the spaces $BV$, $BV_p$, $BV_\phi$, or $\Lambda BV$ into itself, but not $IBV_p^q$.

**1.4. Hammerstein integral equations.** Although Gulgowski's results on integral, composition, and superposition operators have a strong interest on their own, they have been always obtained with applications in mind. A central application is concerned with the *nonlinear Hammerstein equation*

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(s, x(s))\, ds \qquad (0 \le t \le 1), \tag{6}$$

with $\lambda \in \mathbb{R}$, where $g : [0,1] \to \mathbb{R}$, $k : [0,1] \times [0,1] \to \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given functions, and the function $x : [0,1] \to \mathbb{R}$ is unknown. The integral equation (6) may be rewritten equivalently as operator equation

$$x = g + \lambda K S_f x, \tag{7}$$

where $S_f$ is the superposition operator (4) and $K$ the (linear) integral operator (1). In the autonomous case $f = f(u)$ equation (6) takes the simpler form

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(x(s))\, ds \qquad (0 \le t \le 1), \tag{8}$$

or, equivalently as operator equation,

$$x = g + \lambda K C_f x, \tag{9}$$

where $C_f$ is the composition operator (5). Of course, the form of equations (7) and (9) suggests to apply fixed point theory. The basic idea is plane: combine conditions on the nonlinearity $f$ found in the preceding section with suitable conditions on the kernel function $k$ in order to apply the Banach-Caccioppoli fixed point principle for contractions, the Schauder fixed point principle for compact operators, or the Darbo-Sadovskij fixed point principle for condensing operators to the maps $KS_f$ and $KC_f$, respectively. This idea is very old and successful for spaces of continuous or measurable functions. However, imitating this for spaces of functions of bounded variation one encounters some unpleasant new phenomena:

- If one wants to use the Schauder fixed point theorem, ons has to impose some compactness condition. However, compactness criteria in $BV$ and similar spaces which are both necessary and sufficient are not known.[10] Consequently, if one imposes a sufficient compactness condition on the linear operator (1) (as in Theorems 1 or 2), one usually looses a lot.

- If one wants to use the Banach fixed point theorem, one has to impose some Lipschitz condition to the nonlinearity involved. However, such Lipschitz conditions for $S_f$ and $C_f$ in $BV$ and similar spaces may lead to a serious degeneracy. For instance, it was shown by Matkowski and Miś [MM] that the superposition operator (3) satisfies a Lipschitz condition in $BV$ if and only if the generating function $f$ has an affine right regularization.[11] This means, roughly speaking, that the Banach fixed point principle applies only if the underlying problem is linear which is of course quite disappointing.

[10] A certain compactness criterion may be found in the Dunford-Schwartz Bible, but this criterion is extremely difficult to verify.

[11] In case of the composition operator (5) this condition is void.

All this emphasizes the need to apply more sophisticated methods for the solution of (7) or (9). For example, it is a useful device to replace fixed point methods by topological degree theory. An essential tool is here to find invariant balls for the operator $KS_f$ or $KC_f$, respectively. To this end, one has to "control" the growth of the nonlinearity $f$, e.g., by imposing conditions like

$$\text{(C10)} \qquad \lim_{|u|\to\infty} \frac{|f(u)|}{|u|} = 0.$$

Some existence (and in part uniqueness) results in this spirit by the author read as follows.

**Theorem 14** [BiGK]. *Let $g \in BV$, and suppose that conditions* (C1), (C9), and (C10) *are satisfied. Then for every $\lambda \in \mathbb{R}$ there exists a solution $x \in BV$ of equation* (8).

**Theorem 15** [BiCGS]. *Let $g \in \Lambda BV$ and $f \in C^1(\mathbb{R})$, and suppose that conditions* (C2) and (C10) *are satisfied. Then for every $\lambda \in \mathbb{R}$ there exists a solution $x \in \Lambda BV$ of equation* (8).

The following is an existence result for the nonautonomous case of the superposition operator (4).

**Theorem 16** [BiGK1]. *Let $g \in BV$, and let $f$ be a locally bounded Carathéodory function. Suppose that condition* (C1) *is satisfied, and that there exists $R > 0$ such that*

$$\|g\|_{BV} + T_0\alpha(R) < R,$$

*where*

$$T_0 := \|k(0,\cdot)\|_{L_1} + \|m\|_{L_1}, \qquad \alpha(r) := \sup\{|f(t,u)| : 0 \le t \le 1, |u| \le r\}.$$

*Then for $\lambda = 1$ there exists a solution $x \in BV$ of equation* (6) *satisfying $\|x\|_{BV} \le R$.*

Theorem 14 and Theorem 15 has been proved by applying the Schauder fixed point theorem, Theorem 16 by applying the Leray-Schauder degree. If one is interested in finding solutions in the space $IBV_p^q$ one may safely apply the Banach fixed point principle, since in that space global Lipschitz conditions for the operator $S_f$ do not lead to degeneracy of $f$, as Theorem 13 shows. A pleasant consequence is that we do not get only existence, but also uniqueness, at least for small values of the parameter.

**Theorem 17** [BiCGS]. *Let $g \in IBV_1^q$ and $f \in Lip(\mathbb{R})$, and suppose that conditions* (C6) and (C7) *are satisfied. Then there exists $\delta > 0$ such that, for every $\lambda \in \mathbb{R}$ with $|\lambda| < \delta$, there exists a unique solution $x \in IBV_1^q$ of equation* (8).

Along with (6), Gulgowski's result make it also possible to study the *Hammerstein-Volterra equation*

$$\text{(10)} \qquad x(t) = g(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds \qquad (t \ge 0)$$

with variable limit of integration. Equation (10) has nicer properties than (6), because $k(t,s) \equiv 0$ on the upper triangle $0 \le t \le s \le 1$. This has the pleasant consequence that the corresponding integral operator has often spectral radius zero, and so the problem of finding invariant balls for $\tilde{K}S_f$ becomes almost trivial. Indeed, one may make the right-hand side of (10) "arbitrarily small" by either taking $|\lambda|$ small, or by taking $t$ small. In the first case one has a large interval of existence, in the second case a large choice of parameters. We do not go into details; instead, let us briefly sketch the most important application, namely initial and boundary value problems.

**1.5. Boundary value problems.** It is well known that Hammerstein integral equations like (6) ot (8) occur quite naturally in the theory of *initial* or *boundary value problems.* For instance, the fact that Theorem 6 applies to the Riemann-Liouville operator (3) may be used to get existence of solutions to the fractional initial value problem

$$\begin{cases} x^{(\alpha)}(t) = f(x(t)) \ (0 < t \le 1), \\ x^{(\alpha-1)}(0) = x_0. \end{cases}$$

This was done by Gulgowski in his paper [G]. Moreover, in [G1] the author considers the Sturm-Liouville problem

$$\begin{cases} (p(t)x'(t))' + q(t)x(t) = h(t) \ (0 < t < 1), \\ l_0(x(0), x'(0)) = 0, \\ l_1(x(1), x'(1)) = 0, \end{cases} \tag{11}$$

where $l_0$ and $l_1$ are chosen to represent some boundary conditions. If

$$G(s,t) = \begin{cases} c^{-1}x_1(s)x_2(t) & \text{for} \quad 0 < s \le t < 1, \\ c^{-1}x_1(t)x_2(s) & \text{for} \quad 0 < t \le s < 1 \end{cases}$$

denotes the Green's function for (11), where $x_1$ and $x_2$ are appropriate fundamental solutions, Gulgowski gives in [G1] some sufficient conditions which guarantee that the integral operator

$$Kx(t) = \int_0^1 G(s,t)x(s)\,ds$$

maps the space $BV$ into itself and is bounded.

Jacek Gulgowski's application for being conferred with the degree of Dr. hab. is based on six papers [BiCGS,BiGK,BiGK1,G,G1,G2] described above. Apart from those papers, he provides a list of 17 other publications[12] which illustrate his broad range of interests: imbedding theorems, bifurcation theory, approximation theory, wavelets, and path following algorithms. Since those results are not intended for the habilitation procedure, I do not take them into account.

**2. Other activities.** Although this is probably not of utmost importance in judging the scientific quality of the habilitation procedure, I take the opportunity to comment on some additional topics which will be part of Gulgowski's scientific accomplishments and future career.

- **Teaching experience.** I cannot testify a direct experience of Jacek Gulgowski's classroom teaching. However, several colleagues who attended his talks during international conferences and workshops confirm that those talks have always been both clear and stimulating, and proved his qualities as a brilliant lecturer who attaches great importance to conveying ideas, rather than enumerating a boring list of definitions and theorems (which unfortunately is the case for so many other talks). So I take the liberty to claim that, "extrapolating" from this experience, it seems legitimate to presume that his students are also satisfied with his "down-to-earth"-classes, both in contents and style. Since 1997 he has given at Gdańsk University lectures on calculus, linear algebra, functional analysis, analysis on manifolds, differential equations (both ODE's and PDE's), algebraic topology, probability theory, data analysis, mathematical software, and mathematical economy. This impressive list clearly illustrates that he has a broad variety of interests. He also gave regular lectures at the Adam Mickiewicz University in Poznań during the last four years. This probably also included discussions with colleagues from Poznań, since there is a strong group of people working on BV-spaces there. Last but by no means least, Gulgowski gave many popular lectures for students and a non-specialized public, and also organized and co-organized students' competitions and scientific camps. This emphasizes his great efforts in the dissemination of mathematics among a general public.

- **Impact of his research.** Since in the last years I am also working in the field of BV-type spaces and their applications, I am not only perfectly aware of Gulgowski's results, but also know that they are often cited by other authors. Let me point out that I do not attach any importance to rankings expressed through impact factors or citation indices. These are purely formal and do not reflect the quality of a publication.

[12] As a matter of fact, his list also contains 4 contributions to Proceedings of international conferences, but I don't consider such contributions as "real" peer reviewed publications.

- **International visibility.** The fact that Gulgowski's results met, and continue to meet, the interest of an international community is illustrated by invitations to international conferences. He gave talks during meetings in Perugia (Italy), Lausanne (Switzerland), Belfast (Northern Ireland), and Amadora (Portugal). Moreover, he stayed for a period of some weeks in Ghent (Belgium) and Leiden (The Netherlands) in the framework of an international collaboration. Such an international "networking" becomes more and more important these times; although a mathematician often starts thinking on new problems alone (and at home), at a later stage the exchange of ideas on a worldwide scale is crucial. This cannot be done by email, but only in personal face-to-face discussions. As a modest criticism let me remark that for a mathematician of his age it seems strange that he never visited such mathematically active and important countries like Russia or the United States.

- **Academic supervision.** This is an important point: if Gulgowski will be granted with the degree of habilitated doctor, he will have the right to serve as supervisor of PhD theses of doctoral students. Based on the quality of his publications, his broad variety of interests, and his experience with students on any level, I am convinced that Gulgowski will be able to fulfill this task. To be more specific, let me point out that Gulgowski organized a series of "running seminars", a platform where students, either on a Master or PhD level, may present their ongoing work. In this way, students even at an early stage of their career face the criticisms of others, which is not only very important, but also useful, because it often leads to new insight. One student who attended these seminars, Klaudiusz Czudek, won an award for a paper which was published in 2017 in the Journal of Mathematical Analysis and Applications. Another PhD student, Małgorzata Lebiedź, works under Gulgowsk's co-supervision since June 2018.

**3. <u>Overall assessment</u>.** Summarizing, I may definitely confirm that Gulgowski is a renowned mathematician, with a wide variety of interests, a good teaching experience, and a sufficient international background. Based on my own experience after many years abroad, I may assert that the quality of the six papers he presents as a cumulative habilitation would meet the standards of a *Habilitation* in Germany, a *Doctoral Dissertation* in the former Soviet Union, a *Tesi di Libera Docenza* in Italy, or a *Thèse de Docteur Habilité* in France.

As far as I understand, the scientific degree of Dr. hab. is a necessary, but not sufficient, condition to obtain a tenure track position as a professor in Poland. I do hope that, after the positive outcome of this habilitation procedure, Gulgowski will succeed in obtaining such a position. His scientific and personal merits certainly deserve being acknowledged by providing him with a position as a tenure track professor.

**I strongly recommend that the Scientific Committee accepts the papers presented for Gulgowski's habilitation procedure for granting him the degree of habilitated doctor.**

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